Chapter Four

Linear Programming

<u>Objectives:</u>

a) Definition of Linear Programming

b) Forms of Linear Programming (General, Canonical and Standard)

c) Formation of the mathematical model of linear programming

d) Solving the mathematical models using graphical and simplex methods

e) Solving the mathematical models using M-technique and two-phase method

1) <u>Definition of Linear Programming</u>

Linear programming is a scientific method that aims to use the limited resources properly to achieve a specific goal.

The basic requirements for linear programming are:
a) The existence of a specific goal that is required to be achieved (such as the maximum profit or the lowest cost, ... etc).
b) There are different alternatives to reach the goal.
c) Resources used should be limited.
d) Obligation of relationship between variables
e) Expression of the objective function and constraints in linear equations or inequalities.

2) **Forms of Linear Programming**

a) General form: It takes the following form:

$$\max .\text{or} \min . \quad Z = \sum_{j=1}^{n} C_j X_j \qquad \text{Objective function}$$

$$S.t. \qquad \sum_{j=1}^{n} a_{ij} X_j \begin{Bmatrix} \le \\ \ge \\ = \end{Bmatrix} b_i \qquad \text{Constraints}$$

$$i = 1,2,...,m$$

$$j = 1,2,...,n$$

Where,

C_j represents the cost, time, profit or revenue, etc. per unit.

x_j represents the decision variables

a_{ij} represents the technical coefficients

b_i represents the availability amounts

b) Canonical form: The general model of this form is:

$$\max . \quad Z = \sum_{j=1}^{n} C_j X_j \qquad \text{Objective function}$$

$$S.t. \quad \sum_{j=1}^{n} a_{ij} X_j \le b_i \qquad \text{Constraints}$$

$$X_j \ge 0 \qquad \text{nonnegative constraints}$$

This form has the following characteristics:

a) All decision variables are non-negative (**$Xj \ge 0$**).

b) All constrains shall be of a smaller or equal type **($\le$)**.

c) The objective function is only maximized.

The general model can also be converted to canonical form using the following rules:

1) Minimized objective function can be converted into maximized and vice versa by multiplying the objective function by **(-1)**, i.e., **max. Z = min. (-Z)**

2) A constraint of greater than or equal to (≥) can be converted to less than or equal **(≤)** by multiplying the inequality by (-1), i.e,

$$\sum a_{ij} X_j \geq b_i \quad \Leftrightarrow \quad -\sum a_{ij} X_j \leq b_i$$

3) A constraint of equal sign **(=)** can be converted to two constraints of a type smaller than or equal (≤) as follows:

$$\sum a_{ij} X_j = b_i \quad \Leftrightarrow \quad \begin{Bmatrix} \sum a_{ij} X_j \leq b_i \\ -\sum a_{ij} X_j \leq -b_i \end{Bmatrix}$$

4) The absolute value of constraint can be converted to two constraints of a type smaller than or equal **(≤)** as follows:

$$\left|\sum a_{ij} X_j\right| \leq b_i \quad \Leftrightarrow \quad \begin{Bmatrix} \sum a_{ij} X_j \leq b_i \\ -\sum a_{ij} X_j \leq b_i \end{Bmatrix}$$

$$or \quad \left|\sum a_{ij} X_j\right| \geq b_i \quad \Leftrightarrow \quad \begin{Bmatrix} -\sum a_{ij} X_j \leq -b_i \\ \sum a_{ij} X_j \leq -b_i \end{Bmatrix}$$

5) The variable with <u>unrestricted sign</u> can be converted to two non-negative variables, as in the relationship below:

$$X_i = X_i^{'} - X_i^{''} \quad and \quad X_i^{'}, X_i^{''} \geq 0$$

c) Standard form: The general shape of this form takes the following characteristics:

a) All constraints are equations (constraints from a type of equality (=)) except the nonnegative constraint stays inequality from a type of greater than or equal to ($\mathbf{X_j \geq 0}$).

b) The right side of constraints is non-negative (i.e., $\mathbf{b_i \geq 0}$).

c) The objective function is either **min.** or **max.**

The general form can be converted to the standard form and in addition to what is stated in the canonical form, the constraints of the inequalities can be converted to equations and as follows:

$$\sum a_{ij} X_j \leq b_i \quad \Leftrightarrow \quad \sum a_{ij} X_j + S_i = b_i$$
$$\sum a_{ij} X_j \geq b_i \quad \Leftrightarrow \quad \sum a_{ij} X_j - S_i = b_i$$

Where, $\mathbf{S_i}$ represents the slack variables, which are imaginary variables and non-negative (i.e., $\mathbf{S_i \geq 0}$).

Example 1:

Convert the general form of linear programming to the canonical form and standard form:

$$\begin{aligned}
\text{min.} \quad & Z = 2X_1 + 3X_2 + 5X_3 \\
\text{s.t.} \quad & X_1 + X_2 - X_3 \geq -5 \\
& -6X_1 + 7X_2 - 9X_3 = 15 \\
& |19X_1 - 7X_2 + 5X_3| \leq 13 \\
& X_1, X_2 \geq 0, X_3 \text{ unrestricted}
\end{aligned}$$

Solution:

Assuming that $X_3 = X_3^{'} - X_3^{''}$

a) Canonical form:-

$$
\begin{aligned}
min. \quad & Z=-2X_1-3X_2-5(X_3^{'}-X_3^{''}) \\
s.t. \quad & -X_1-X_2+(X_3^{'}-X^{''})\leq 5 \\
& -6X_1+7X_2-9(X_3^{'}-X_3^{''})\leq 15 \\
& 6X_1-7X_2+9(X_3^{'}-X_3^{''})\leq -15 \\
& 19X_1-7X_2+5(X_3^{'}-X_3^{''})\leq 13 \\
& -19X_1+7X_2-5(X_3^{'}-X_3^{''})\leq 13 \\
& X_1,X_2,X_3^{'},X_3^{''}\geq 0
\end{aligned}
$$

b) Standard form:-

$$
\begin{aligned}
max. \quad & Z=2X_1+3X_2+5(X_3^{'}-X_3^{''}) \\
s.t. \quad & -X_1-X_2+(X_3^{'}-X^{''})+S_1=5 \\
& -6X_1+7X_2-9(X_3^{'}-X_3^{''})=15
\end{aligned}
$$

$$
\begin{aligned}
& 19X_1-7X_2+5(X_3^{'}-X_3^{''})+S_3=13 \\
& -19X_1+7X_2-5(X_3^{'}-X_3^{''})+S_4=13 \\
& X_1,X_2,X_3^{'},X_3^{''},S_1,S_2,S_3,S_4\geq 0
\end{aligned}
$$

3) Formulation of the mathematical model of linear programming

The mathematical model of linear programming can be formulated according to the data available to the researcher, as shown in the following example:

Example 2:

A factory produces three products A, B and C, and each product is achieved through three different processes. The time taken (min) to produce one unit of each product, the available capacity per process (min/day) and the profit per unit per product (thousand dinars) were as follows:

Process	**Taken time (min)**			**The available capacity (min/day)**
	A	***B***	***C***	
I	***1***	***2***	***1***	***430***
II	***3***	***0***	***2***	***460***
III	***1***	***4***	***0***	***420***
Profit	***3***	***2***	***5***	**-**

Required: **(i)** Formulate first the mathematical model of the linear programming of the above question to maximize the total profit. **(ii)** Then, reformulate the model for each of the following cases:

(A) Assuming a fourth product was added to the production process, the time taken in the three operations is (3, 5 and 1), respectively, the profit per unit is 6 thousand dinars, and the capacity available for the third process is utilized completely.

(B) Assuming that the sum of available non-utilized capacity for the three processes should not exceed 10 minutes / day.

(C) Assuming that the market studies indicated that the ratio of the number of units produced from the product A to the number of units produced from the products B and C must be not less than 0.4.

Solution:

(i) Assuming that X_1, X2 and X3 represent the number of units produced daily from products A, B and C, respectively. The mathematical model would be:

$$\begin{aligned} \max. \quad & Z = 3X_1 + 2X_2 + 5X_3 \\ s.t. \quad & X_1 + 2X_2 + X_3 \leq 430 \\ & 3X_1 + 2X_3 \leq 460 \\ & X_1 + 4X_2 \leq 420 \\ & X_1, X_2, X_3 \geq 0 \end{aligned}$$

(ii)

(A) The mathematical model would be:

$$\begin{aligned} \max. \quad & Z = 3X_1 + 2X_2 + 5X_3 + 6X_4 \\ s.t. \quad & X_1 + 2X_2 + X_3 + 3X_4 \leq 430 \\ & 3X_1 + 2X_3 + 5X_4 \leq 460 \\ & X_1 + 4X_2 + X_4 = 420 \\ & X_1, X_2, X_3, X_4 \geq 0 \end{aligned}$$

(B)

$$430 - (X_1+2X_2+X_3) + 460 - (3X_1+2X_3) + 420 - (X_1+4X_2) \leq 10$$

$$\rightarrow \quad 5X_1 + 6X_2 + 3X_3 \geq 1300$$

The mathematical model would be:

$$\begin{aligned} \max. \quad & Z = 3X_1 + 2X_2 + 5X_3 \\ s.t. \quad & X_1 + 2X_2 + X_3 \leq 430 \\ & 3X_1 + 2X_3 \leq 460 \\ & X_1 + 4X_2 \leq 420 \\ & 5X_1 + 6X_2 + 3X_3 \geq 1300 \\ & X_1, X_2, X_3 \geq 0 \end{aligned}$$

(C)

$$\frac{X_1}{X_2 + X_3} \geq 0.4 \Rightarrow \quad X_1 - 0.4X_2 - 0.4X_3 \geq 0$$

The mathematical model would be:

$$\begin{aligned} \max. \quad & Z = 3X_1 + 2X_2 + 5X_3 \\ s.t. \quad & X_1 + 2X_2 + X_3 \leq 430 \\ & 3X_1 + 2X_3 \leq 460 \\ & X_1 + 4X_2 \leq 420 \\ & X_1 - 0.4X_2 - 0.4X_3 \geq 0 \\ & X_1, X_2, X_3 \geq 0 \end{aligned}$$

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4 - Solving the linear programming model:

There are two basic ways to solve the linear programming models:

(a) Graphical method: This method is used in the case of a specified number of variables (two or three only) but does not give us the practical way to solve linear programs because most of the problems of linear programming include a large number of variables.

This method is based on drawing these constraints from their two intersection points with the coordinate's axes, and then defining the common region between these constraints called the **Feasible Solutions Region (FSR).** The angles of this region represent the **Extreme points** from which we obtain the optimal values for the two variables so that they achieve the objective function. This method is considered the basic method to reach the Simplex method.

Example 3:

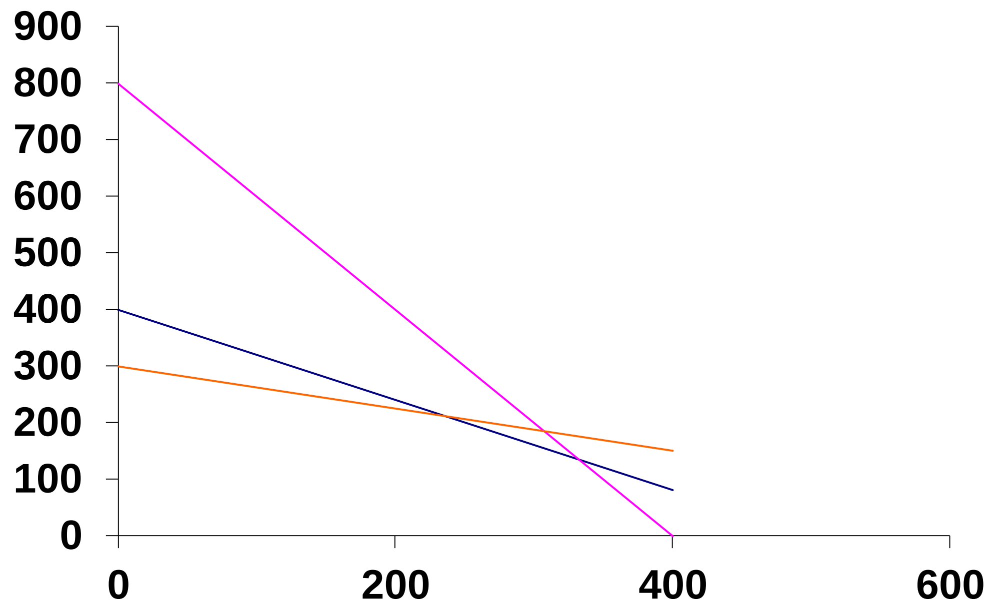
Solve the following linear programming model:

$$\begin{aligned} \max.\quad & Z = 120X + 100Y \\ s.t.\quad & 2X + 2.5Y \le 1000 \\ & 3X + 1.5Y \le 1200 \\ & 1.5X + 4Y \le 1200 \\ & X, Y \ge 0 \end{aligned}$$

Solution:

$$\begin{array}{llllllll} 1. & 2X + 2.5Y = 100 & if & X = 0 & then & Y = 400 & \Rightarrow & (0,400) \\ & & if & Y = 0 & then & X = 500 & \Rightarrow & (500.0) \\ 2 & 3X + 1.5Y = 1200 & if & X = 0 & then & Y = 800 & \Rightarrow & (0,800) \\ & & if & Y = 0 & then & X = 400 & \Rightarrow & (400,0) \\ 3. & 1.5X + 4Y = 1200 & if & X = 0 & then & Y = 300 & \Rightarrow & (0,300) \\ & & if & Y = 0 & then & X = 800 & \Rightarrow & (800,0) \\ 4. & X = 0 & & & & & & \\ 5. & Y = 0 & & & & & & \end{array}$$

These points are fixed graphically to determine the region of acceptable solutions (F.S.R.).



From the above diagram, we note that the extreme points are: A, B, C, D and E.

By solving equations 1 and 3, we obtain point C (4000/17, 3600/17).
By solving equations 1 and 2, we obtain point D (1000/3, 400/3).
By compensating these extreme points in the objective function, we find the optimal point:

Points	***Z = 120 X +100 Y***
A(0,0)	***Z = 0 + 0 = 0***
B(0,300)	***Z = 0 +100*300 = 30000***
C(4000/17, 3600/17)	***Z=120*(4000/17)+100*(3600/17)=840000/17***
D(1000/3,400/3)	***Z=120*(1000/3)+100*(400/3)=160000/3 →max.***
E(400,0)	***Z = 120* 400 + 0 = 48000***

Thus, the optimal solution is **X** = 1000/3 and **Y** = 400/3, and the value of the objective function **Z** = 16000/3.

Homework:

Find the optimal solution for the following linear programming to maximize as well as minimize the objective function.

$$\begin{aligned} & Z = 4X + 5Y \\ s.t. \quad & 2X + Y \le 6 \\ & X + 2Y \le 5 \\ & X - 2Y \le 2 \\ & -X + Y \le 2 \\ & X + Y \ge 1 \\ & X, Y \ge 0 \end{aligned}$$

(b) Simplex Method: This method is one of the most efficient mathematical means to find the optimal solutions to the problems of linear programming in general, as this method corresponds to the graphical method when **Si = 0**, and in the case of **Si** > **0**, any constraint will move down so that it becomes parallel to the same line when **Si = 0**.

This method does not look for all the possible basic solutions but it generates successive basic acceptable solutions so that each new solution has the potential to improve the objective function.

It is worth noting that this method is used only when all the constraints are from type smaller than or equal to **(≤)** provided that **bi** > **0**, except the non-negative constrain remains greater than or equal to (≥).

The main steps of the solution are:
i) Convert the mathematical model into the **standard form**.
ii) Choose the **Starting Basic Solution Feasible Solution (S.B.F.S.)**, as shown in the table below:

		C_1	C_2	...	C_n	0	0	...	0	0
B.C.	**B.V.**	X_1	X_2	...	X_n	S_1	S_2	...	S_m	**R.H.S.**
0	S_1	a_{11}	a_{12}	...	a_{1n}	1	0	...	0	b_1
0	S_2	a_{21}	a_{22}	...	a_{2n}	0	1	...	0	b_2
:	:	:	:	...	:	:	:	...	:	:
0	S_m	a_{m1}	a_{m2}	...	a_{mn}	0	0	...	1	b_m
$Z_j - C_j$		$-C_1$	$-C_2$	...	$-C_n$	0	0	...	0	0

iii) A new acceptable basic solution is selected so that the objective function improves by entering **a non-basic variable** whose value in the **Zj-Cj** row (coefficients of the objective function) is the **most negative** if the objective function is of the **max** type and the **most positive** value if the objective function is a **min** type (**Optimality condition** with ensuring that the values of the **R.H.S**. column are not negative).

While, the **Leaving variable** is determined as the lowest ratio resulted from dividing the **R.H.S** column by the corresponding positive values only from the **Entering Variable** column (**the Feasibility condition**), and the meeting element of the leaving variable row with the entering variable column is called the **Pivot element.**

iv) The entering variable is deleted from all the equations in the table except for the equation associated with the leaving variable, so the leaving variable row is divided by the pivot element and replaced by the entering variable. To delete the entering variable from the rest of the equations, the row conversions are done by multiplying the new row of entering variable by the element corresponding to the pivot element with a reverse sign for each row of the basic variables rows and gathered with the elements of old rows for each basic variable to obtain the new rows for them.

iiv) We continue to repeat the previous steps until all **Zj-Cj** row values become non negative if the objective function is of the **max** type, or is not positive if the function is of the min type. , That is, we stop when the value of the objective function cannot be improved and thus we have obtained the optimal solution to the problem.

Example 4: Solve example 3 by using the Simplex method.

Solution:

$$\begin{aligned} \max. \quad & Z = 120X + 100Y \\ s.t. \quad & 2X + 2.5Y + S_1 = 1000 \\ & 3X + 1.5Y + S_2 = 1200 \\ & 1.5X + 4Y + S_3 = 1200 \\ & X, Y, S_1, S_2, S_3 \geq 0 \end{aligned}$$

		120	100	0	0	0	0	
B.C.	B.V.	X	Y	S_1	S_2	S_3	R.H.S.	Ratio
0	S_1	2	2.5	1	0	0	1000	500
←0	S_2	3	1.5	0	1	0	1200	400 → min.
0	S_3	1.5	4	0	0	1	1200	800
$Z_j - C_j$		-120↑	-100	0	0	0	0	-
← 0	S_1	0	1.5	1	-2/3	0	200	133.3→min.
120	X	1	0.5	0	1/3	0	400	800
0	S_3	0	3.25	0	-0.5	1	600	184.6
$Z_j - C_j$		0	-40↑	0	40	0	48000	-
100	Y	0	1	2/3	-4/9	0	400/3	
120	X	1	0	-1/3	5/9	0	1000/3	
0	S_3	0	0	-13/6	17/18	1	500/3	
$Z_j - C_j$		0	0	80/3	200/9	0	160000/3	

Since all **Zj-Cj** row values are non-negative and the objective function is **max** type, therefore the solution is optimal, so **X** = 1000/3 and **Y** = 400/3 and the value of the objective function **Z** = 16000/3, which is the same solution we obtained in solving the example 3.

The calculations performed for the first repetition in the table above are:

Since **120** is the most negative value in the **Zj-Cj** row, so the entering variable is **X**, and since the lowest ratio is **400** in the **R.H.S.** column, so the leaving variable is $\mathbf{S_2}$, and to obtain the new row of **X**, we divide the old $\mathbf{S_2}$ row by **3**. And, to obtain the new $\mathbf{S_1}$ and $\mathbf{S_3}$ rows, we follow the following operations:

-2	-1	0	-2/3	0	-800	**New X row * - 2**
2	2.5	1	0	0	1000	**Old $\mathbf{S_1}$ row**
By addition						
0	1.5	1	-2/3	0	200	**New $\mathbf{S_1}$ row**

-1.5	-0.75	0	-0.5	0	-600	**New X row * -1.5**
1.5	4	0	0	1	1200	**Old $\mathbf{S_3}$ row**
By addition						
0	3.25	0	-0.5	1	600	**New $\mathbf{S_3}$ row**

Note: If at least one constrain is appeared from type of larger than (≥) or equal to (=), the simplex method cannot be applied, so the following methods can be used:

5) M - technique:

It may also be called the **Penalty method**, and as we mentioned earlier, this method is used when not all the constraints of a type smaller than or equal to (≤) provided that $\mathbf{b_i \geq 0}$, the basic steps of this method are :

i) The model is written in **standard form**.
ii) Artificial variables ($\mathbf{R_i}$) are added to the constraints of larger than or equal to (≥) or equality (=). The values of these variables in the final solution (optimal) must be equal to zero, in another meaning:

- If the constraint of a type smaller than or equal to (≤), the idle variable $\mathbf{S_i}$ is added.

- If the constraint is of a type larger than or equal to (≥), the idle variable $\mathbf{S_i}$ is subtracted and the variable R_i is added

- If the constraint is of equality (=), $\mathbf{R_i}$ is added the variable **Ri**. While, coefficients of the Artificial variables ($\mathbf{\underline{R_i}}$) in the objective function are (**-M**) in case of **max** and (**+M**) in case of **min**, considering that the value of **M** is too large. Whereas, coefficients of the idle variables $\mathbf{S_i}$ in the objective function remain **zero**.

C) Artificial variables ($\mathbf{\underline{R_i}}$) are used as basic variables for the constraints that exist in the primary acceptable basic solution (**S.B.F.S**.).

D) Continue to solve as in the simplix method taking into account that the value of **M** is too large and larger than the values in the table when determining the variables involved.

<u>Example 5</u>: Solve the following mathematical model:

$$\begin{aligned} min. \quad & Z = 5X_1 - 6X_2 - 7X_3 \\ s.t. \quad & X_1 + 5X_2 - 3X_3 \geq 15 \\ & 5X_1 - 6X_2 + 10X_3 \leq 20 \\ & X_1 + X_2 + X_3 = 5 \\ & X_1, X_2, X_3 \geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} min. \quad & Z = 5X_1 - 6X_2 - 7X_3 + MR_1 + MR_2 \\ s.t. \quad & X_1 + 5X_2 - 3X_3 - S_1 + R_1 = 15 \\ & 5X_1 - 6X_2 + 10X_3 + S_2 = 20 \\ & X_1 + X_2 + X_3 + R_2 = 5 \\ & X_1, X_2, X_3, S_1, S_2, R_1, R_2 \geq 0 \end{aligned}$$

		5	-6	-7	0	0	M	M	R.H.S.	Ratio
B.C.	**B.V.**	X_1	X_2	X_3	S_1	S_2	R_1	R_2		
←M	R_1	1	5	-3	-1	0	1	0	15	3min.
0	S_2	5	-6	10	0	1	0	0	20	-
M	R_2	1	1	1	0	0	0	1	5	5
$Z_j - C_j$		2M-5	6M+6↑	-2M+7	-M	0	0	0	20M	-
−6	X_2	1/5	1	-3/5	-1/5	0	1/5	0	3	-
0	S_2	31/5	0	32/5	-6/5	1	6/5	0	38	5.9
←M	R_3	4/5	0	8/5	1/5	0	-1/5	1	2	1.25min
$Z_j - C_j$		4/5M-31/5	0	8/5M+53/5↑	1/5M+6/5	0	-6/5M-6/5	0	2M-18	-
−6	X_2	1/2	1	0	-1/8	0	1/8	3/8	15/4	
0	S_2	3	0	0	-2	1	2	-4	30	
−7	X_3	1/2	0	1	1/8	0	-1/8	5/8	5/4	
$Z_j - C_j$		-23/2	0	0	-1/8	0	-M+1/8	-M-53/8	-125/4	

Therefore, **X1** = 0, **X2** = 15/4 and **X3** = 5/4, and the value of the objective function at its lower limit **Z** = -125/4.

Homework:

Solve the mathematical model for the following linear programming:

$$\min. \quad Z = X_1 - 3X_2 - 2X_3$$
$$3X_1 - X_2 + 2X_3 \le 7$$
$$-2X_1 + 4X_2 \le 12$$
$$-4X_1 + 3X_2 + 8X_3 \le 10$$
$$X_1, X_2, X_3 \ge 0$$

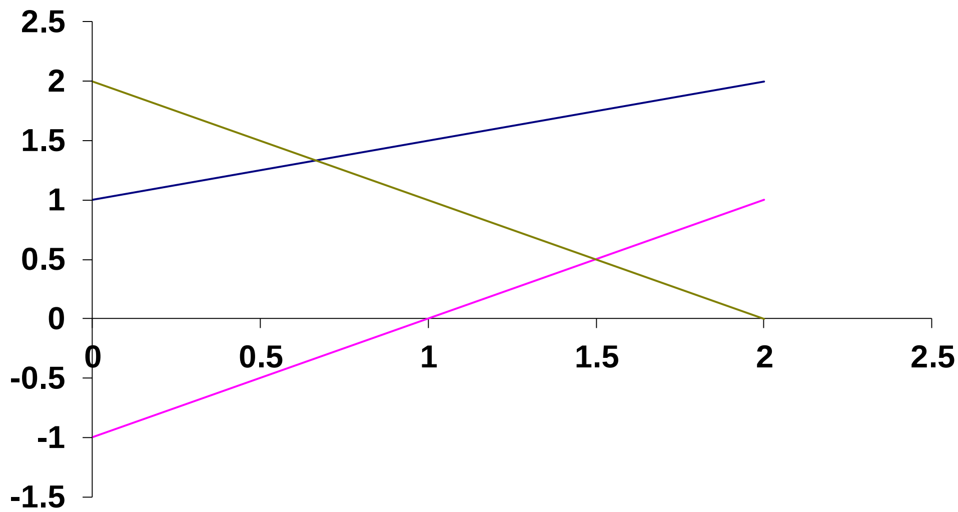
Tutorial No. 5

Q.: Solve the following linear programming model by the **Graphical Method:**

$$\begin{aligned} max. \quad & Z = 2X + 4Y + 8 \\ s.t. \quad & -X + 2Y \le 2 \\ & X - Y \le 1 \\ & X + Y = 2 \\ & X, Y \ge 0 \end{aligned}$$

Solution:

$$\begin{array}{llllllll}
1. & -X + 2Y = 2 & if & X = 0 & then & Y = 1 & \Rightarrow & (0,1) \\
 & & if & Y = 0 & then & X = -2 & \Rightarrow & (-2,0) \\
2 & X - Y = 1 & if & X = 0 & then & Y = -1 & \Rightarrow & (0,-1) \\
 & & if & Y = 0 & then & X = 1 & \Rightarrow & (1,0) \\
3. & X + Y = 2 & if & X = 0 & then & Y = 2 & \Rightarrow & (0,2) \\
 & & if & Y = 0 & then & X = 2 & \Rightarrow & (2,0) \\
4. & X = 0 & & & & & & \\
5. & Y = 0 & & & & & &
\end{array}$$



The **F.R.S.** is the line **AB**

The intersection of equations **2 and 3** gives point **B (3/2, 1/2).**

The intersection of equations **1 and 3** gives point **C (2/3, 4/3).**

Points	***Z = 2X + 4Y + 8***
O(0,0)	***Z =0***
A(1,0)	***Z = 10***
B(3/2 , 1/2)	***Z= 3 + 2 + 8 = 13***
C(2/3 , 4/3)	***Z= 4/3 + 16/3 + 8 = 44/3 → max.***
D(0,1)	***Z= 12***

The optimal solution is X = 2/3, Y = 4/3 and Z = 44/3

Chapter Five

Work Study

Objectives:

- **a)** Advantages and main steps of work study
- **b)** Determination of work content
- **c)** Determination of standard time

1. Advantages of work study

a) Adopting the scientific bases of rules of control and accuracy on the organization of payment of wages and incentives.

b) Success of controlling process on progress of works and costs.

c) Ensuring the satisfaction and achieving the stability of workers, which positively creates fruitful cooperation and serious interaction between workers at various levels of employment.

2. Determination of work content

Work study involves two main subjects:

First: Analyzing the existing working procedure and thinking to eliminate uneconomical movements in order to simplify and ease the work and then suggesting the most effective procedure of performance.

Second: Measuring the necessary time to perform the suggested procedure.

The main steps of determination of work study include the following:

a) Selection

b) Recording

c) Inspection

d) Development

e) Determination

f) Measuring

g) Measuring

h) Application

i) Following up

3. Determination of standard time

In order to determine the required time to perform the work by a worker with a **standard skill level**, the following steps are used:

a) Selecting the work that must be studied.

b) Record all information related to the work and the conditions under which it is done.

c) Inspecting, analyzing and testing the recorded data to remove the nonproductive elements as much as possible.

d) Measuring all the required elements to perform each stage and for a number of sufficient units in order to obtain the number of times that represents the realistic picture in the work.

e) Determining or describing the series of stages and details of proposed method of performance and the used devices and equipment.

Example:

The technological route to turn a workpiece with 10 mm diameter and 200 mm length consists of the following stages:

A: Fixing the workpiece in the lathe.

B: Approaching the cutting tool to the workpiece.

C: Running the machine.

D: Removing the cutting tool from the workpiece and stopping the machine to lift the workpiece.

E: Measuring the dimensions of the turned part.

Determine the standard time to perform this work with 90% confidence level.

Solution:

i) Determination of the observed time:

The work that was done by the worker was observed, the time that consumed to do each stage was recorded, this process was repeated for 10 times, and the following data were recorded:

Work stages	Observed time (second)										Mean	Range
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*		
A	*50*	*60*	*55*	*55*	*55*	*55*	*50*	*50*	*60*	*60*	*55*	*10*
B	*36*	*34*	*25*	*25*	*30*	*25*	*25*	*30*	*30*	*30*	*29*	*11*
C	*125*	*115*	*115*	*115*	*115*	*120*	*120*	*125*	*125*	*125*	*120*	*10*
D	*40*	*35*	*36*	*36*	*36*	*35*	*35*	*35*	*36*	*36*	*36*	*5*
E	*30*	*30*	*30*	*25*	*25*	*35*	*35*	*28*	*32*	*30*	*30*	*10*

Therefore, mean of the observed time to turn the workpiece is:

55 + 29 +120 + 36 + 30 = 270 seconds = 4.5 minute

ii) Determination of the required number of work cycles:

In order to obtain the required number of work cycles to be adopted to achieve the required accuracy to calculate the standard time, the following equation can be used:

$$m = \left(\frac{\alpha \sqrt{n \sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i \right)^2}}{\sum_{i=1}^{n} X_i} \right)^2$$

Where,

m = Number of work cycles

α = Level of required confidence to accept the results. At 90% confidence level, α = 20 and at 95% confidence level, $\alpha = 40$.

n = Number of observations.

X_i = The recorded time for the work stage that is characterized by a larger range.

If two stages or more have the same range, the range is taken for the stage that contains larger times.

And, since stage **B** has a larger range = 11, then

$$\sum_{i=1}^{10} X_i = 290 \quad and \quad \sum_{i=1}^{10} X_i^2 = 8552$$

$$m = \left(\frac{20 * \sqrt{10 * 8552 - (290)^2}}{290} \right)^2 = 6.75 \cong 7 < n = 10$$

Thus, the determined sample size by 10 readings is enough to express the reality and ensure the required accuracy to determine the standard time.

iii) Determination of the natural time:

Natural time is the time that the normal worker can perform the work with a performance rate of **100%**within the natural working conditions and without high stress. It is calculated by the following relation:

The natural time = The observed time x The estimated ratio of efficiency

There are **three estimated ratios of efficiency** in the industrial reality and these are:

Case one: If the worker performs at a **standard skill level**, the estimated efficiency ratio is 100% or 1.

Thus, **the natural time = the observed time = 4.5 minute**

Case two: If the worker performs at a level **less than the standard skill**, he cannot perform the work in the observed time for the normal worker,

and thus this time must increase by 20% as a maximum. According to the conditions and nature of the work and the degree of difficulty or lack both of them, so the natural time would be:

$$4.5 * \frac{120}{100} = 5.4\, min.$$

Case three: If the worker performs at a level **higher than the standard skill,** certainly he will be able to complete the work in less than the observed time for the normal worker, so the estimated efficiency ratio drops to 80% as a minimum, so the natural time would be:

$$4.5 * \frac{80}{100} = 3.6\, min.$$

iv) Addition of allowances:

In order to achieve the standard time accurately, it is necessary to take into account some of the allowances that the worker needs to compensate for his natural needs during performance, stress and delays that may occur during the implementation of the production stages. Therefore, the allowances are generally calculated as a percentage of the observed time not to exceed 15%. Therefore, the standard time for each case is as follows:

Case one (Standard skill): $$4.5 + 4.5 * \frac{15}{100} = 5.175 \, min.$$

Case two (Below standard skill): $$5.4 + 4.5 * \frac{15}{100} = 6.075 \, min.$$

Case three (Above standard skill): $$3.6 + 4.5 * \frac{15}{100} = 4.275 \, min.$$

<u>**Note:**</u> It is preferred to summarize the results by using a table at the end of the solution, as shown in this table:

Level of skill	Observed time (min)	Natural time (min)	Standard time (min)
Standard	4.5	4.5	5.175
Below standard	4.5	5.4	6.075
Above standard	4.5	3.6	4.275

Chapter Seven:

Transportation and Assignment Model

Objective:

(a) Finding the primary solution using the starting basic feasible solution (**S.B.F.S.**) methods

(b) Finding the optimal solution using Multipliers method

(c) Solving the assignment models in maximized and minimized objective function

First: Transportation Model

The transport model is one of the most important models of linear programming in industrial establishments. It is considered as a complement to the production process in order to supply it with the required production requirements at the specified time and place.

This model searches the transfer of a certain commodity from a number of sources, represented by centers of supply of raw materials for establishments, to different locations represented by demand centers (industrial establishments) at the lowest or lowest cost, provided that the supply at each source, the demand at each location and the cost of transporting one unit (or the taken time to transport the units) from each source to each specific and limited location.

(1) The least cost transportation problem

Assuming there are **m** of sources and **n** of locations and that:

S_i represents the number of units displayed at the source **i.**

D_j represents the number of units required at the location **j**.
C_{ij} represents the cost of transporting one unit at the path (**i, j**) that links the source **i** to the location **j.**

X_{ij} represents the number of units transported from the source **i** to the location **j**.

Therefore, the main objective is to determine the number of units transported from the source **i** to the location **j** so that the total transportation cost (**TTC**) is as minimum as possible.

And assuming that the costs are linear, the linear programming model for the transportation problem is:

$$\begin{aligned} \textbf{min.} \quad & Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} X_{ij} \\ \textbf{s.t.} \quad & \sum_{j=1}^{n} X_{ij} = a_i \\ & \sum_{i=1}^{m} X_{ij} = b_j \\ & X_{ij} \geq 0 \end{aligned}$$

Sometimes, sum of the supply at the sources and sum of the demand at the locations may be unequal. In this case, the model is unbalanced, and to achieve the balance, we follow:

i) If the demand is greater than the supply, we add an imaginary source so that it supplies the shortage of quantity that amounts $\sum_j b_j - \sum_i a_i$.

ii) If the demand is smaller than the offer, we add an imaginary location to absorb the excess quantity that amounts $\sum_i a_i - \sum_j b_j$.

The cost of transporting one unit from these imaginary sources or to these imaginary locations is equal to zero.

The main steps used to solve the transportation model at the lowest cost are:

1- Determine the starting basic feasible solution (**S.B.F.S.**)

2- Determine the entering variable among the non-basic variables, if all variables achieve the optimality condition, then stop, and otherwise go to the next step.

3- Determine the leaving variable (using the feasibility condition) among the variables of the current basic solution, then find the new basic solution and return to the previous step.

(2) Methods of the starting basic feasible solution (S.B.F.S.)

They give us a solution can be launched from it to reach the optimal solution, namely:

a) Northwest Corner Method:

This method is the simplest methods, which starts by specifying the highest allowed amount of supply and demand to X_{11} (in the far north-west corner of the table), i.e. $X_{11} = \min (a_1, b1)$. Then, we exclude the verified column (row) and then equalize the remaining variables of the excluded column (row) by zero. After adjusting the supply and demand quantities for all non-excluded rows and columns, we specify the maximum accepted cell of the first element that is not excluded in the new row. This process is completed when exactly one row or one column is not excluded.

b) Least cost method

It is better than the previous method because it takes the costs into account. The procedure used in this method is to determine the available quantity of the least expensive variable per unit, exclude the verified column (row), then adjust the supply and demand for all non-excluded elements, repeat the process by determining the available quantity of the non-excluded least expensive variable per unit and we continue to solve until we have one row (column) is not excluded.

c) Vogel's Approximation Method (V.A.M.)

This method is better than the previous ones since it give us a solution closer to the optimality because it takes the costs of penalty into consideration, as explained in the following steps:

i) Estimating the cost of penalty for each column and for each row by subtracting the value of less than two consecutive costs of the same row or column.

ii) Determine the row or column that has the largest cost of penalty, allocate the available quantity of the least expensive variable in the selected row or column and then modify the supply and demand after deleting the verified row (column).

iii) (1) If we have only one row (column) not deleted, we determine the basic variables in the row (column) by the least expensive way.

(2) If all rows and columns that are not deleted have zero offer and demand, the zero basic variables will be determined by the least cost way.

(3) Otherwise, we recalculate the cost of penalty for the non-deleted rows and columns and then return to step (**ii**) (noting that the penalty costs of rows and columns, which their offer and demand are zero, are not calculated).

Note that if the maximum penalties are equal, we choose the row (column) which has the lowest transportation cost, if the lowest transfer costs are equal, we choose the row (column) which transports the largest amount and if the largest amounts of transportation are equal, we choose the row (column) randomly.

d) Russel's Approximation Method (R.A.M.)

This method is better than the previous ones because it gives us a preliminary solution closer to the optimal solution (especially for large matrices) and its steps are:

i) Determine the highest transportation cost for each row (symbolized by $\overline{a_i}$) and for each column (symbolized $\overline{b_j}$).

ii) Form a new matrix commissioned: $\Delta_{ij} = C_{ij} - \overline{a_i} - \overline{b_j}$

iii) Determine the cell that has the lowest transportation cost Δ_{ij} and give its variable as much as possible which is equal to min. (a_i, b_j).

iv) By deleting the verified row (column) and changing quantity of supply of the row or demand of the column in which the cell is located to the amount of difference between the quantities of supply and demand corresponding to them.

iiv) (1) If one row (column) is remained, we give the remaining row (column) the remaining demand and supply quantities.

(2) If more than one row (column) is remained, we return to step (**a**).

General Note: For all previous methods, if a column and row are verified together, we delete one of them only and make the other zero, and that ensures specifying zero values for the basic variables.

(3) Finding the optimal solution using Multipliers method

This method is used to test and improve the primary solution (**S.B.F.S.**) for reaching the optimal solution after verifying the basic condition: the number of basic cells equals **to m+n-1**, where **n** is the number of columns and **m** is the number of rows. Also, this method is called Modified Distribution Method (MODI) and estimates the cells of basic variable.

In transportation table, the Multiplier $\mathbf{U_i}$ shares with each row **i** and the Multiplier **j** with each column $\mathbf{V_j}$ **,** and the equation for each basic variable $\mathbf{X_{ij}}$ is written in the current solution:

$$U_i + V_j = C_{ij}$$

(m+n-1) of equations will form (due to existence of (m+n-1) basic variables) which have (m+n) of unknowns. We can estimate the multipliers values from these equations by assuming a random value for one multiplier (usually we assume $U_1 = 0$) and then solve the equations, which their numbers will be equal to their unknowns and then we estimate the new cost $\bar{C}_{pq}$ for each non-basic variable X_{pq} is,

$$\bar{C}_{pq} = C_{pq} - (U_p + V_q)$$

So, we choose the entering variable to be the <u>largest negative</u> value to $\bar{C}_{pq}$ (optimality condition in the simplex method) and by using the closed loop of the entering variable as explained previously and determine the leaving variable which has the least expensive for the cells that take the negative sign in the loop (feasibility condition in the simplex method).

<u>Example:</u>

Three reservoirs S_1, S_2, S_3 can pump 15, 20 and 25 million liter of clear water daily, supplying four cities C_1, C_2, C_3, C_4 that need 8, 10, 12 and 15 million liter of clear water per day. It is required to reach an arrangement to transport the pure water between the three reservoirs and the four cities with the lowest total costs of transportation (assuming that storing the excess water does not cause any cost) according to the transportation costs (per million liter) shown in the table below:

	C_1	C_2	C_3	C_4
S_1	2	3	4	5
S_2	3	2	5	2
S_3	4	1	2	3

<u>Solution:</u>

Due to the imbalance because the sum of pumping quantities (25 + 20 + 15 = 60) is greater than the sum of the demand quantities (8 + 10 + 12 + 15 = 45), so we add an imaginary city C_5, where the costs of transporting clear water to it are equal to zero and the amount of its supply (60 - 45 = 15) million liter of clear water.

(1) <u>Finding the primary solution (S.B.F.S.) by using the following four methods:</u>

a) Using the north-west corner method

	C_1	C_2	C_3	C_4	C_5	Supply
S_1	[2] 8	[3] 7	[4]	[5]	[0]	15
S_2	[3]	[2] 3	[5] 12	[2] 5	[0]	20
S_3	[4]	[1]	[2]	[3] 10	[0] 15	25
Demand	8	10	12	15	15	60

Therefore, the total transport cost (**TTC**) is:

$$T.T.C. = 2 * 8 + 3 * 7 + 2 * 3 + 5 * 12 + 2 * 5 + 3 * 10 + 0 * 15 = 143$$

b) Using the least cost method:

	C_1	C_2	C_3	C_4	C_5	Supply
S_1	[2] 0	[3]	[4]	[5]	[0] 15	15
S_2	[3] 5	[2]	[5]	[2] 15	[0]	20
S_3	[4] 3	[1] 10	[2] 12	[3]	[0]	25
Demand	8	10	12	15	15	60

Therefore, the total transport cost is:

$$T.T.C. = 2*0 + 0*15 + 3*5 + 2*15 + 4*3 + 1*10 + 2*12 = 91$$

c) Using the Vogel Approximation Method (VAM)

	C_1	C_2	C_3	C_4	C_5	Supply	P.C.
S_1	2 *0*	3	4	5	0 *15*	*15*	<u>2</u> 1 1 <u>3</u>
S_2	3 *5*	2	5	2 *15*	0	*20*	2 0 0 1 1
S_3	4 *3*	1 *10*	2 *12*	3	0	*25*	1 1 2 1 1
Demand	*8*	*10*	*12*	*15*	*15*	*60*	
P.C.	1 1 1 1 1	1 1 1	2 <u>2</u> <u>2</u>	1 1 1 1 1	0		

Therefore, the total transport cost is:

$$T.T.C. = 2*0 + 0*15 + 3*5 + 2*15 + 4*3 + 1*10 + 2*12 = 91$$

d) Using Russel's Approximation Method (RAM)

	C_1	C_2	C_3	C_4	C_5	Supply
S_1	2 *8*	3	4	5	0 *7*	*15*
S_2	3	2	5	2 *15*	0 *5*	*20*
S_3	4	1 *10*	2 *12*	3	0 *3*	*25*
Demand	*8*	*10*	*12*	*15*	*15*	*60*

The final table for this method is extracted from the tables below:

	C_1	C_2	C_3	C_4	C_5
S_1	**-7**	**-5**	**-6**	**-5**	**-5**
S_2	**-6**	**-6**	**-5**	**<u>-8</u>**	**-5**
S_3	**-4**	**-6**	**-7**	**-6**	**-4**

Fill the cell $\mathbf{X_{24}}$ and delete the location $\mathbf{C_4}$:

	C_1	C_2	C_3	C_5
S_1	**-6**	**-4**	**-5**	**-4**
S_2	**-6**	**-6**	**-5**	**-5**
S_3	**-4**	**-6**	**<u>-7</u>**	**-4**

Fill the cell $\mathbf{X_{33}}$ and delete the location $\mathbf{C_3}$:

	C_1	C_2	C_5
S_1	**-5**	**-3**	**-3**
S_2	**-4**	**-4**	**-3**
S_3	**-4**	**<u>-6</u>**	**-4**

Fill the cell $\mathbf{X_{32}}$ and delete the location $\mathbf{C_2}$:

	C_1	C_5
S_1	**-4**	**-2**
S_2	**-4**	**-3**
S_3	**-4**	**-4**

Fill the cell $\mathbf{X_{35}}$ and delete the location $\mathbf{S_3}$:

	C_1	C_5
S_1	**-3**	**-2**
S_2	**-3**	**-3**

Fill the cell $\mathbf{X_{25}}$ and delete the location $\mathbf{S_2}$**:**

So, the residual values of the remaining two cells are given X_{15}, X_{11} for stay a single row.

Therefore, the total transport cost is:

$$T.T.C. = 2*8 + 0*7 + 2*15 + 0*5 + 1*10 + 2*12 + 0*3 = 80$$

From the above, we note that the total cost of transportation using the four methods was different and as follows:

Northwest-corner (**143**) > Least cost (**91**) $\geq$ Vogel VAM (**91)** > Russell RAM (**80**).

So the Russell RAM method is usually the best, followed by the **VAM** method.

Based on the primary solution (**S.B.F.S.**), which we obtained in the third way **VAM** (although it is better to use the fourth method RAM as the best method, but because of the review of optimal solution method, this method is chosen) and for the purpose of reaching to the optimal solution, the following method must be used to test and improve the solution and after verifying the basic condition:

No. of basic cells = m + n - 1 = 5 + 3 - 1 = 7

(2) Finding the optimal solution by using multipliers method:

As we mentioned earlier, we find the values of $\mathbf{U_i}$ and $\mathbf{V_j}$ by the following relationship are: $\mathbf{U_i + V_j = C_{ij}}$ for the basic cells, assuming that: $U_1 = 0$, and based on the primary solution obtained by VAM method,

	C_1	C_2	C_3	C_4	C_5	Supply
S_1	[2] 0	[3]	[4]	[5]	[0] 15	15
S_2	[3] 5	[2]	[5]	[2] 15	[0]	20
S_3	[4] 3	[1] 10	[2] 12	[3]	[0]	25
Demand	8	10	12	15	15	60

T.T.C. = 91 and no. of basic cells = 7

$$C_{11} = U_1 + V_1 = 2 \overset{U_1=0}{\Rightarrow} V_1 = 2$$
$$C_{15} = U_1 + V_5 = 0 \overset{U_1=0}{\Rightarrow} V_5 = 0$$
$$C_{21} = U_2 + V_1 = 3 \overset{V_1=2}{\Rightarrow} U_2 = 1$$
$$C_{24} = U_2 + V_4 = 2 \overset{U_2=1}{\Rightarrow} V_4 = 1$$
$$C_{31} = U_3 + V_1 = 4 \overset{V_1=2}{\Rightarrow} U_3 = 2$$
$$C_{32} = U_3 + V_2 = 1 \overset{U_3=2}{\Rightarrow} V_2 = -1$$
$$C_{33} = U_3 + V_3 = 2 \overset{U_3=2}{\Rightarrow} V_3 = 0$$

As for the non-basic cells, we find $\overline{C}_{ij}$ for them the relationship $\overline{C}_{ij} = C_{ij} - (U_i + V_j)$ and as follows:

$$\overline{C}_{12} = C_{12} - (U_1 + V_2) = 3 - (0 + (-1)) = 4$$
$$\overline{C}_{13} = C_{13} - (U_1 + V_3) = 4 - (0 + 0) = 4$$
$$\overline{C}_{14} = C_{14} - (U_1 + V_4) = 5 - (0 + 1) = 4$$
$$\overline{C}_{22} = C_{22} - (U_2 + V_2) = 2 - (0 - 1) = 2$$
$$\overline{C}_{23} = C_{23} - (U_2 + V_3) = 5 - (1 + 0) = 4$$
$$\overline{C}_{25} = C_{25} - (U_2 + V_5) = 0 - (1 + 0) = -1$$
$$\overline{C}_{34} = C_{34} - (U_3 + V_4) = 3 - (2 + 1) = 0$$
$$\overline{C}_{35} = C_{35} - (U_3 + V_5) = 0 - (2 + 0) = -2 \text{ most negative}$$

The entering variable will be the most negative variable for the values of $\overline{C}_{ij}$ and it is X_{35}, while the leaving variable is determined by the stepping path of the entering variable $\overset{+}{X}_{35} \rightarrow \overset{-}{X}_{15} \rightarrow \overset{+}{X}_{11} \rightarrow \overset{-}{X}_{31}$. And the cell with the least transportation quantity from the negative cells will be determined as a leaving variable, i.e. X_{31}. While the new table will be:

	C_1	C_2	C_3	C_4	C_5	Supply
S_1	[2] 3	[3]	[4]	[5]	[0] 12	15
S_2	[3] 5	[2]	[5]	[2] 15	[0]	20
S_3	[4]	[1] 10	[2] 12	[3]	[0] 3	25
Demand	8	10	12	15	15	60

T.T.C. = 6 + 0 + 15 + 30 + 10 + 24 + 0 = 85

No. of basic cells = m + n – 1 = 3 + 5 – 1 = 7

Calculations can be performed to extract values of $\overline{C}_{ij}$ directly on the table and as shown in the bottom box of each non-basic cell in the table below:

		V_1=2	V_2=1	V_3=2	V_4=1	V_5=0	
		C_1	C_2	C_3	C_4	C_5	***Supply***
U_1=0	S_1	2	3	4	5	0	
		3	2	2	4	12	15
U_2=1	S_2	3	2	5	2	0	
		5	0	2	15	-1	20
U_3=0	S_3	4	1	2	3	0	
		2	10	12	2	3	25
Demand		8	10	12	15	15	60

Thus, the entering variable is $\mathbf{X_{25}}$**,** considering that it has a negative value $\overline{C}_{ij}$, while the leaving variable $\mathbf{X_{21}}$ is determined by the stepping path of this entering variable. The new table will be:

		V_1=2	V_2=1	V_3=2	V_4=2	V_5=0	***Supply***
		C_1	C_2	C_3	C_4	C_5	
U_1=0	S_1	2	3	4	5	0	
		8	2	2	3	7	15
U_2=0	S_2	3	2	5	2	0	
		1	1	3	15	5	20
U_3=0	S_3	4	1	2	3	0	
		2	10	12	1	3	25
Demand		8	10	12	15	15	60

T.T.C. = 16 + 0 + 30 + 0 + 10 + 24 + 0 = 80

Because there is no negative value for $\overline{C}_{ij}$ values (whose values are fixed in the lower box of the non-basic cells in the table above), thus the solution is optimal. Accordingly:

The first reservoir supplies the first city 8 million liter of clear water. The second reservoir supplies the fourth city 15 million liter of clear water. The third reservoir supplies the second and third cities with 10 and 12 million liters of clear water, respectively.

Homework:

Find the optimal solution for the following transport problem:

Sou.	*Dest.*				*Sup.*
	D1	***D2***	***D3***	***D4***	
S1	***10***	***20***	***5***	***7***	***10***
S2	***13***	***9***	***12***	***8***	***20***
S3	***4***	***15***	***7***	***9***	***30***
S4	***14***	***7***	***1***	***0***	***40***
S5	***3***	***12***	***5***	***19***	***50***
Dem.	***60***	***60***	***20***	***10***	***150***

Chapter 8:

Network Planning

1. Critical Path:

These networks are widely used to control the stages of project construction and implementation as well as in the stages of manufacturing or assembling the commodities. This is done by analyzing and coordinating the activities and events necessary for production in the form of interconnected business networks and schedules to guide the implementation of these works.

In general, the elements for drawing the network plans and setting up the follow-up and control schedules are:

- **Event:** It is referred to by a circle having a special number that should not be repeated, and indicates the order of the event only. Each network has one start event and one end event, and the event does not need time or resources to implement.

- **Activity:** It is referred to as one arrow. Any activity should not be represented by more than one arrow. Any activity needs time and resources for its implementation and the time to perform the activity is usually set over each arrow, noting that there is no relationship between the length of the arrow and the time required to perform it. Each activity has a start event and end event. Two activities may be involved in the same start event, but the end event is different for each, or two activities can be involved in the same end event, but the start event is different for each, and two activities should not participate in the same start event and the same end event.

- **Path:** It represents a series of successive arrows beginning with the start event and ending with the end event, each path is usually distinguished by the numbers of events that pass, and **the path that takes the longest time is called Critical Path (CP).** The activities of this path are critical ones, since any delay occurs during implementing any of its activities leads to delay the implementation of the work. Therefore, the time of critical path determines the time required to complete the work.

Calculation of the **critical path time** has two stages:

The first stage - It is called the **forward pass,** where the calculations start from the starting point towards the ending point, and at each point, the earliest time **(ES_j)** is calculated from the relationship:

$$ES_j = \max_i .\{ES_j + D_{ij}\} \quad \forall (i, j) activities$$

Considering that **$ES_1 = 0$, D_{ij}** represents the time required to complete the activity **(i, j)**, and the value is placed in the square shape.

The second stage – It is called the **backward pass**, where the calculations start from the ending point towards the starting point, and at each point, the latest time **(LC_i)** is calculated from the relationship:

$$LC_i = \min_j .\{LC_i - D_{ij}\} \quad \forall (i, j) activities$$

Considering that **$LC_n = ES_n$** , the value is placed in the triangle shape.
Each activity **(i, j)** located on the critical path must achieve:

$$ES_j - ES_i = LC_j - LC_i = D_{ij}$$

The Free Float time (F.F.) represents the time surplus available to reach a particular event and is calculated from the relationship:

$$FF_{ij} = ES_j - ES_i - D_{ij}$$

Example 1: Draw the network planning for the following projects:

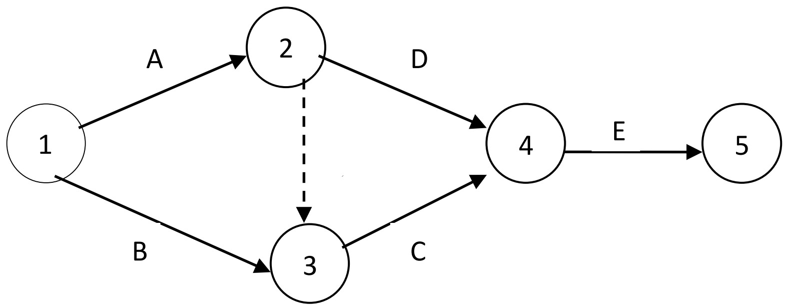
a)

Act.	Pre-activity
A	----
B	----
C	A,B
D	A
E	C,D

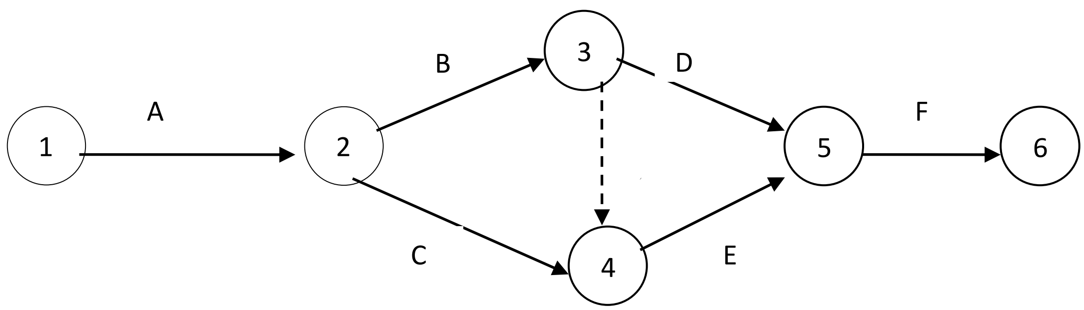
b)

Act.	Pre-activity
A	----
B	A
C	A
D	B
E	B,C
F	D,E

a)



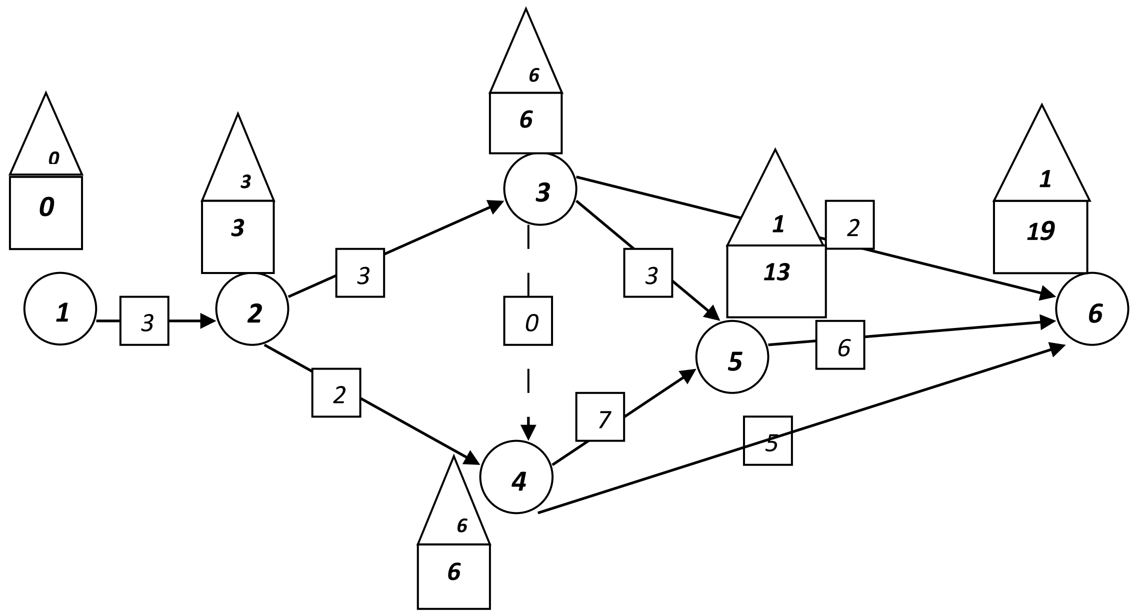
b)



Example 2: The following table represents the requirements for the manufacture of a particular commodity with nine activities. Find the critical path to manufacturing this commodity:

Activity	***1-2***	***2-3***	***2-4***	***3-4***	***3-5***	***3-6***	***4-5***	***4-6***	***5-6***
D_{ij}	***3***	***3***	***2***	***0***	***3***	***2***	***7***	***5***	***6***

Solution:



Forward pass	*Backward pass*
$ES_1 = 0$	$LC_6 = 19$
$ES_2 = 0 + 3 = 3$	$LC_5 = 19 - 6 = 13$
$ES_3 = 3 + 3 = 6$	$LC_4 = \min. \{ 13-7, 19-5 \} = 6$
$ES_4 = \max. \{ 3+2, 6+0 \} = 6$	$LC_3 = \min. \{ 6-0, 13-3, 19-2 \} = 6$
$ES_5 = \max. \{ 6+3, 6+7 \} = 13$	$LC_2 = \min. \{ 6-3, 6-2 \} = 3$
$ES_6 = \max. \{ 6+2, 6+5, 13+6 \} = 19$	$LC_1 = 3 - 3 = 0$

Therefore, the critical path to manufacture the commodity is: **1-2-3-4-5-6** with the critical activities: **(1,2) , (2,3) , (3,4) , (4,5) , (5,6)**, the critical time is **19**.

2- <u>Program Evaluation and Review Technique (PERT):</u>

The PERT method is one of the most modern management methods to control the stages of manufacturing and derives its importance in the practical life because it identifies the critical activities that necessarily need attention, more observation than others, and care to provide all the necessary supplies and requirements for their implementation on the limited time. In addition, the calculation of the free float time between the activities helps to guide and transfer the surplus financial and human resources from the non-critical activities to critical ones. In this case, the best ways to shorten the time and labor costs are prepared. In this way, three types of times are studied:

- **Optimistic time (a)**, which considers that the implementation will be done very good.
- **Pestimistic time (b)**, which considers that the implementation will be done very poor.
- **Most likely time (m)**, which considers that the implementation will be done normally.

While, **the expected time for the activity (i, j)** is calculated from the relationship:

$$\overline{D} = \frac{a + b + 4m}{6}$$

While, the **Variance (V)** for each activity is calculated from the relationship:

$$V = \left(\frac{b - a}{6} \right)^2$$

Therefore, the probability of implementing the project on time will be:

$$Pr\left(Z \le \frac{ST_i - CT_i}{\sqrt{V(\mu_i)}} \right)$$

Where,

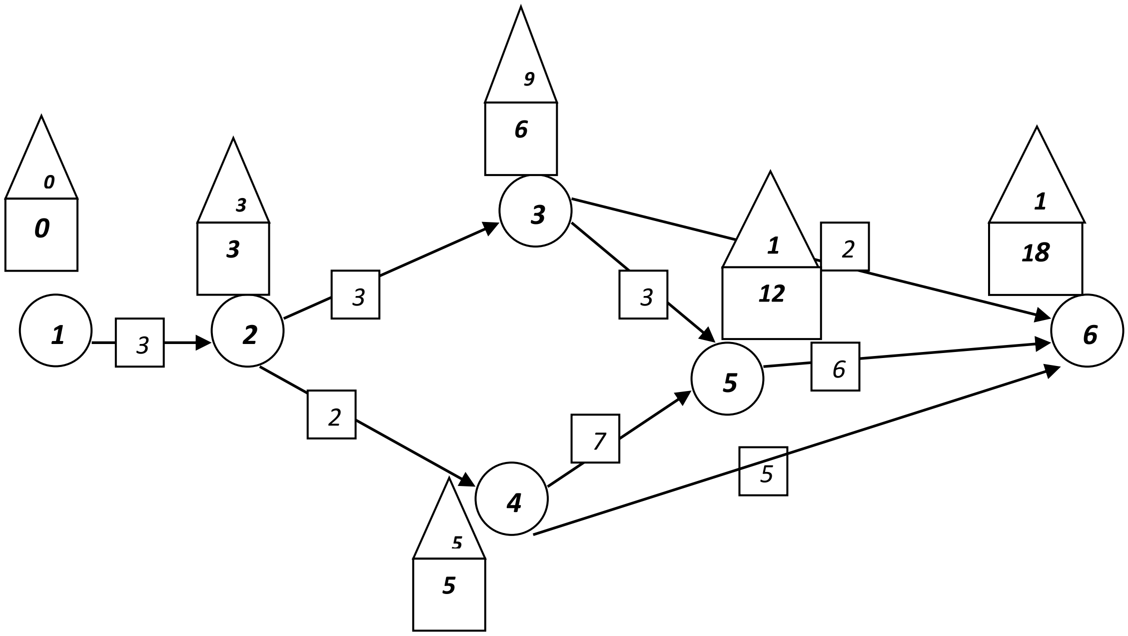
ST_i represents the limited time to complete the project.
CT_i represents the critical time of the project.
V (μ) represents the sum of variances of the critical activities of the project.
The probability value above can be found from the **normal distribution table**.
Note that the **standard deviation** is the square root of the **variance**.

<u>**Example 3:**</u> The following data show the implementation times of each activity of an industrial project. Calculate the probability of project implementation within 20 months.

Activity	a	b	m
1,2	2	8	2
2,3	1	11	1.5
2,4	0.5	7.5	1
3,5	1	7	2.5
3,6	1	3	2
4,5	6	8	7
4,6	3	11	4
5,6	4	8	6

Solution:

Activity	$\overline{D}$	V
1,2	3	1
2,3	3	---
2,4	2	1.36
3,5	3	----
3,6	2	----
4,5	7	0.11
4,6	5	----
5,6	6	0.44
	V(μ)	2.91



Thus, the critical path is represented by the activities: **(1,2), (2,4), (4,5), (5.6),** and the critical time: **CT = 18**.

$$Pr\left(Z_i \le \frac{20-18}{\sqrt{2.91}} \right) = Pr(Z \le 1.17) = 0.879$$

In other words, the probability of completing the project in **20** months is approximately **88%**.

Homework:

1) Draw the network planning for the data given in the following table.

2) Find the critical path for this network.

3) Find the critical time.

Activity	*Pre. Act.*	*Duration*		*Activity*	*Pre. Act.*	*Duration*
R	**----**	***24***		***D***	***C , B***	***6***
E	***R***	***16***		***C***	***A***	***8***
H	***G***	***16***		***B***	***A***	***16***
N	***P, Q, U, S***	***8***		***U***	***F***	***8***
M	***L , K***	***8***		***Q***	***E***	***12***
K	***H***	***16***		***A***	***R***	***16***
P	***E , D***	***36***		***F***	***R***	***40***
S	***T , M***	***16***		***G***	***R***	***24***
L	***+*** ***H***	***24***		***T***	***G***	***4***

Chapter 9:

Sequencing Models

Sequencing (consecutive) models generally aim to find the optimal sequence to implement various jobs as they pass through **m** machines (**m** = 1, 2, 3,...) in addition to obtain the least total implementation time and find (idle time) for each machine.

The general assumptions that the sequencing models rely on are:
1. Each job has a beginning and an end.
2. Only one job can be performed on a particular machine at a specific time.
3. The job must be completed before starting the next job.
4- There is only one machine of every kind.
5. The job must be fully prepared when the time of implementation starts.
6. The time required to transfer the job from machine to machine can be neglected.
7 - It is assumed that there is no failure that would disrupt or stop the work such as maintenance or change in work shifts or lack of any of the factors of production.

So, these models will take the following cases:

1- Processing n jobs through one machine:

In this case, **n** jobs are performed through only one machine under the following algorithm:
(A) Arranging the jobs according to the time taken ascending or descending.
(B) We find the shortest processing time (S.p.t.) by dividing the sum of the jobs finishing times for the **ascending order** by the number of jobs.
(C) We find the longest processing time (L.p.t) by dividing the sum of the jobs finishing times for the **descending order** by the number of jobs.

Example 1: Six jobs are performed on one machine and their taken times (hours) for each job are:

Jobs	A	B	C	D	E	F
Time	8	6	2	7	10	4

Find the least time to perform all jobs according to the two measures:

(**a**) The shortest processing time (S.p.t.)

(b) The longest processing time (L.p.t.)

Solution:

1. According to the ascending order:

Sequence	Jobs	Time	Processing	
			Start	Finish
1	C	2	0	2
2	F	4	2	6
3	B	6	6	12
4	D	7	12	19
5	A	8	19	27
6	E	10	27	37
∑				103

S.p.t = 103/6 = 17.16 hrs.

2. According to the descending order:

Sequence	Jobs	Time	Processing	
			Start	Finish
1	E	10	0	10
2	A	8	10	18
3	D	7	18	25
4	B	6	25	31
5	F	4	31	35
6	C	2	35	37
∑				156

L.p.t = 156/6 = 26 hrs

Note: The optimal sequence of jobs can be found if there are different weights for each job by finding the modified time ($\bar{t}$) by dividing the time taken for each job $\mathbf{t_i}$ by the weights corresponding to that job $\mathbf{W_i}$ and the **ascending order** of the modified time is the optimal sequence.

Example 2: Find the optimal sequence of the following jobs performed on one machine and the operating times (hours) are:

Jobs	**A**	**B**	**C**	**D**	**E**	**F**
Time (t_i)	**10**	**6**	**5**	**4**	**2**	**8**
Weight (W_i)	**5**	**10**	**5**	**1**	**3**	**5**

Solution:

The modified time is:

Modified time ($\bar{t}$)	**Jobs**
10/5 = 2	**A**
6/10 = 0.6	**B**
5/5 = 1	**C**
4/1 = 4	**D**
2/3 = 0.67	**E**
8/5 = 1.6	**F**

Therefore, the optimal sequence is: B - E - C - F - A – D

2- Processing n jobs through two machines:

This case takes the following algorithm:

1. Determine the minimum time of each job.
2. In the sequence of jobs, start with the **ascending sequence** of the first machine (i.e. from the lower time to the higher time). In the case of the least two equal times, first select the time that has a **greater difference** with the other time of the second machine or select the time that has the **smallest difference** with the other time of the first machine.
3. Continue the sequence of jobs according to the **descending sequence** of the second machine (i.e. from the higher time to the lower time).
4. Based on the sequence of jobs, find the starting and finishing time of each job of the first machine.
5. For the same sequence of jobs, find the starting and finishing time of each for the second machine. The starting time depends on the greater value

between the finishing time of the previous job on the second machine and the finishing time of the present job on the first machine.

6. Calculate the least total time required to perform all jobs on the two machines which is the time of performing the final job on the second machine.
7. The lost (**idle time**) of the first machine is the difference between the finishing times on both machines. While, the idle time of the second machine is the sum of the differences between the starting and finishing time of each job on the second machine.

Example 3: Six jobs are performed on two machines **A** and **B,** the sequence of work is **A** and then **B**, and the taken time (hour) for each job is:

Jobs	**1**	**2**	**3**	**4**	**5**	**6**
Mach. A	**3**	**12**	**5**	**2**	**9**	**11**
Mach. B	**8**	**10**	**9**	**6**	**3**	**1**

Required: Find **(a)** the optimal sequence, **(b)** the least total time required to perform all the jobs and **(c)** the idle time for both machines.

Solution:

1	**2**	**3**	**4**	**5**	**6**
2 **3**	**12**	3 **5**	1 **2**	**9**	**11**
8	4 **10**	**9**	**6**	5 **3**	6 **1**

(a) The optimal sequence is: **4 – 1 – 3 – 2 – 5 – 6**.

Jobs	**Mach. A**			**Mach. B**			
	Time	**Start**	**Finish**	**Time**	**Start**	**Finish**	**Idle time**
4	**2**	**0**	**2**	**6**	**2**	**8**	**2**
1	**3**	**2**	**5**	**8**	**8**	**16**	**0**
3	**5**	**5**	**10**	**9**	**16**	**25**	**0**
2	**12**	**10**	**22**	**10**	**25**	**35**	**0**
5	**9**	**22**	**31**	**3**	**35**	**38**	**0**
6	**11**	**31**	**42**	**1**	**42**	**43**	**4**

∑		**6**

(b) The least total time required to perform all the jobs is **43** hours.

(c) The idle time for machine **A** is: **43 - 42** = **1** hr.

The idle time for machine **B** is: **6** hrs.

Example 4: Seven jobs are performed on two machines **A** and then **B**, and the taken time (hour) for each job is:

Jobs	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Mach. A	**3**	**12**	**15**	**6**	**10**	**11**	**9**
Mach. B	**8**	**10**	**10**	**6**	**12**	**1**	**3**

Required: Find **(a)** the optimal sequence, **(b)** the least total time required to perform all the jobs and **(c)** the idle time for both machines.

Solution:

1	**2**	**3**	**4**	**5**	**6**	**7**
1 **3**	**12**	**15**	2 **6**	3 **10**	**11**	**9**
8	5 **10**	4 **10**	**6**	**12**	7 **1**	6 **3**

(a) The optimal sequence is: **1 – 4 – 5 – 3 – 2 – 7 – 6**.

Jobs	**Mach. A**			**Mach. B**			
	Time	**Start**	**Finish**	**Time**	**Start**	**Finish**	**Idle time**
1	**3**	**0**	**3**	**8**	**3**	**11**	**3**
4	**6**	**3**	**9**	**6**	**11**	**17**	**0**
5	**10**	**9**	**19**	**12**	**19**	**31**	**2**
3	**15**	**19**	**34**	**10**	**34**	**44**	**3**
2	**12**	**34**	**46**	**10**	**46**	**56**	**2**
7	**9**	**46**	**55**	**3**	**56**	**59**	**0**
6	**11**	**55**	**66**	**1**	**66**	**67**	**7**
∑							**17**

(b) The least total time required to perform all the jobs is **67** hours.

(c) The idle time for machine **A** is: **67 - 66** = **1** hr.

The idle time for machine **B** is: **17** hrs.

3. Processing n jobs through 3 machines:

In this case, you must check at least one of the two conditions:

(a) The least time on the first machine ≥ The greatest time on the second machine.
Or
(b) The least time on the third machine ≥ The greatest time on the second machine.

The solution algorithm will be:
(1) Convert the three machines into two imaginary machines **G** and **H**, and their operating times are:

$$G_i = A_i + B_i \quad , \quad H_i = B_i + C_i$$

(2) Find the optimal sequence of the two machines **H** and **G**.
(3) Use the sequence of jobs according to the optimal sequence, and find the starting and finishing time of each process for each machine from the original machines according to the previous method.

(4) The idle time for both the first and third machines (**A** and **C**) is calculated in the same previous way, but the difference is in the calculation of the idle time on machine **B**, as it is calculated from the relationship:

Finishing time of the final job (according to the optimal sequence of jobs) on the third machine **-** Finishing time of the final job on the second machine **+** the idle time calculated for the second machine.

Example 5: Six jobs are performed on three machines **A, B, C**, according to the **ABC** sequence. Find **(a)** the optimal sequence for performing all the jobs, **(b)** the least total time and **(c)** idle time for each machine. The time taken (**hour**) for each operation on each machine is:

Jobs	1	2	3	4	5	6
Mach. A	3	12	5	2	9	11
Mach. B	8	6	4	6	3	1
Mach. C	13	14	9	12	8	13

Solution:

Check the second condition: the least time on the third machine ≥ the largest time on the second machine.

So, we can solve the model using the above algorithm:

Assuming that: $\mathbf{G_i = A_i + B_i}$, $\mathbf{H_i = B_i + C_i}$

Jobs	1	2	3	4	5	6
Mach. G	11 3	18 5	9 2	8 1	12	12 4
Mach. H	21	20	13	18	11 6	14

1. The optimal sequence is: **4 – 3 – 1 – 6 – 2 – 5**.

Jobs	Mach. A			Mach. B				Mach. C			
	T.	S.	F.	T.	S.	F.	I.	T.	S.	F.	I.
4	2	0	2	6	2	8	2	12	8	20	8
3	5	2	7	4	8	12	0	9	20	29	0
1	3	7	10	8	12	20	0	13	29	42	0
6	11	10	21	1	21	22	1	13	42	55	0
2	12	21	33	6	33	39	11	14	55	69	0
5	9	33	42	3	42	45	3	8	69	77	0
∑							17				8

2. The least total time required to perform all the jobs is **77** hrs.
3. The idle time for machine **A** is: **77 - 42 = 35** hrs. .
   The idle time for machine **B** is: **77 - 45 +17 = 49** hrs.
   The idle time for machine **C** is: **8** hrs.

Homework:

Five jobs are performed on three machines **A, B, C**, according to the **ABC** sequence. Find:-

(a) The optimal sequence for performing the jobs.

(b) The least total time to perform all the jobs.

(c) The idle time for each machine.

The time taken (**hour**) for each operation on each machine is:

Job	**1**	**2**	**3**	**4**	**5**
Mach. A	**3**	**8**	**7**	**5**	**4**
Mach. B	**4**	**5**	**1**	**2**	**3**
Mach. C	**7**	**9**	**5**	**6**	**10**

Q.: For the data given in the following frequency distribution table, compute if the nearest error in the data is 1:

(a) Arithmetic mean **(b) Median** **(c) Mode**

(d) Range **(e) Variance** **(f) Standard deviation**

Classes	150-158	159-167	168 -	177 -	186 -	195 -	204-	213-
Frequency	5	15	20	21	20	15	3	1

Tetorial No. 3:

Q.: (I) Change the following linear programming model to:

(a) Canonical form

(b) Standard form

$$\min. Z = X_1 - X_2 + 2X_3$$

$$\text{S. t.} \quad 2X_1 + X_2 + 2X_3 \leq 20$$

$$X_1 + 2X_2 - X_3 \geq 30$$

$$|X_1 - 3X_2 + 2X_3| \leq 10$$

$$X_1, X_2 \geq 0, X_3 \text{ unrestricted in sign}$$

(II) A company produces three cutting machines (**M1, M2, and M3**) and needs two types of raw materials to manufacture these machines as shown in the following table:

Raw material	**Machines**		
	M1	**M2**	**M3**
A	**6**	**10**	**14**
B	**4**	**6**	**10**
Profit of one machine ($)	**12000**	**17000**	**22000**

Tutorial No.4

Q.4: The technology procedure for drilling a workpiece with 5 mm diameter consists of the following stages shown in the table below. This work was observed for 5 times and the results are also in this table:

Work stages	Observation time (second)				
	1	**2**	**3**	**4**	**5**
Fixing the workpiece on the machine table	63	61	57	63	66
Approaching the drill to the workpiece	16	17	15	18	19
Turn on the drilling machine	101	101	99	103	106
Removing the drill and stop the machine	21	23	19	23	24

Calculate:

(1) The average observation time for drilling the workpiece.

(2) Standard times for the three cases for a worker performing the work with: **(i)** lower skill, **(ii)** normal skill and **(iii)** higher skill according to the estimated percentage for efficiency for each case.

(3) Number of work cycles that must be done for reaching the required accuracy to calculate the standard time if **α = 40**.

Use this equation in your solution.

$$m = \left(\frac{\alpha \sqrt{n \sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i \right)^2}}{\sum_{i=1}^{n} X_i} \right)^2$$