ME 144: Heat Transfer
Convection Relations for External Flows

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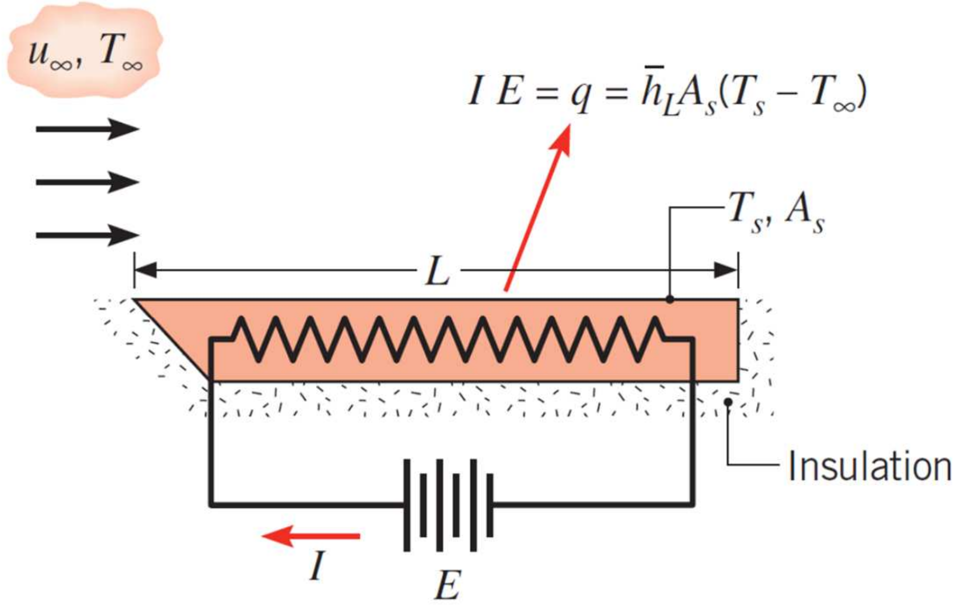
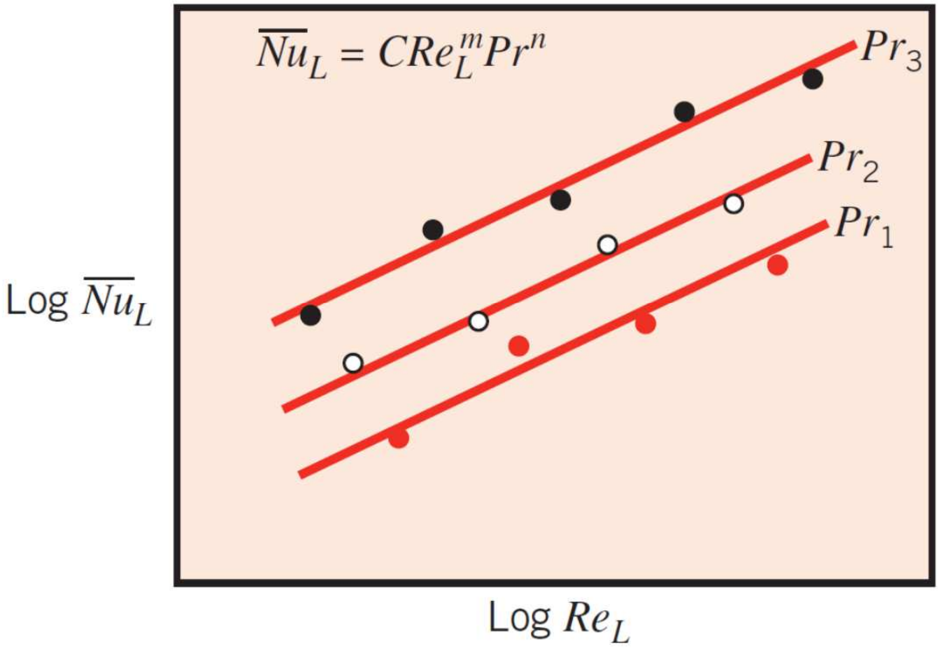
Empirical Correlations

- Generally, convection correlations for external flows are determined experimentally using controlled lab conditions

- Correlations for the average Nusselt number are sought of the form: $\overline{\mathrm{Nu}}_L = C\mathrm{Re}_L{}^m\mathrm{Pr}^n$

- Here material properties are evaluated at the <u>film temperature:</u> $T_f \equiv \dfrac{T_s - T_\infty}{2}$





Flat Plate: Laminar Boundary Layer Flow

Velocity Boundary Layer

Mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

x-Momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

ν ≡kinematic viscosity

Energy:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

Recall our assumptions/simplifications:

⇨ $\frac{\partial P}{\partial x}$ Related to variations in U_∞ due to changes in geometry

- think of Bernoulli-like effects
- **Note**: $\partial P^*/\partial x^* = 0$ for a flat plate!

⇨ $$\frac{\partial P}{\partial y} = 0$$

⇨ $$\frac{\partial^2 u}{\partial y^2} \gg \frac{\partial^2 u}{\partial x^2}$$

⇨ $$\frac{\partial^2 T}{\partial y^2} \gg \frac{\partial^2 T}{\partial x^2}$$

⇨ Neglect viscous dissipation which is an adequate assumption for low speed incompressible flows

Flat Plate: Laminar Boundary Layer Flow

Velocity Boundary Layer

Mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

x-Momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

ν ≡kinematic viscosity

Energy:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

- These are the governing equations for an isothermal flat plate (T_s) with a uniform flow U_∞, T_∞
- Solution Approach: Since there is no intrinsic length scale in the problem we seek the solution in terms of a similarity variable.

Flat Plate: Laminar Boundary Layer Flow

Velocity Boundary Layer

- The solution procedure for this problem was first obtained by Blasius (1908)
- This was done by first introducing the stream function as defined by (such that continuity is still satisfied):

$$u = \frac{\partial \psi}{\partial y} = \psi_y \qquad v = -\frac{\partial \psi}{\partial x} = -\psi_x$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

- The momentum equation in terms of ψ then becomes:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}}$$

- Base on the dimensional analysis we performed in last class we saw that: $\delta \sim \frac{x}{\sqrt{\mathrm{Re}_x}}$
- This allowed Blasius to reason that the similarity variable η would be given by:

$$\eta = \frac{y}{x}\sqrt{\mathrm{Re}_x} = y\sqrt{\frac{U_\infty}{\nu x}}$$

Flat Plate: Laminar Boundary Layer Flow

Velocity Boundary Layer

- Further, he reasoned that the stream function could be defined in terms of η with proper scaling:

$$\psi(x,y) \sim U_\infty \delta(x) f(\eta) = \sqrt{\upsilon x U_\infty} f(\eta)$$

- Substitution of the stream function into the momentum equation leads to the nonlinear ODE:

$$2f''' + ff'' = 0$$

- This equation is subject to boundary conditions:

$$\psi_x(x,0) = \psi_y(x,0) = 0$$

At the wall

$$\lim_{y\to\infty} \psi_y(x,y) = U_\infty$$

At the BL edge

Flat Plate: Laminar Boundary Layer Flow

Velocity Boundary Layer

- The numerical solution of the Blasius equation yields, by defining the height at which:

$$\frac{u}{U_\infty} = 0.99$$

$$\Downarrow$$

$$\delta(x) = \frac{5}{\sqrt{\mathrm{Re}_x}}$$

- And the wall friction coefficient:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho {U_\infty}^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

Flat Plate: Laminar Boundary Layer Flow

Thermal Boundary Layer

- Knowledge of the velocity field then permits the direct calculation of the thermal field

- Defining again our dimensionless temperature:

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

- Assuming that $\theta = \theta(\eta)$ one can derive another ODE for the temperature field:

$$\theta'' + \frac{\mathrm{Pr}}{2} f \theta' = 0$$

- Which is subject to:

$$\theta(\eta = 0) = 0$$

$$\theta(\eta = \infty) = 1$$

- Evidently this solution depends on Pr

Flat Plate: Laminar Boundary Layer Flow

Thermal Boundary Layer

- Numerical results for $Pr \geq 0.6$ (typical for gases) shows that:

$$\theta'\Big|_{\eta=0} = 0.332\mathrm{Pr}^{1/3}$$

- In terms of the convection coefficient (or Nusselt number):

$$h = \frac{1}{T_s - T_\infty} k \frac{\partial T}{\partial y}\Big|_{y=0} \quad \Rightarrow \quad h = k\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$$

Local solution (Eq 7.23): $\mathrm{Nu}_x \equiv \dfrac{h_x x}{k} = 0.33\mathrm{Re}_x{}^{1/2}\mathrm{Pr}^{1/3}$ $\quad \mathrm{Pr} \geq 0.6$

Average solution (Eq. 7.30): $\overline{\mathrm{Nu}}_x \equiv \dfrac{\bar{h}_x x}{k} = 0.664\mathrm{Re}_x{}^{1/2}\mathrm{Pr}^{1/3}$ $\quad \mathrm{Pr} \geq 0.6$

- It also follows from the ODE solution that: $\dfrac{\delta}{\delta_t} \simeq \mathrm{Pr}^{1/3}$

Flat Plate: Turbulent Boundary Layer

- The results for turbulent boundary layers are essentially obtained from experiments
- The key findings for the plate are:

$$\mathrm{Re}_{crit} \sim 5 \times 10^5$$

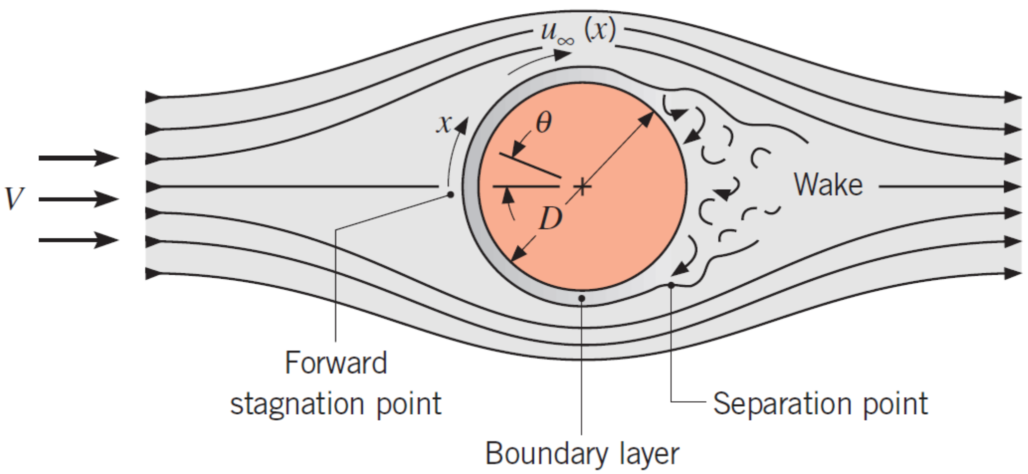
$$\frac{\delta}{x} = 0.37 \mathrm{Re}_x{}^{-1/5}$$

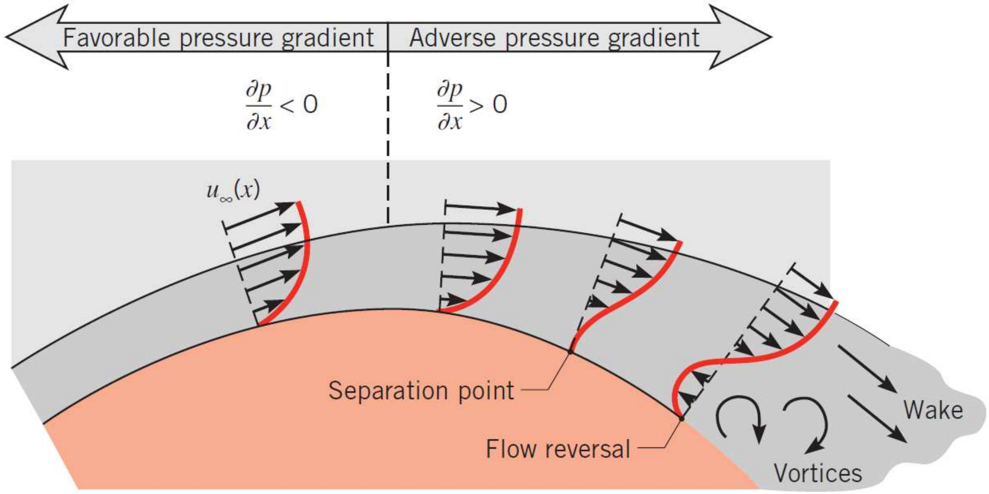
$$\mathrm{Nu}_x = 0.0296 \mathrm{Re}_x{}^{4/5} \mathrm{Pr}^{1/3}$$

Recall

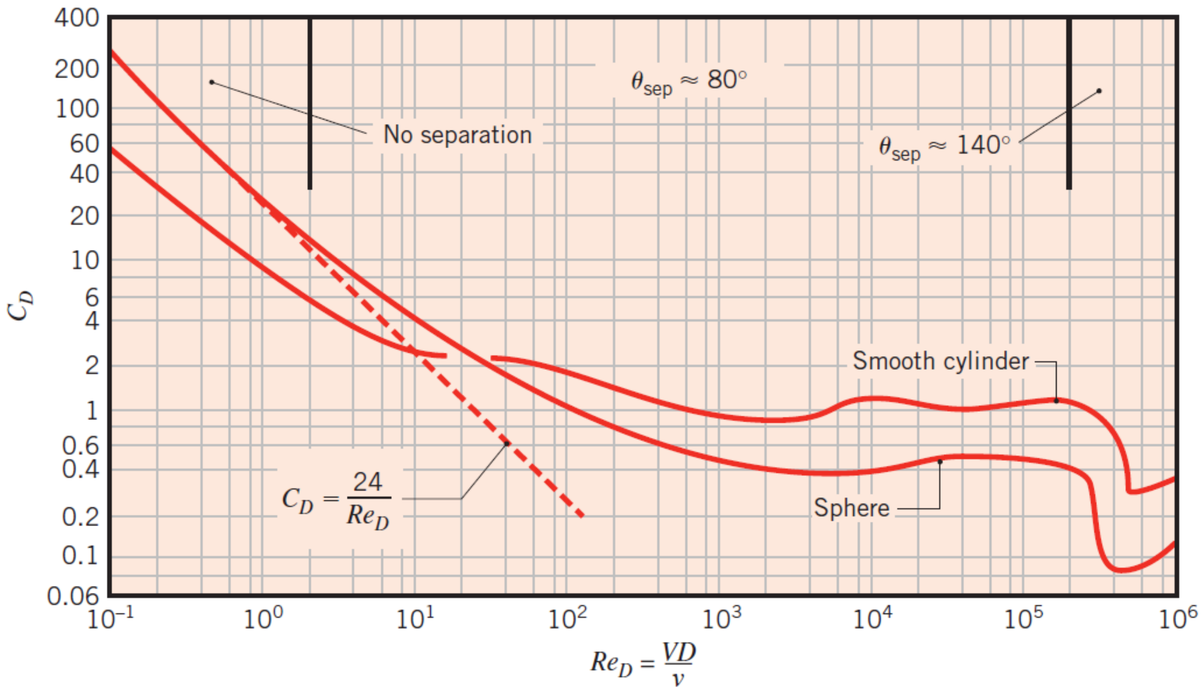
Prandtl number (Pr)	$\frac{c_p \mu}{k} = \frac{\nu}{\alpha}$	Ratio of the momentum and thermal diffusivities
Reynolds number (Re_L)	$\frac{VL}{\nu}$	Ratio of the inertia and viscous forces

Convection for a Cylinder in a Cross Flow

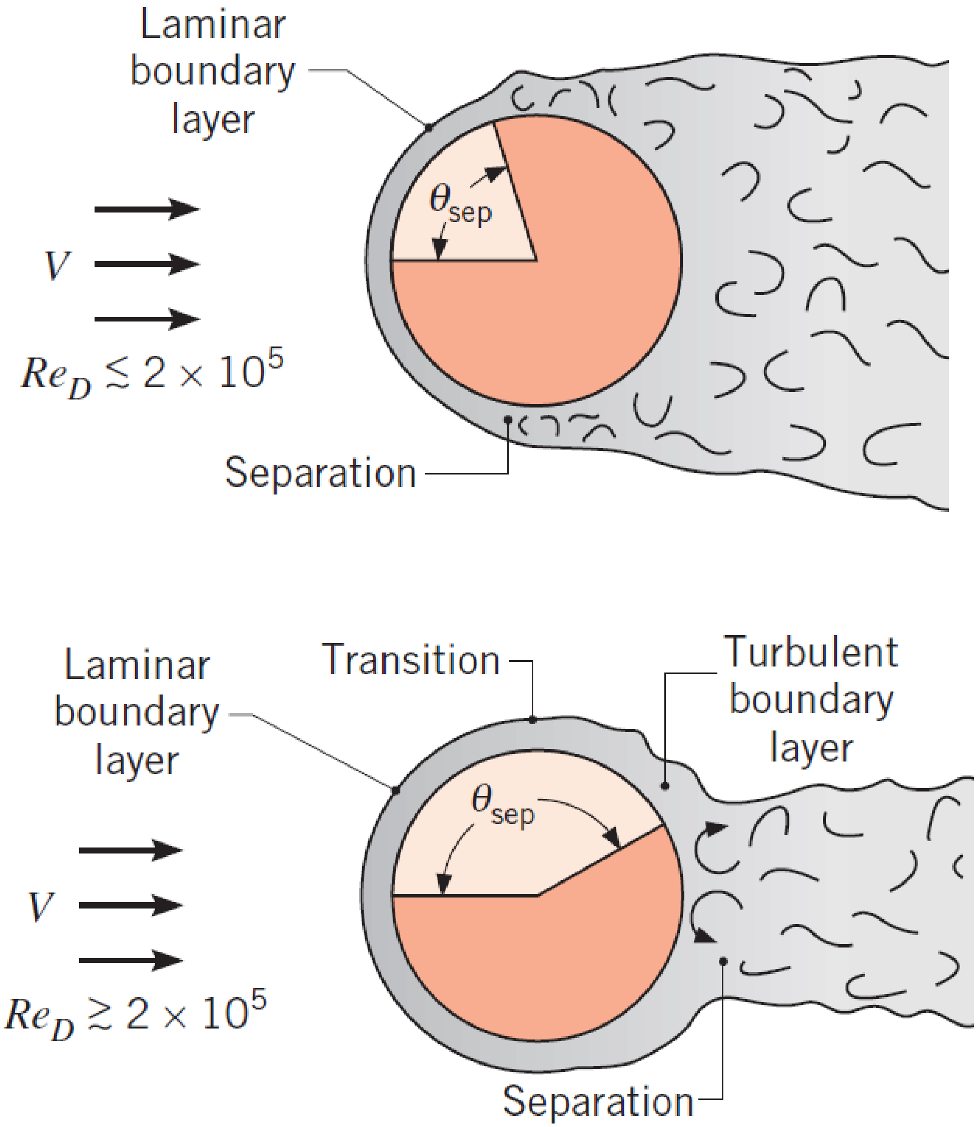


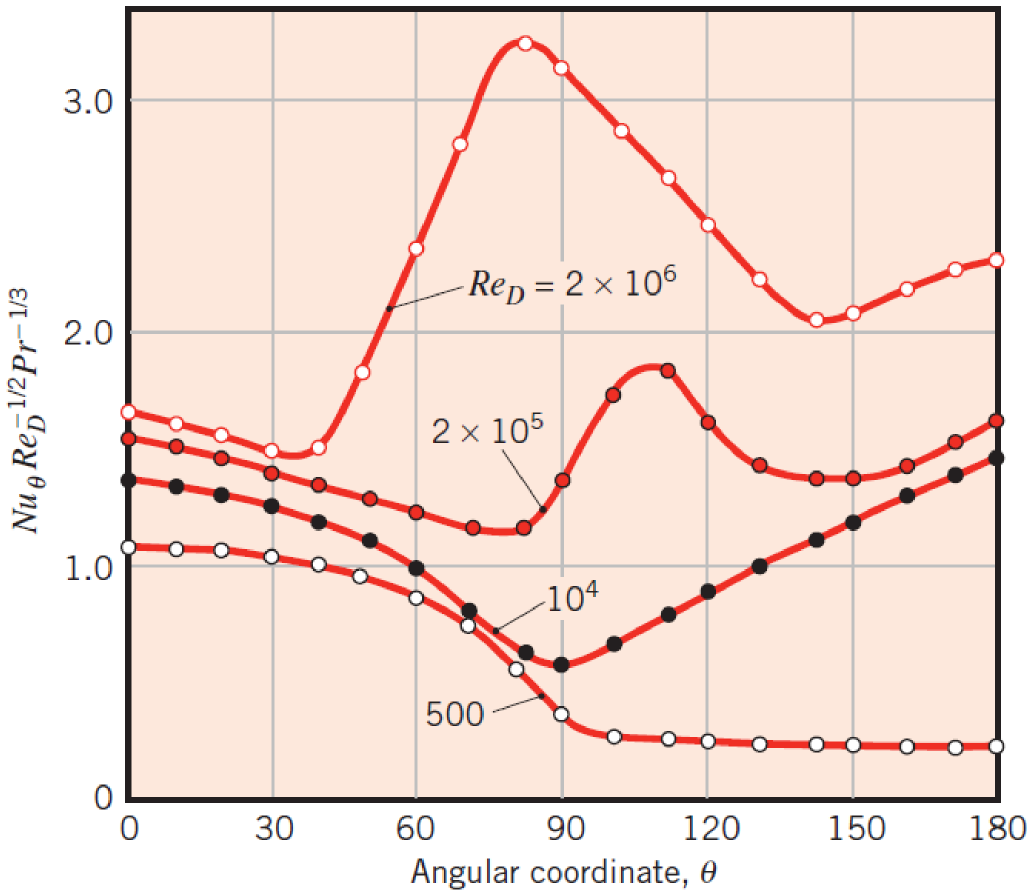


$$C_D = \frac{F_D}{A_f(\rho V^2/2)}$$



Convection for a Cylinder in a Cross Flow





Convection for a Cylinder in a Cross Flow

As engineers we are generally interested in average conditions

Hilpert correlation (**Eq. 7.52**):

$$\overline{\mathrm{Nu}}_D = C\mathrm{Re}_D{}^m\mathrm{Pr}^{1/3}$$

$$\mathrm{Pr} \geq 0.7$$

- Values of C and m can be found in Table 7.2 for various Re_D
- all properties are evaluated at the film temperature

$$T_f = \frac{T_s + T_\infty}{2}$$

Zukauskas correlation (**Eq. 7.53**):

$$\overline{\mathrm{Nu}}_D = C\mathrm{Re}_D{}^m\mathrm{Pr}^n\left(\frac{\mathrm{Pr}}{\mathrm{Pr}_s}\right)^{1/4}$$

$$0.7 \leq \mathrm{Pr} \leq 500$$
$$1 \leq \mathrm{Re}_D \leq 10^6$$

- All properties evaluated at T_∞ except for Pr_s which is evaluated at T_s
- Values of C and m can be found in Table 7.4 for various Re_D
- If $\mathrm{Pr} \leq 10, n = 0.37$; if $\mathrm{Pr} \geq 10, n = 0.36$

Churchill and Bernstein correlation (**Eq. 7.54**):

$$\overline{\mathrm{Nu}}_D = 0.3 + \frac{0.62\mathrm{Re}_D{}^{1/2}\mathrm{Pr}^{1/3}}{[1 + (0.4/\mathrm{Pr})^{2/3}]^{1/4}}\left[1 + \left(\frac{\mathrm{Re}_D}{282{,}000}\right)^{5/8}\right]^{4/5}$$

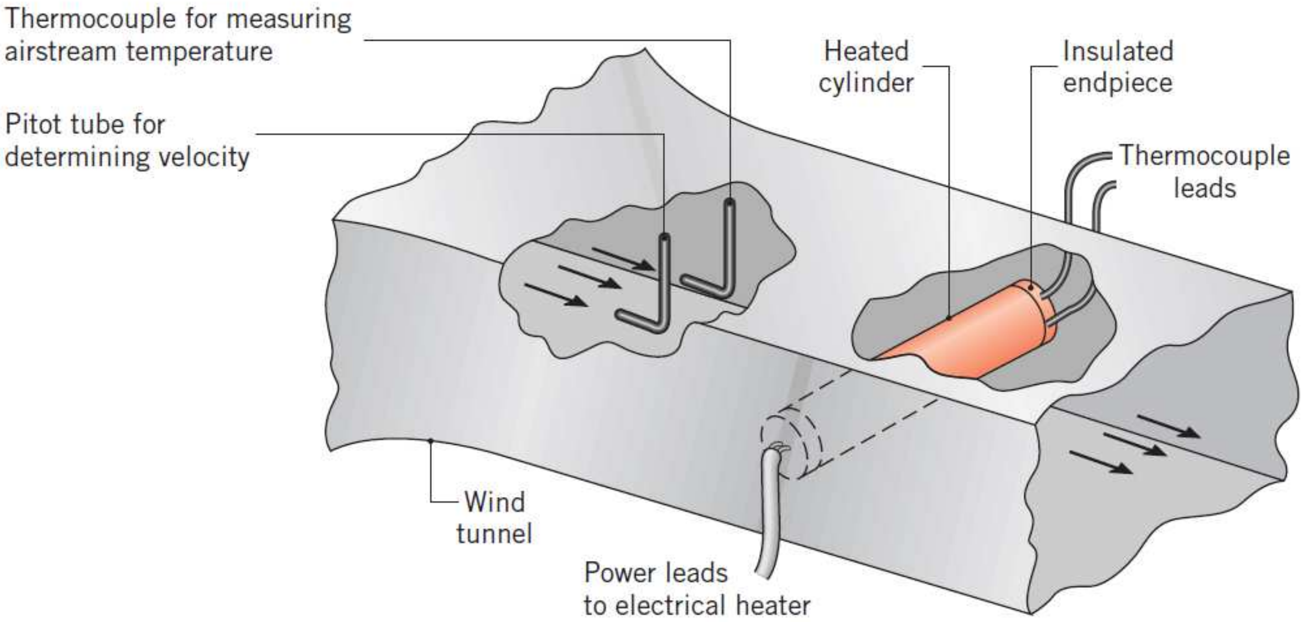
$$\mathrm{Re}_D\mathrm{Pr} \geq 0.2$$

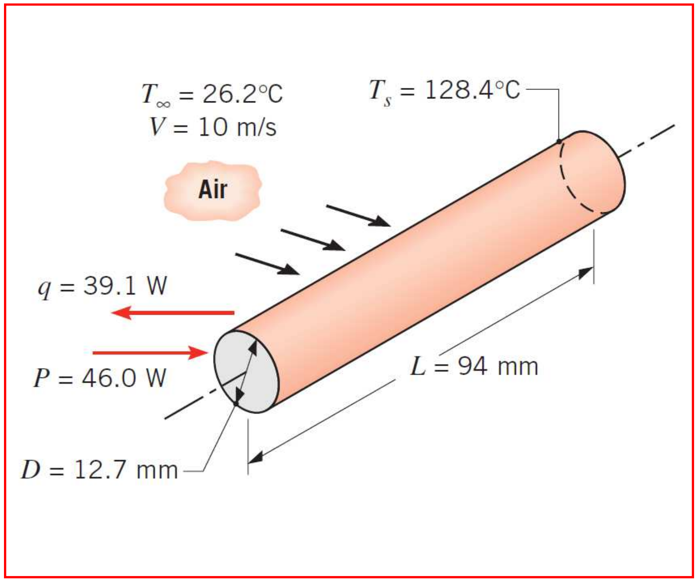
- all properties are evaluated at the film temperature:

Convection for a Cylinder in a Cross Flow

Example 7.4

- Experiments have been conducted on a metallic cylinder ($D = 12.7$ mm, $L = 94$ mm). The cylinder is heated internally by an electrical heater and is subjected to a cross flow of air in a low-speed wind tunnel.
- Under a specific set of operating conditions for which the upstream air velocity and temperature were maintained constant ($U_\infty = 10$ m/s and T_∞ =26.2 C).
- The heater power dissipation was measured to be 46 W, while the average cylinder surface temperature was determined to be 128.4 C.
- It is estimated that 15% of the power dissipation is lost through the cumulative effect of surface radiation and conduction through the end pieces.





Convection for a Cylinder in a Cross Flow

Example 7.4

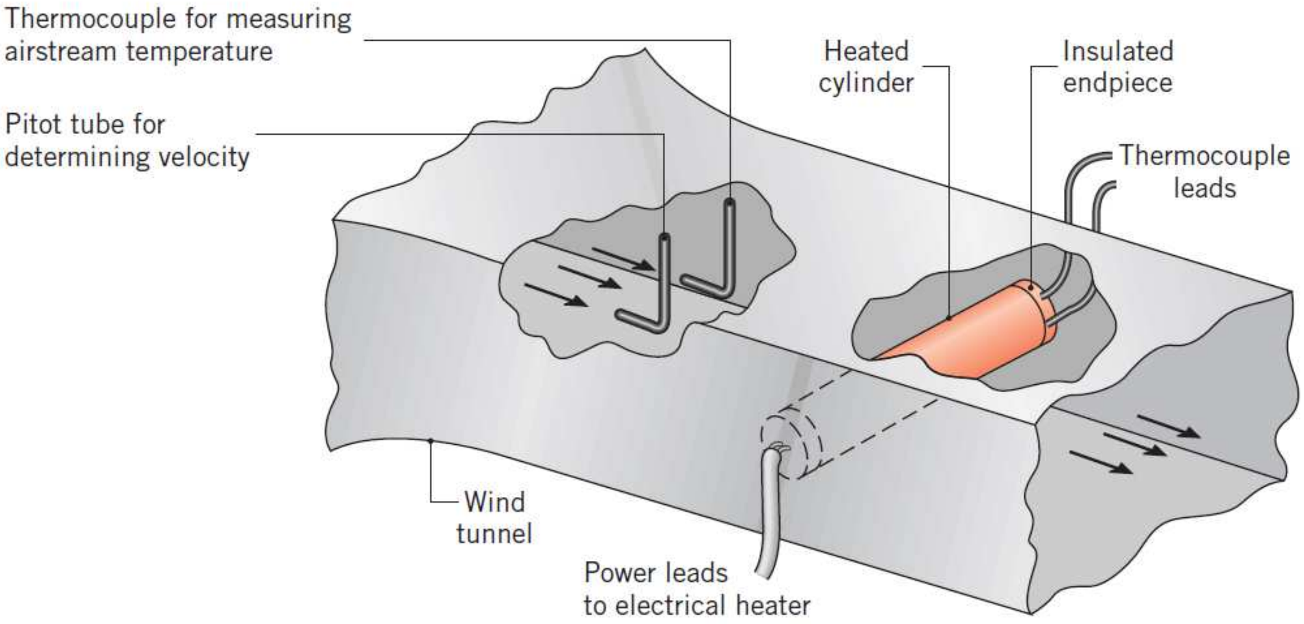
Find

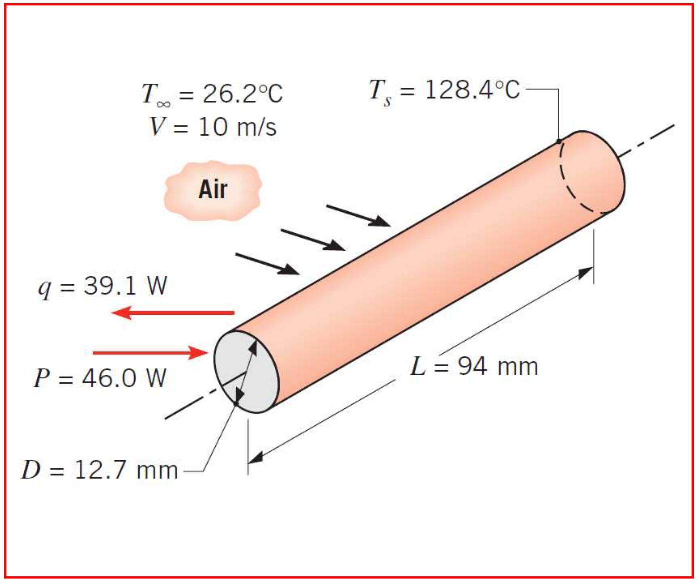
1. Determine the convection heat transfer coefficient from the experimental observations.
2. Compare the experimental result with the convection coefficient computed from an appropriate correlation.

Assume

1. Steady-state, incompressible flow conditions.
2. Uniform cylinder surface temperature.

Care must be taken when using these (or any empirically-based) correlations!





Convection for a Cylinder in a Cross Flow

Example 7.4

Determine flow properties and non-dimensional parameters from Table A.4 for air:

$T_\infty = 26.2\ C = 304.35\ K \simeq 300\ K$:

$$\nu = 15.89 \times 10^{-6}\frac{m^2}{s} \qquad k = 26.3 \times 10^{-3}\frac{W}{m \cdot K} \qquad \mathrm{Pr} = 0.707 \qquad \mathrm{Re}_D = \frac{U_\infty D}{\nu} = 7992$$

$$T_f = \frac{(26.2 + 128.4)}{2} C = 355.45\ K \simeq 350\ K:$$

$$\nu = 20.92 \times 10^{-6}\frac{m^2}{s} \qquad k = 30 \times 10^{-3}\frac{W}{m \cdot K} \qquad \mathrm{Pr} = 0.700 \qquad \mathrm{Re}_D = \frac{U_\infty D}{\nu} = 6071$$

$T_s = 128.4\ C = 406.55\ K \simeq 400\ K$:

$$\mathrm{Pr}_s = 0.690$$

Convection for a Cylinder in a Cross Flow

Example 7.4: Analysis

Newton's law of cooling

The convection heat transfer coefficient may be determined from the data by using:

$$\bar{h} = \frac{q}{A \cdot (T_s - T_\infty)}$$

As 15% of the power is lost through the end pieces, thus: $q = 0.85P$

And the area is: $A = \pi DL$

$$\bar{h} = \frac{0.85 \times 45 \text{ W}}{\pi \times 0.0127 \text{ m} \times 0.094 \text{ m}(128.4 - 26.2)\text{C}}$$

$$\boxed{\bar{h} = 102 \ \frac{\text{W}}{\text{m}^2 \cdot \text{K}}}$$

Convection for a Cylinder in a Cross Flow

Example 7.4

Hilpert correlation (**Eq. 7.52**):

$$\overline{\mathrm{Nu}}_D = C\mathrm{Re}_D{}^m \mathrm{Pr}^{1/3} \qquad \mathrm{Pr} \geq 0.7$$

All properties evaluated at the film temperature:

$$\mathrm{Pr} = 0.700 \qquad \mathrm{Re}_D = \frac{U_\infty D}{\nu} = 6071$$

Hence, from Table 7.2, $C = 0.193$ and $m = 0.618$

$$\overline{\mathrm{Nu}}_D = 0.193(6071)^{0.618}(0.700)^{1/3} = 37.3$$

$$\bar{h} = \overline{\mathrm{Nu}}_D \frac{k}{D} = 37.3 \frac{0.030\ \mathrm{W/m{\cdot}K}}{0.0127\ \mathrm{m}}$$

$$\boxed{\bar{h} = 88\ \frac{\mathrm{W}}{\mathrm{m^2{\cdot}K}}}$$

Convection for a Cylinder in a Cross Flow

Example 7.4

Zukauskas correlation (**Eq. 7.53**):

$$\overline{\mathrm{Nu}}_D = C\mathrm{Re}_D{}^m\mathrm{Pr}^n\left(\frac{\mathrm{Pr}}{\mathrm{Pr}_s}\right)^{1/4}$$

$$0.7 \lesssim \mathrm{Pr} \lesssim 500$$
$$1 \lesssim \mathrm{Re}_D \lesssim 10^6$$

$$\mathrm{Pr} = 0.707 \qquad \mathrm{Re}_D = \frac{U_\infty D}{\nu} = 7992$$ Evaluated at ∞

$$\mathrm{Pr}_s = 0.690$$ Evaluated at s

From Table 7.4, $C = 0.26$ and $m = 0.6$ and since $\mathrm{Pr} < 10$, $n = 0.37$

$$\overline{\mathrm{Nu}}_D = 0.26(7992)^{0.6}(0.707)^{0.37}\left(\frac{0.707}{0.690}\right)^{1/4} = 50.5$$

$$\bar{h} = \overline{\mathrm{Nu}}_D\frac{k}{D} = 50.5\,\frac{0.0263\ \mathrm{W/m{\cdot}K}}{0.0127\ \mathrm{m}}$$

$$\boxed{\bar{h} = 105\ \frac{\mathrm{W}}{\mathrm{m^2{\cdot}K}}}$$

Convection for a Cylinder in a Cross Flow

Example 7.4

Churchill and Bernstein correlation (**Eq. 7.54**):

$$\overline{\mathrm{Nu}}_D = 0.3 + \frac{0.62\mathrm{Re}_D{}^{1/2}\mathrm{Pr}^{1/3}}{[1+(0.4/\mathrm{Pr})^{2/3}]^{1/4}}\left[1+\left(\frac{\mathrm{Re}_D}{282{,}000}\right)^{5/8}\right]^{4/5}$$

$\mathrm{Re}_D\mathrm{Pr} \geq 0.2$

All properties evaluated at the film temperature:

$$\mathrm{Pr} = 0.700 \qquad \mathrm{Re}_D = \frac{U_\infty D}{\nu} = 6071$$

No table look-up for this correlation

$$\overline{\mathrm{Nu}}_D = 0.3 + \frac{0.62(6071)^{1/2}(0.70)^{1/3}}{[1+(0.4/0.70)^{2/3}]^{1/4}}\left[1+\left(\frac{6071}{282{,}000}\right)^{5/8}\right]^{4/5} = 40.6$$

$$\bar{h} = \overline{\mathrm{Nu}}_D\,\frac{k}{D} = 40.6\,\frac{0.030\ \mathrm{W/m\cdot K}}{0.0127\ \mathrm{m}}$$

$$\boxed{\bar{h} = 96.0\ \frac{\mathrm{W}}{\mathrm{m^2\cdot K}}}$$

Convection for a Cylinder in a Cross Flow

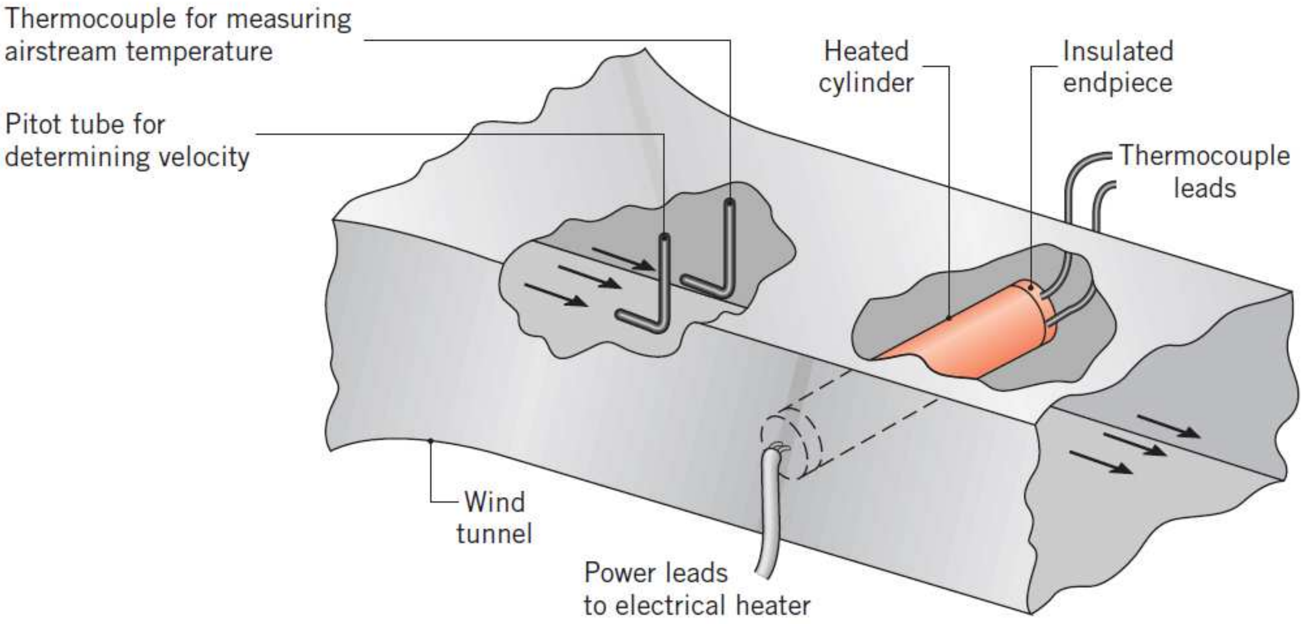
Example 7.4

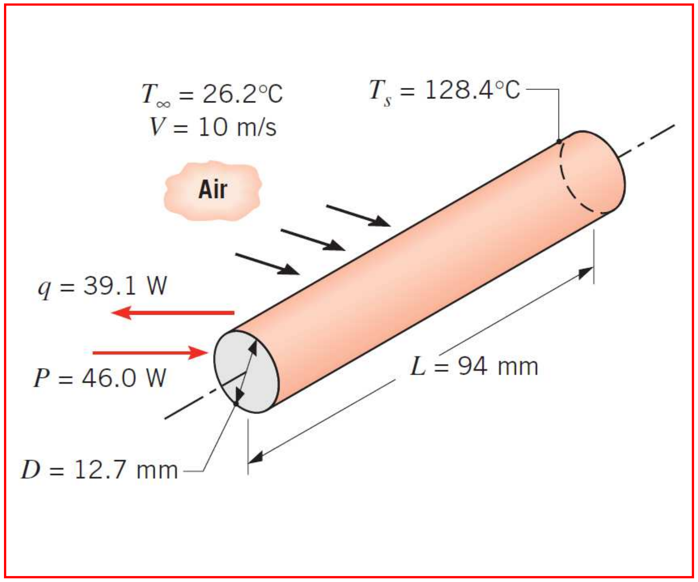
Comments

- Uncertainties associated with measuring the air velocity, estimating the heat loss from cylinder ends, and averaging the cylinder surface temperature, which varies axially and circumferentially, render the experimental result accurate to no better than 15%.

- Accordingly, calculations based on each of the three correlations are within the experimental uncertainty of the measured result.

- Recognize the importance of using the proper temperature when evaluating fluid properties.

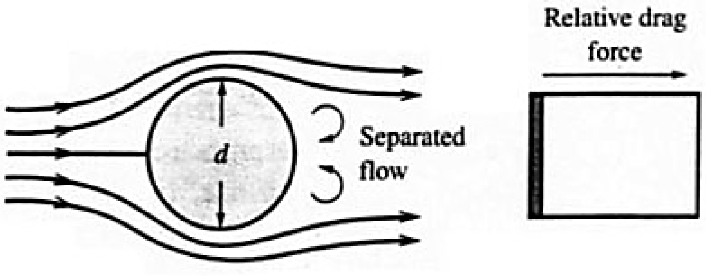




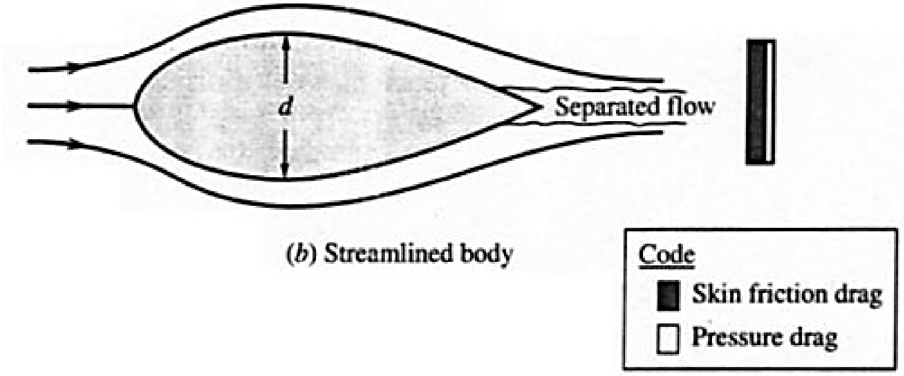
Convection for a Sphere in a Cross Flow

Boundary layer effects associated with flow over a sphere are much like those for the circular cylinder, with transition and separation playing prominent roles.

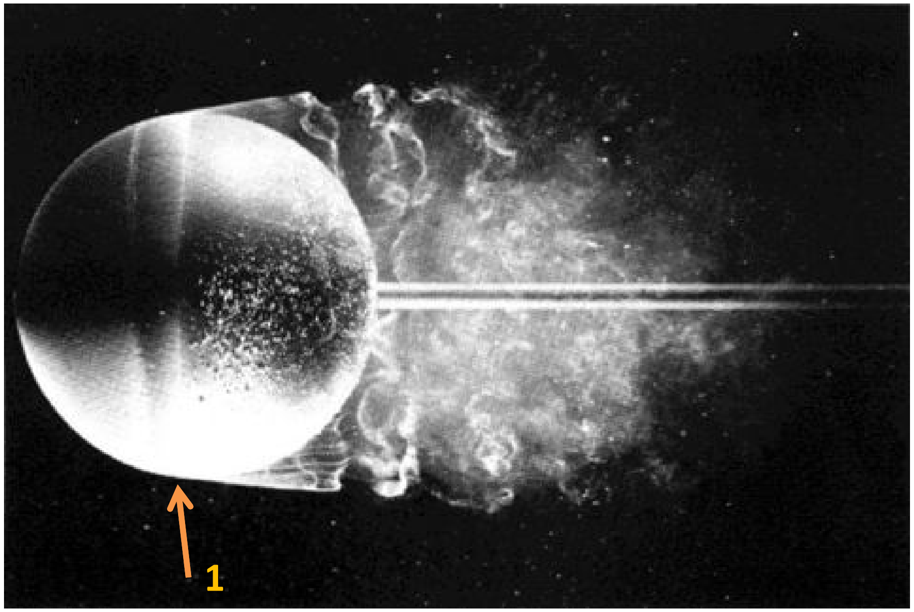
1) **Laminar separation**
2) **Turbulent BL transition**
3) **Turbulent separation**

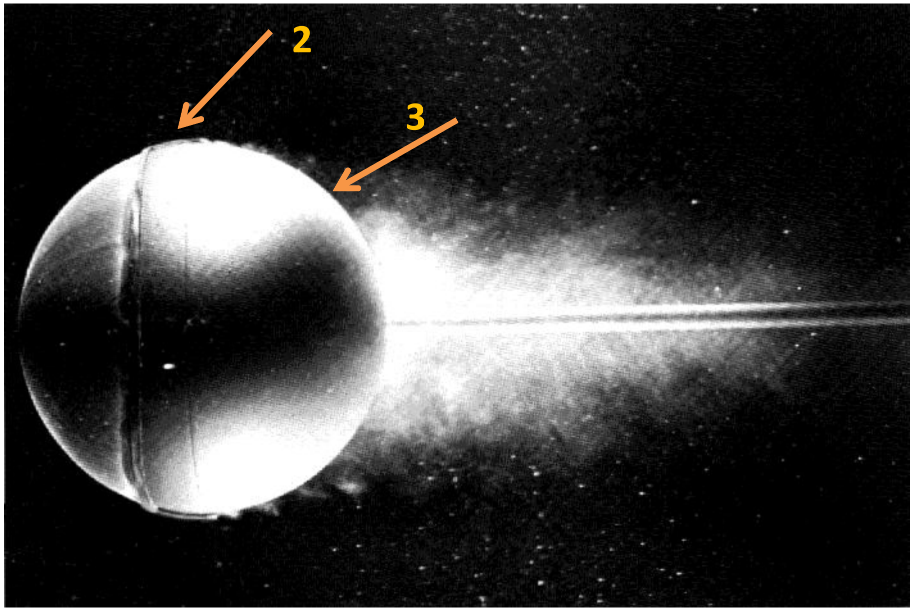


(*a*) Blunt body



(*b*) Streamlined body





Convection for a Sphere in a Cross Flow

- Numerous heat transfer correlations have been proposed

- The Whitaker correlation is an expression of the form:

$$\overline{\mathrm{Nu}}_D = 2 + \left(0.4\mathrm{Re}_D{}^{1/2} + 0.06\mathrm{Re}_D{}^{2/3}\right)\mathrm{Pr}^{0.4}\left(\frac{\mu}{\mu_s}\right)^{1/4}$$ **(Eq. 7.52)**

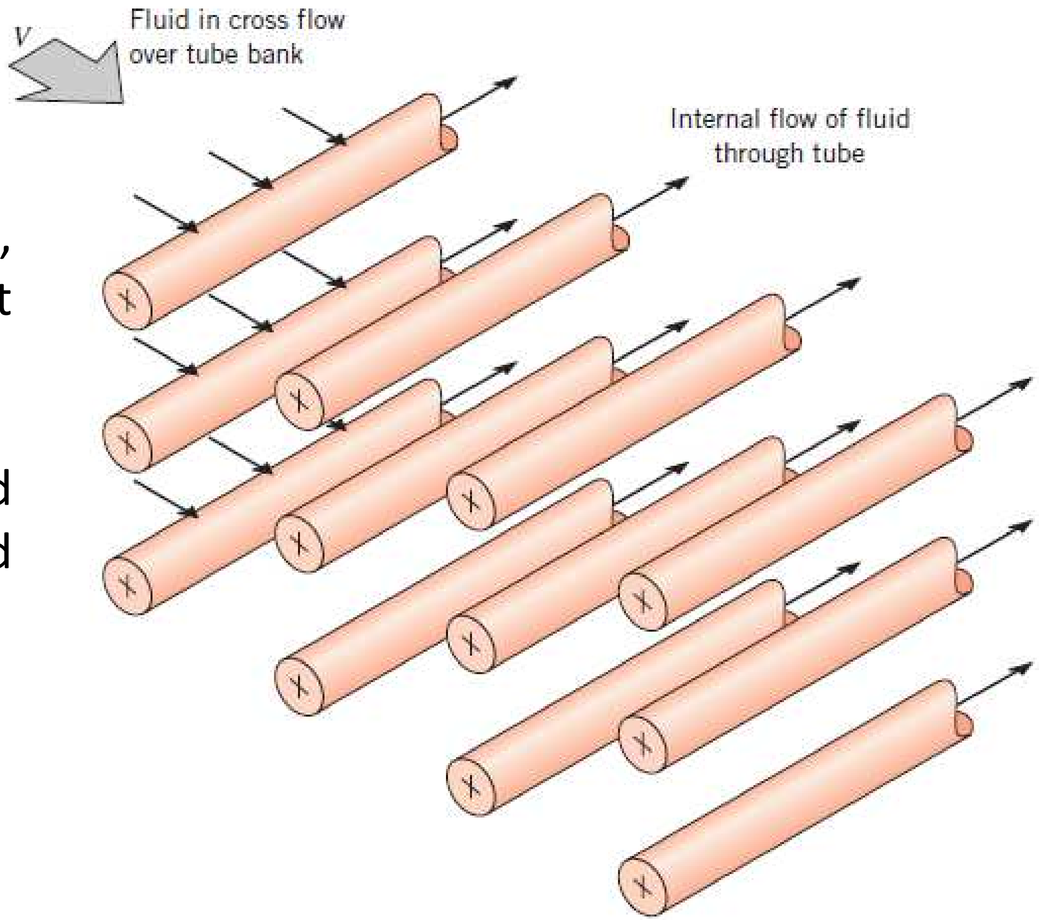
All properties evaluated at T_∞

$$\begin{bmatrix} 0.71 \le \mathrm{Pr} \le 380 \\ 3.5 \le \mathrm{Re}_D \le 7.6 \times 10^4 \\ 1.0 \le \left({}^{\mu}/_{\mu_s}\right) \le 3.2 \end{bmatrix}$$

Convection for Tube Banks in a Cross Flow

Heat transfer to or from a bank (or bundle) of tubes in cross flow is relevant to numerous industrial applications, such as steam generation in a boiler or air cooling in the coil of an air conditioner.

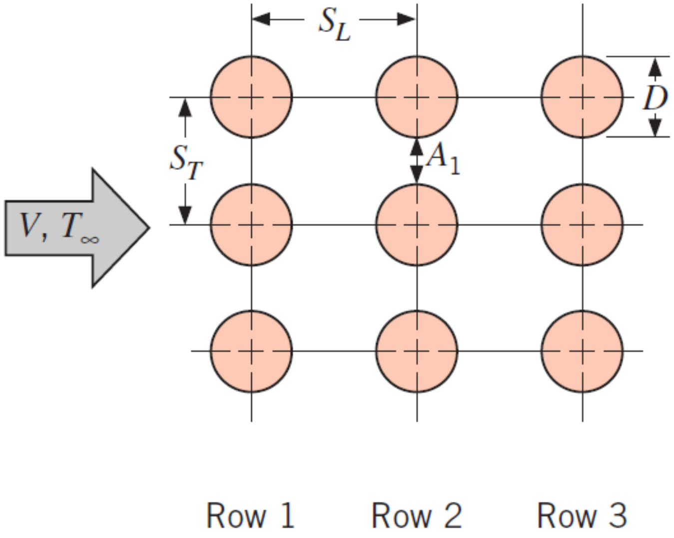
- Typically, one fluid moves over the tubes, while a second fluid at a different temperature passes through the tubes.
- In this section we are specifically interested in the convection heat transfer associated with cross flow over the tubes.



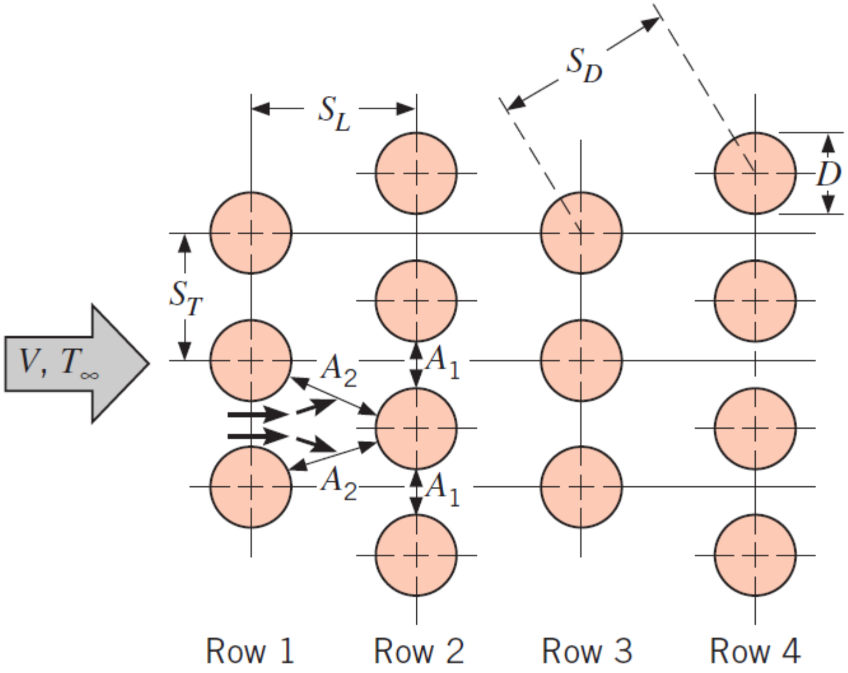
Convection for Tube Banks in a Cross Flow

- The tube rows of a bank can be either *aligned* or *staggered*
- Flow around the tubes in the first row of a tube bank is similar to that for a single (isolated) cylinder in cross flow
- For downstream rows, flow conditions depend strongly on the tube bank arrangement

$$\bar{h} = \overline{\mathrm{Nu}}_D \frac{k}{D}$$



Aligned



Staggered

Convection for Tube Banks in a Cross Flow

Typically, we wish to know the *average* heat transfer coefficient for the *entire* tube bank. Zukauskas has proposed a correlation of the form:

$$\overline{\mathrm{Nu}}_D = C_1\left(\mathrm{Re}_{D,max}\right)^m \mathrm{Pr}^{0.36}\left(\frac{\mathrm{Pr}}{\mathrm{Pr}_s}\right)^{1/4} \qquad \textbf{(Eq. 7.58)}$$

$$\left[\begin{array}{c} N_L \geq 20 \\ 10 \leq \mathrm{Re}_{D,max} \leq 2 \times 10^6 \\ 0.7 \leq \mathrm{Pr} \leq 500 \end{array}\right]$$

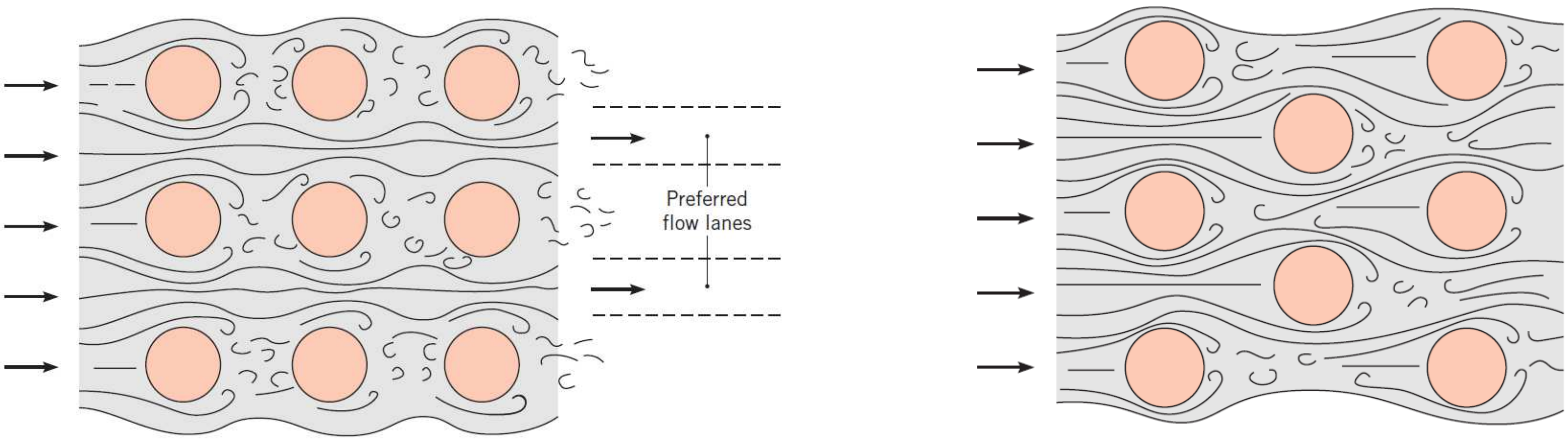
All properties evaluated at the average of bank inlet (T_∞) and outlet temperatures (T_0)

$N_L$represents number of tube bank rows

Empirically derived constants C_1 and m can be found from Table 7.5

$$\bar{h} = \overline{\mathrm{Nu}}_D\,\frac{k}{D}$$

Convection for Tube Banks in a Cross Flow



$$\overline{\mathrm{Nu}}_D = C_1\left(\mathrm{Re}_{D,max}\right)^m \mathrm{Pr}^{0.36}\left(\frac{\mathrm{Pr}}{\mathrm{Pr}_s}\right)^{1/4}$$

$$\bar{h} = \overline{\mathrm{Nu}}_D \frac{k}{D}$$

Staggering will increase heat flux BUT at a penalty of flow losses (head loss!!!)

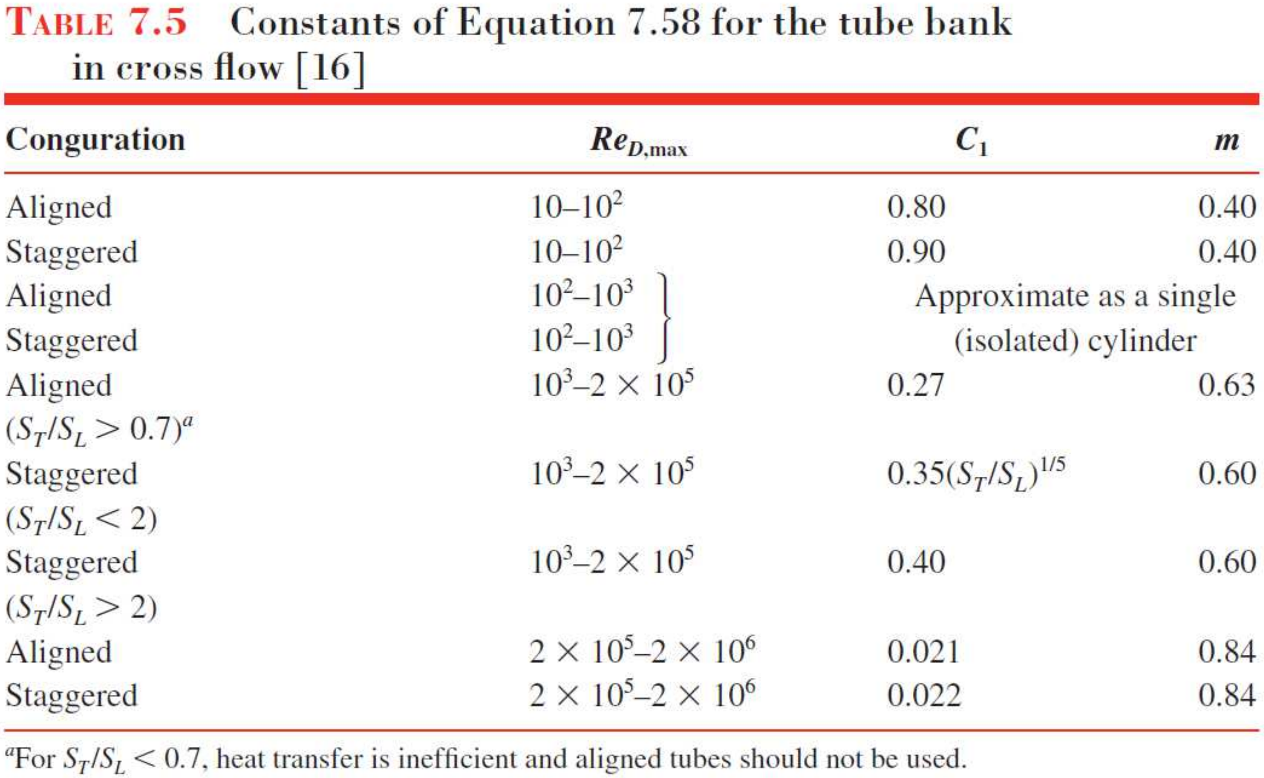


TABLE 7.5 Constants of Equation 7.58 for the tube bank in cross flow [16]

Configuration	$Re_{D,\max}$	C_1	m
Aligned	10–10^2	0.80	0.40
Staggered	10–10^2	0.90	0.40
Aligned	10^2–10^3	Approximate as a single (isolated) cylinder	
Staggered	10^2–10^3		
Aligned $(S_T/S_L > 0.7)^a$	10^3–2×10^5	0.27	0.63
Staggered $(S_T/S_L < 2)$	10^3–2×10^5	$0.35(S_T/S_L)^{1/5}$	0.60
Staggered $(S_T/S_L > 2)$	10^3–2×10^5	0.40	0.60
Aligned	2×10^5–2×10^6	0.021	0.84
Staggered	2×10^5–2×10^6	0.022	0.84

[a] For $S_T/S_L < 0.7$, heat transfer is inefficient and aligned tubes should not be used.

Convection for Tube Banks in a Cross Flow

- The previous correlation is generally good for tube banks of 20 or more rows
- If less than 20 rows are present then a simple correction may be applied:

$$\overline{\mathrm{Nu}}_D\Big|_{(N_L<20)} = C_2\,\overline{\mathrm{Nu}}_D\Big|_{(N_L\geq 20)} \qquad \text{(Eq. 7.59)}$$

- The coefficient C_2 can be found in Table 7.6

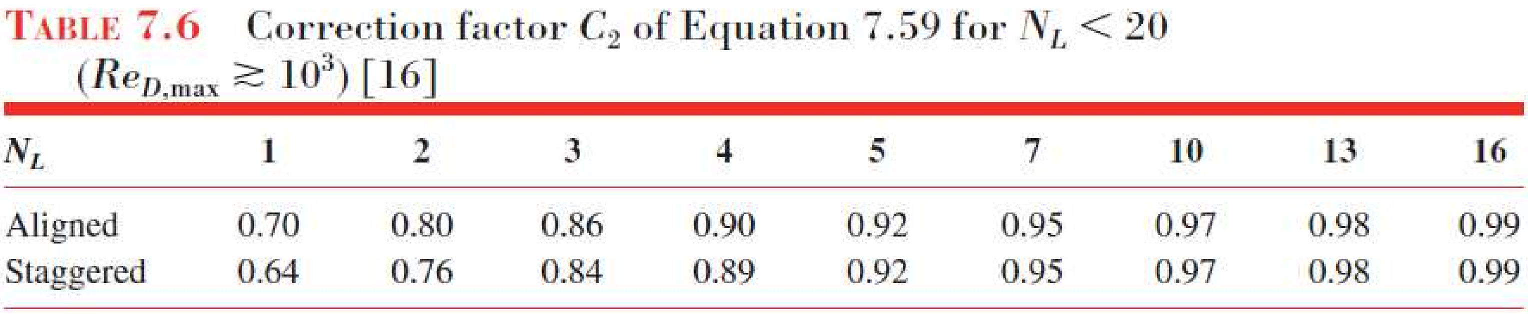


TABLE 7.6 Correction factor C_2 of Equation 7.59 for $N_L < 20$ ($Re_{D,\max} \gtrsim 10^3$) [16]

N_L	1	2	3	4	5	7	10	13	16
Aligned	0.70	0.80	0.86	0.90	0.92	0.95	0.97	0.98	0.99
Staggered	0.64	0.76	0.84	0.89	0.92	0.95	0.97	0.98	0.99

Convection for Tube Banks in a Cross Flow

- **Again, staggering will increase heat flux BUT at a penalty of flow losses**

- there is generally as much interest in the pressure drop associated with flow across a tube bank as in the overall heat transfer rate

- The power required to move the fluid across the bank is often a major operating expense and is directly proportional to the pressure drop, which may be expressed as:

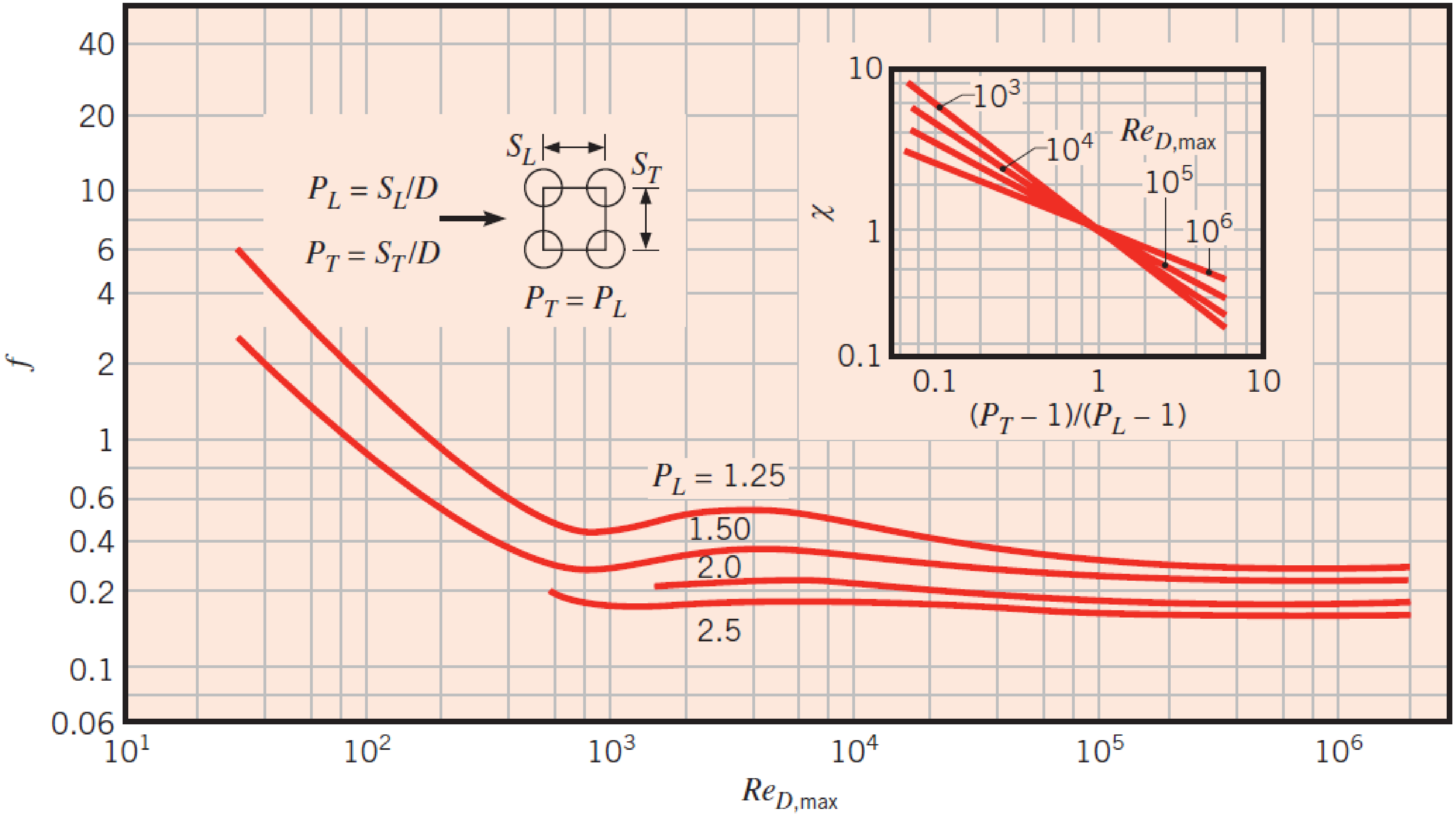
$$\Delta p = N_L \chi \left(\frac{\rho {V_{max}}^2}{2} \right) f \qquad \text{(Eq. 7.65)}$$

- Here, Δp is the head loss of the bank, χ is the correction factor, and f is the friction factor

- Both χ and f can be found in Tables 7.14 1nd 7.15 for aligned and staggered banks, respectively

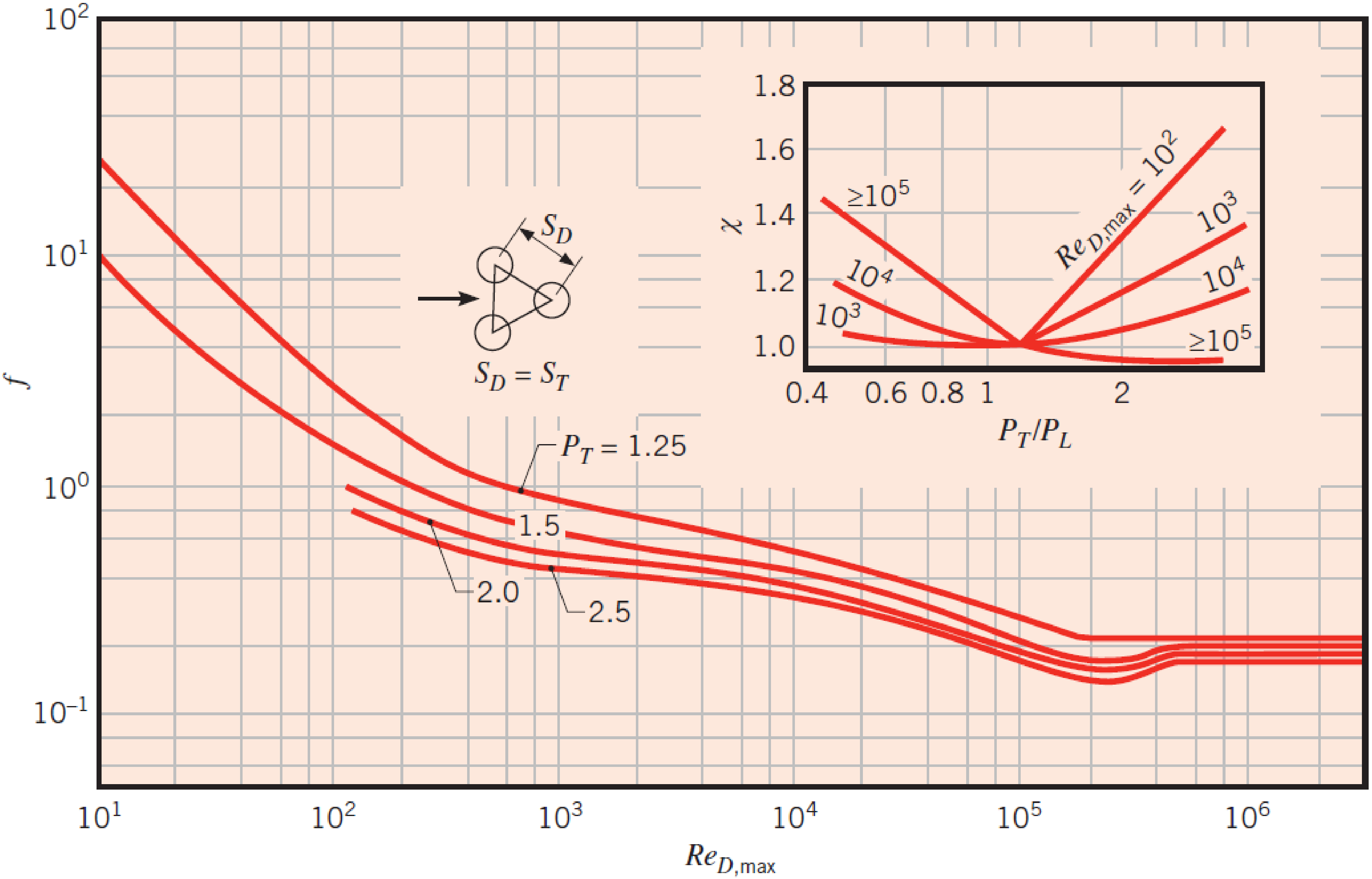
Convection for Tube Banks in a Cross Flow

$$\Delta p = N_L \chi \left(\frac{\rho V_{max}{}^2}{2}\right) f$$



Convection for Tube Banks in a Cross Flow

$$\Delta p = N_L \chi \left(\frac{\rho V_{max}{}^2}{2} \right) f$$



Convection for Tube Banks in a Cross Flow

For flows through tube banks, the total heat transfer is often sufficient to change the temperature of the free stream low by a significant amount.

For overall convection calculations, the `log mean temperature difference' ΔT_{lm} is used to characterize the mean temperature difference between the surface and free stream:

$$\Delta T_{\mathrm{lm}} = \frac{(T_s - T_i) - (T_s - T_o)}{\ln\left(\dfrac{T_s - T_i}{T_s - T_o}\right)} \tag{7.62}$$

where T_i and T_o are temperatures of the fluid as it enters and leaves the bank, respectively. The outlet temperature, which is needed to determine ΔT_{lm}, may be estimated from

$$\frac{T_s - T_o}{T_s - T_i} = \exp\left(-\frac{\pi D N \bar{h}}{\rho V N_T S_T c_p}\right) \tag{7.63}$$

where N is the total number of tubes in the bank and N_T is the number of tubes in each row.

Convection for Tube Banks in a Cross Flow

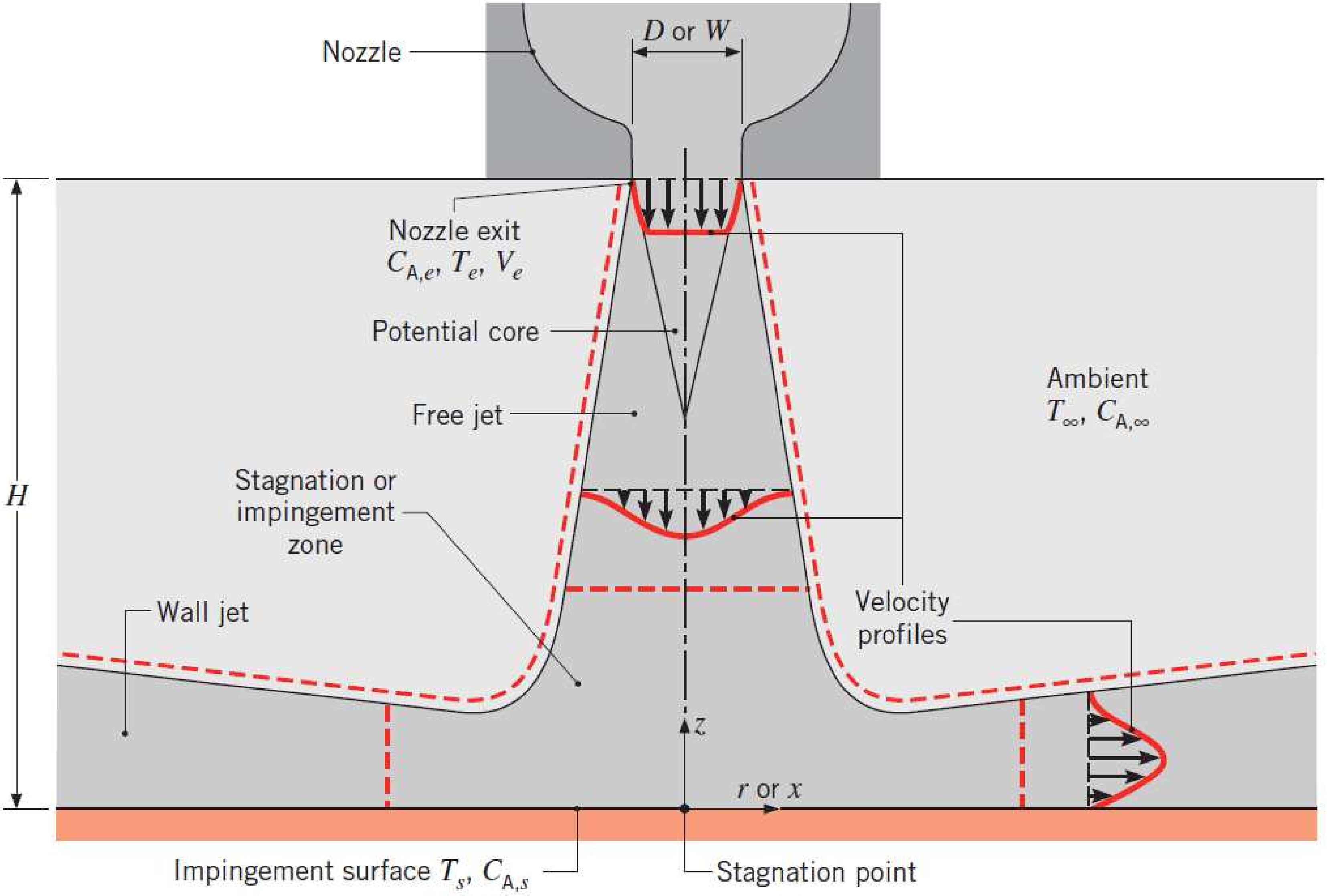
Example (Problem 7.90)

A preheater involves the use of condensing steam at 100°C on the inside of a bank of tubes to heat air that enters at 1 atm and 25°C. The air moves at 5 m/s in cross flow over the tubes. Each tube is 1 m long and has an outside diameter of 10 mm. The bank consists of 196 tubes in a square, aligned array for which $S_T = S_L = 15$ mm.

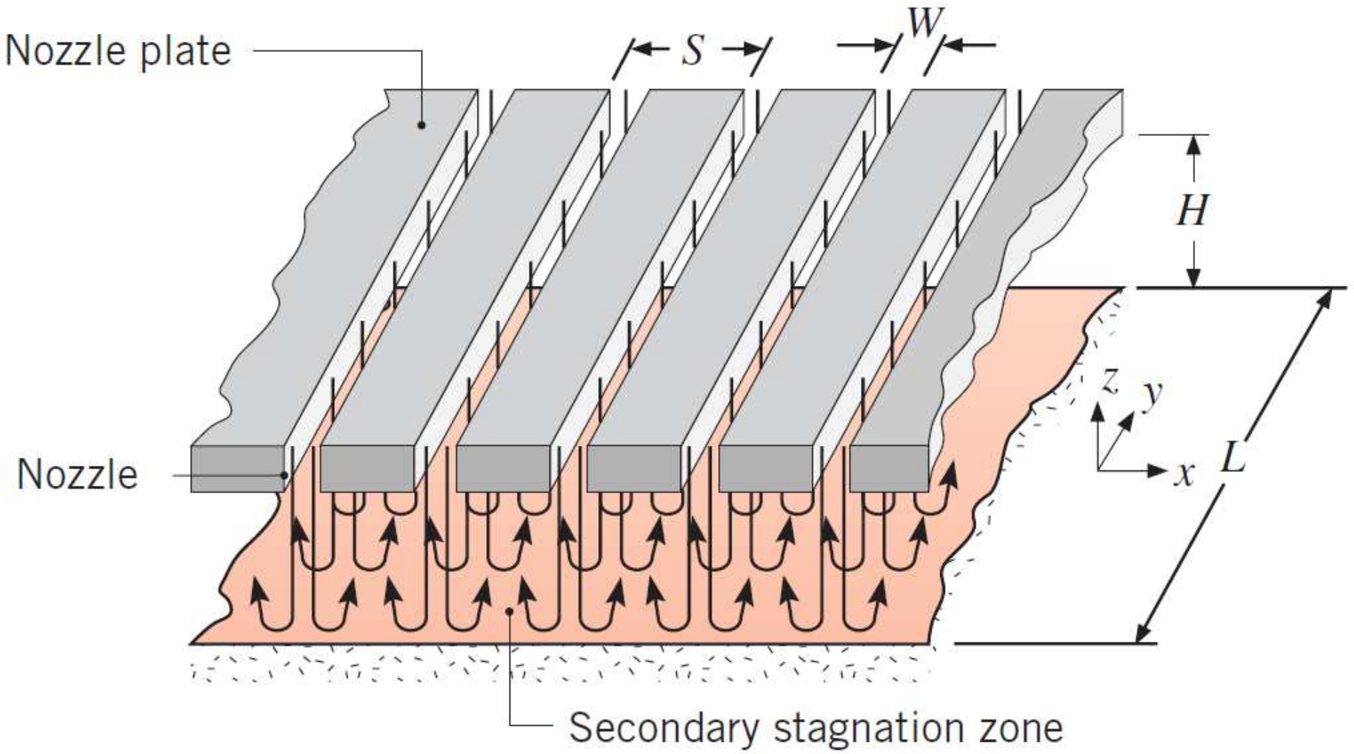
What is the total rate of heat transfer to the air?

What is the pressure drop associated with the airflow?

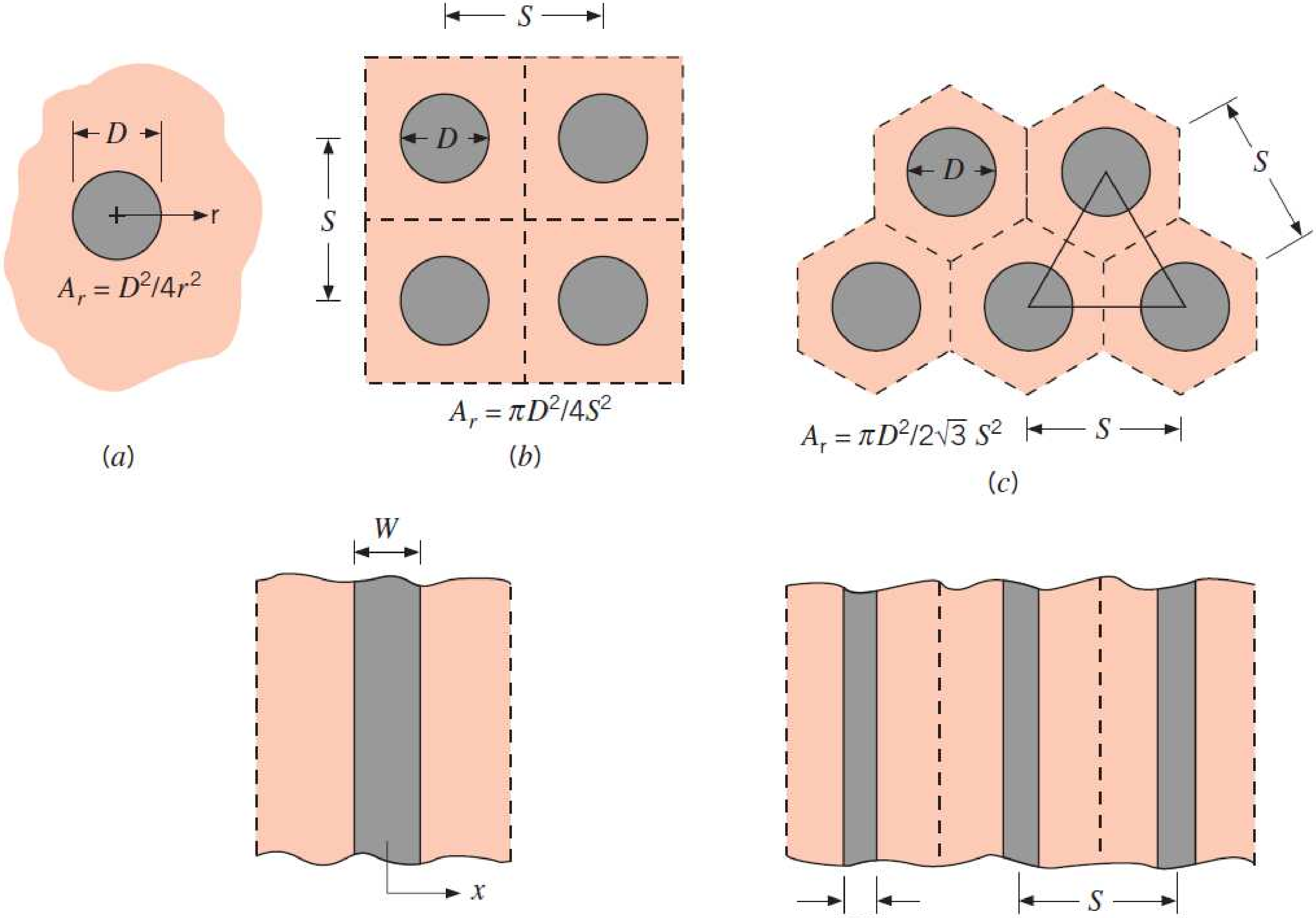
Convection for Impinging Jets



Convection for Impinging Jets



Convection for Impinging Jets



$A_r = D^2/4r^2$

(*a*)

$A_r = \pi D^2/4S^2$

(*b*)

$A_r = \pi D^2/2\sqrt{3}\, S^2$

(*c*)

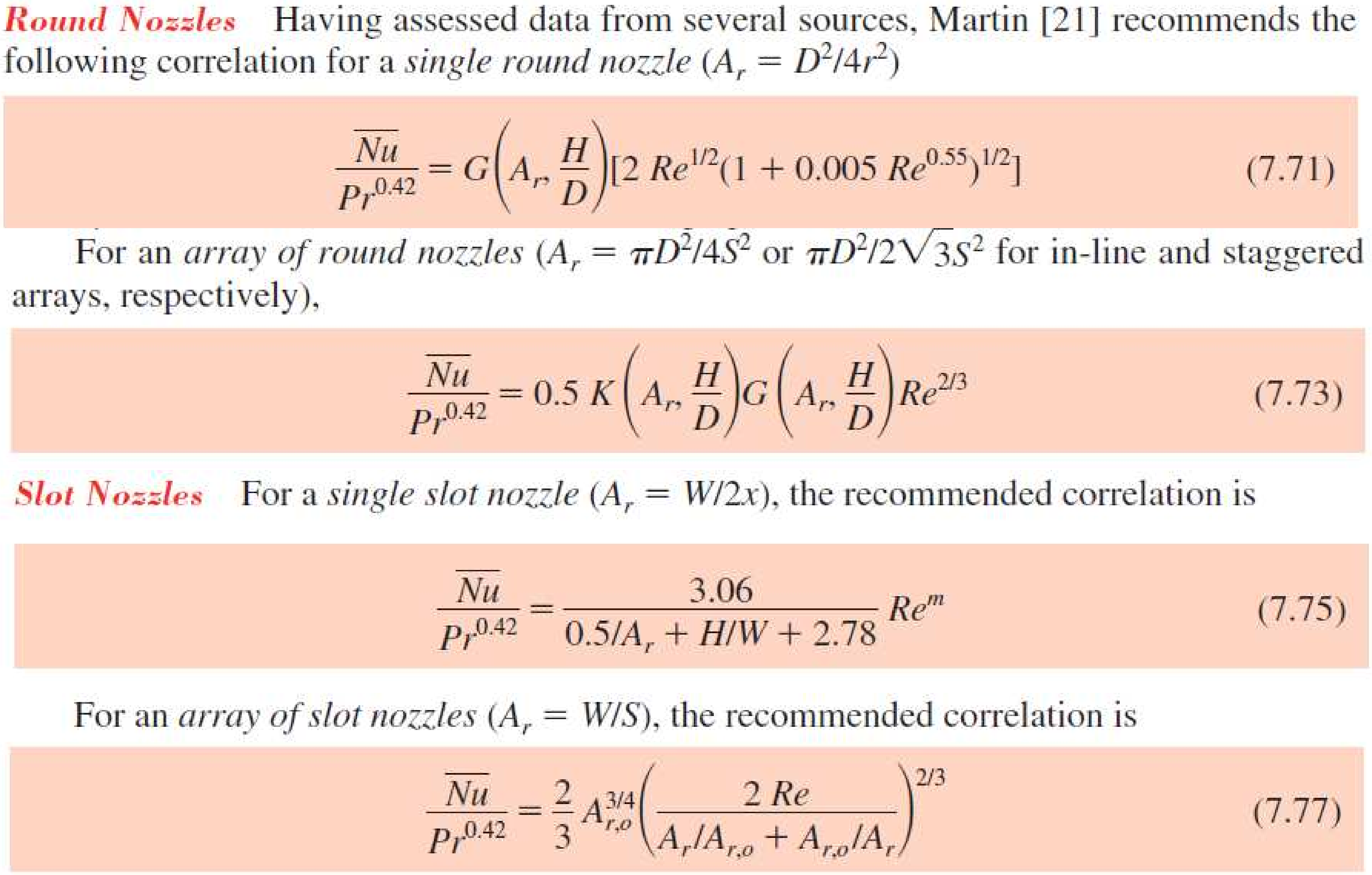
$A_r = W/2x$

(*d*)

$A_r = W/S$

(*e*)

Convection for Impinging Jets



Round Nozzles Having assessed data from several sources, Martin [21] recommends the following correlation for a *single round nozzle* ($A_r = D^2/4r^2$)

$$\frac{\overline{Nu}}{Pr^{0.42}} = G\left(A_r, \frac{H}{D}\right)\left[2\,Re^{1/2}\left(1 + 0.005\,Re^{0.55}\right)^{1/2}\right] \qquad (7.71)$$

For an *array of round nozzles* ($A_r = \pi D^2/4S^2$ or $\pi D^2/2\sqrt{3}S^2$ for in-line and staggered arrays, respectively),

$$\frac{\overline{Nu}}{Pr^{0.42}} = 0.5\,K\left(A_r, \frac{H}{D}\right)G\left(A_r, \frac{H}{D}\right)Re^{2/3} \qquad (7.73)$$

Slot Nozzles For a *single slot nozzle* ($A_r = W/2x$), the recommended correlation is

$$\frac{\overline{Nu}}{Pr^{0.42}} = \frac{3.06}{0.5/A_r + H/W + 2.78}Re^m \qquad (7.75)$$

For an *array of slot nozzles* ($A_r = W/S$), the recommended correlation is

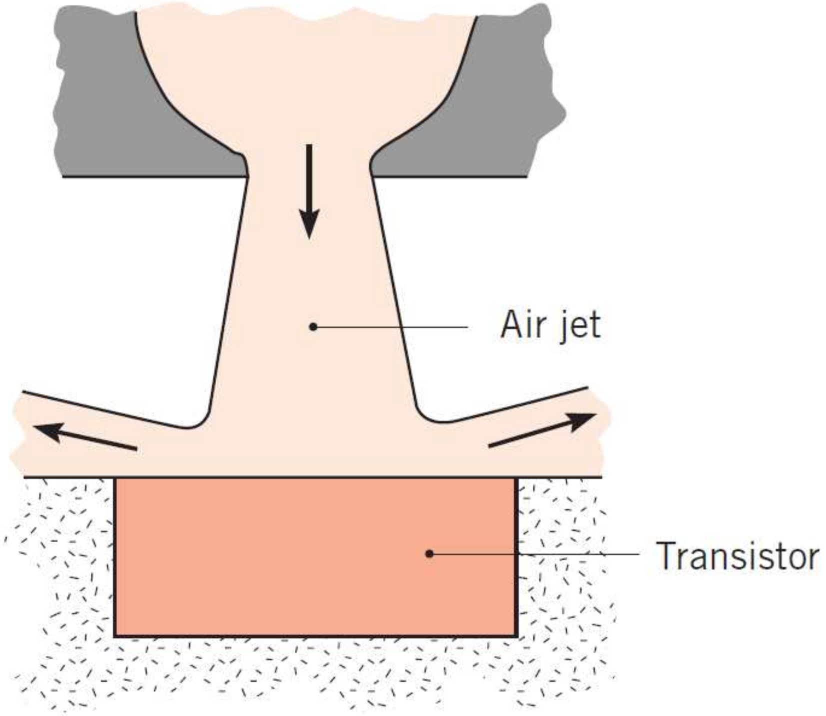
$$\frac{\overline{Nu}}{Pr^{0.42}} = \frac{2}{3}A_{r,o}^{3/4}\left(\frac{2\,Re}{A_r/A_{r,o} + A_{r,o}/A_r}\right)^{2/3} \qquad (7.77)$$

Convection for Impinging Jets

Example (Problem 7.100)

A circular transistor of 10-mm diameter is cooled by impingement of an air jet exiting a 2-mm-diameter round nozzle with a velocity of 20 m/s and a temperature of 15C. The jet exit and the exposed surface of the transistor are separated by a distance of 10 mm.

If the transistor is well insulated at all but its exposed surface and the surface temperature is not to exceed 85C, what is the transistor's maximum allowable operating power?



Convection for Impinging Jets

Example (Problem 7.100)

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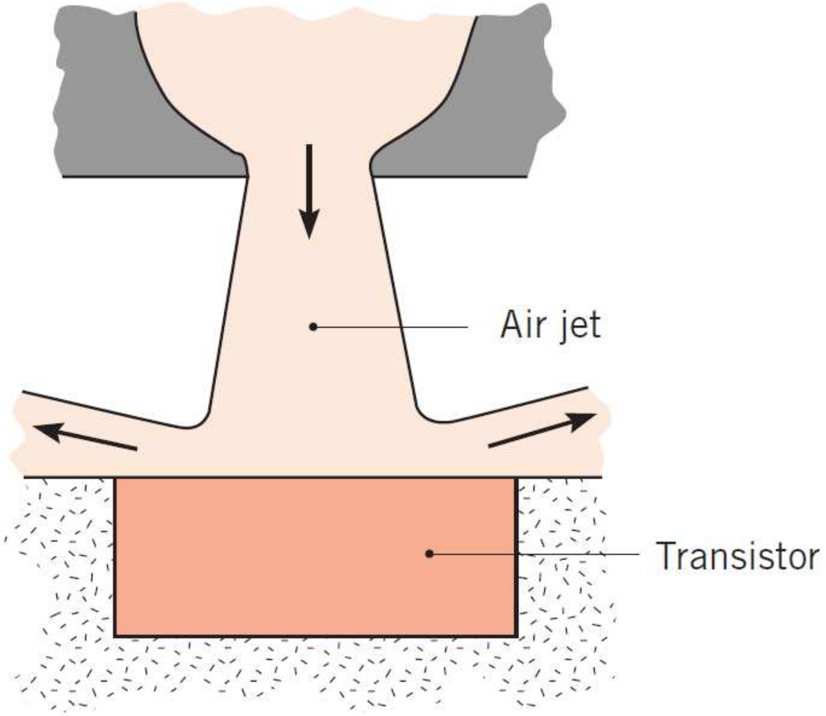


Approaching These External Flow Problems

Application of a convection correlation for *any flow situation* are facilitated by following a few simple rules.

1. ***Become immediately cognizant of the flow geometry.*** Flat plate? Cylinder? Sphere? Tube bank? Impinging jets?

2. ***Specify the appropriate reference temperature and evaluate the pertinent fluid properties at that temperature.*** May need to refer to material properties table (Table A.4)

3. ***Calculate the Reynolds number.*** Boundary layer conditions are strongly influenced by this parameter… determine whether the flow is laminar or turbulent.

4. ***Decide whether a local or surface average coefficient is required.***

5. ***Select the appropriate correlation.***

References

- Bergman, Lavine, Incropera, and Dewitt, “*Fundamentals of Heat and Mass Transfer, 7th Ed.,*” Wiley, 2011
- D. E. Hitt, “*External Convection Relations*,” ME 144 Lecture Notes, University of Vermont, Spring 2008
- Chapman, “*Heat Transfer, 3rd Ed.,*” MacMillan, 1974
- Y. A. Çengel and A. J. Ghajar, “*Heat and Mass Transfer, 5th Ed.,*” Wiley, 2015

ME 144: Heat Transfer
Internal Convection (v 1.0)

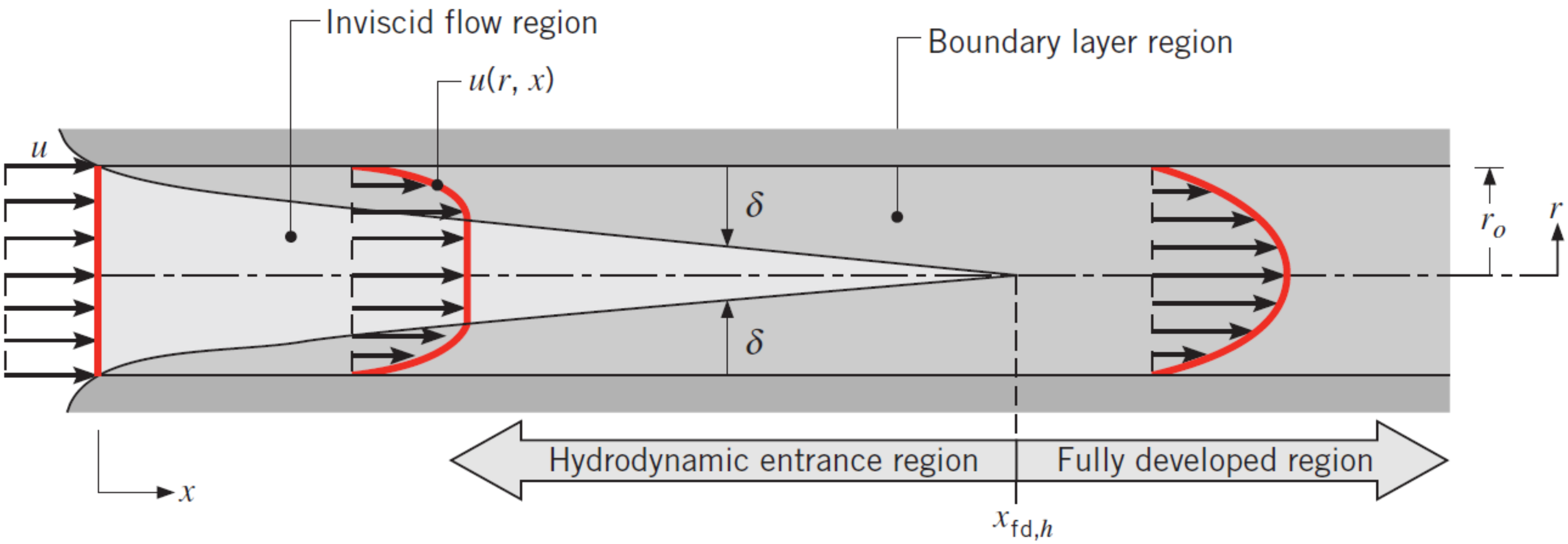
J. M. Meyers

Initial Remarks

- Here we examine the convective heat transfer that occurs in internal flows (i.e. pipe flow)

- There are several equivalent concepts much like the fluid dynamic pipe flow problem:
  - Entry lengths
  - Flow development
  - Notion of fully developed flow

- We will assume that you are already quite familiar with the fluid dynamic aspects allowing us to focus more on the heat transfer



Mean Temperature

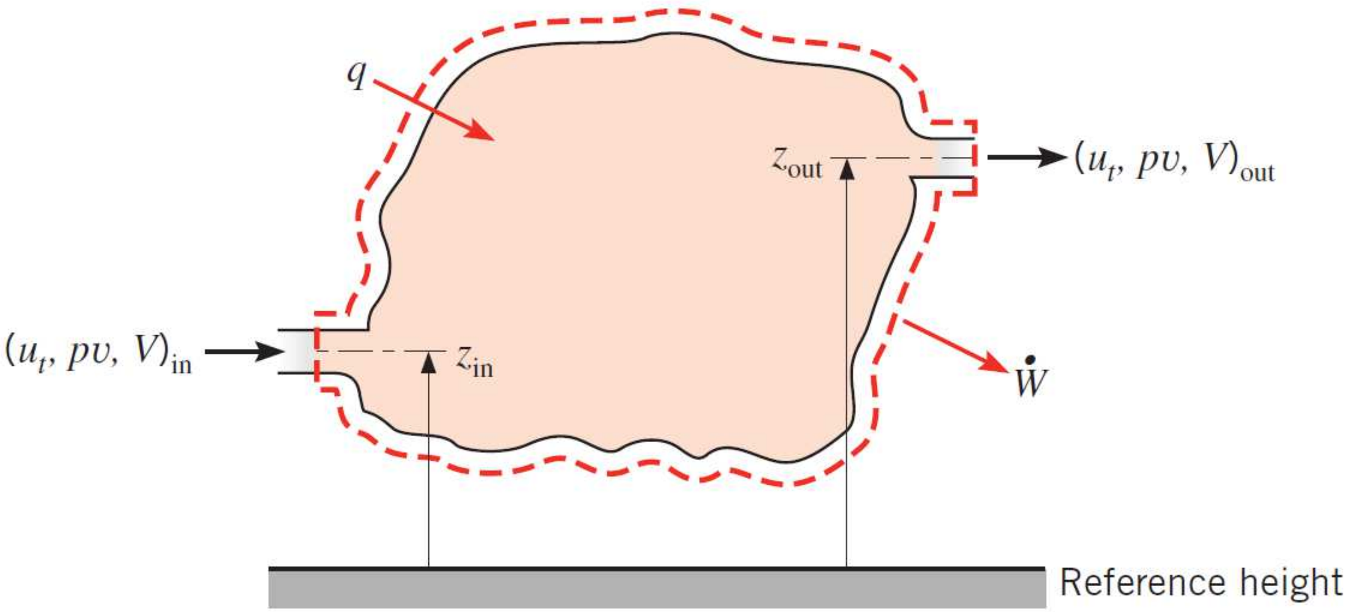
- There is no fixed free stream temperature in these problems- owing to this, another temperature is introduced… recall Newton's Law of cooling:

$$q" = h(T_s - T_\infty)$$

- Consider the following simplified steady-flow thermal energy equation from Chpt. 1:

$$q = \dot{m}C_p(T_{out} - T_{in})$$

- There is no uniform temperature profile (especially in pipe flow) so it is common to use a mean temperature, T_m



Mean Temperature

- This is an important quantity of pipe flow defined for a pipe cross section A at a downstream location x

- The mean temperature is **defined** so as to be equivalent to the integrated convective heat transport:

$$\dot{m} C_p T_m = \int_A \rho C_p u T dA$$

Eq 8.24

(equivalent heat flux) (actual energy flux)

- Noting that: $\dot{m} = \int_A \rho u dA$ ➡ $T_m = \dfrac{\int \rho C_p u T dA}{C_p \int \rho u dA}$

- If C_p and ρ are constant:

$$\boxed{T_m = \frac{\int uTdA}{\int udA}}$$

Temperature averaged by area-weighted velocity

Mean Temperature

- For a circular pipe with fully developed flow (using $\dot{m} = \rho u_m A_c$ where u_m is the average velocity and A_c is the tube cross-section) it follows:

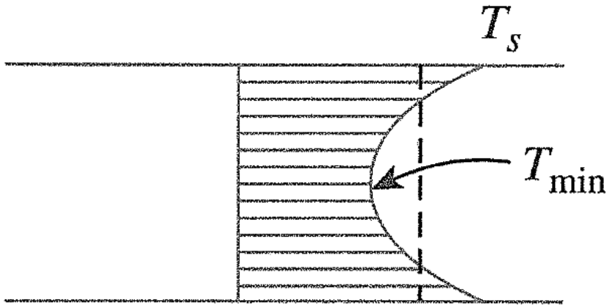
$$\dot{m} C_p T_m = \int_{A_c} \rho C_p u T dA_c$$

$$\Downarrow$$

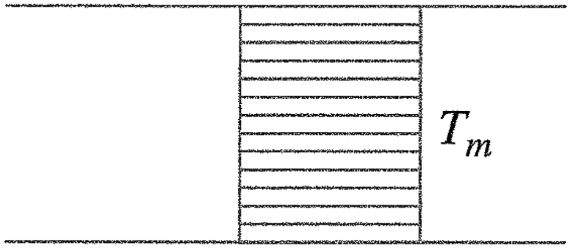
$$T_m = \frac{2}{u_m {r_0}^2} \int_0^{r_0} uTrdr$$

Eq 8.26

Poiseuille flow



(a) Actual



(b) Idealized

Newton’s Law of Cooling for a Pipe

- For pipe flow, we modify Newton’s Law of Cooling to account for the mean temperature based on a *local* convection heat transfer coefficient, h

$$q_s" = h(T_s - T_m)$$

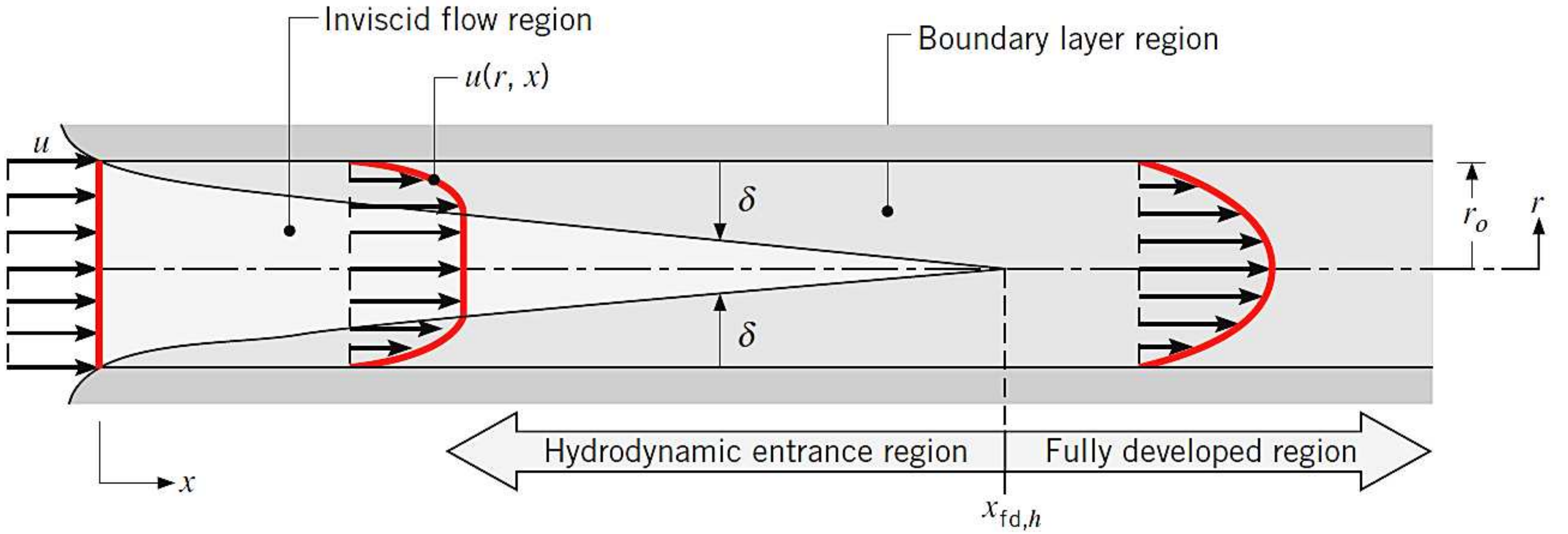
- Where q_s and T_s are the surface flux and temperature, respectively.

- We should note that where T_∞ is constant for free flows, T_m will vary in the x-direction such that:

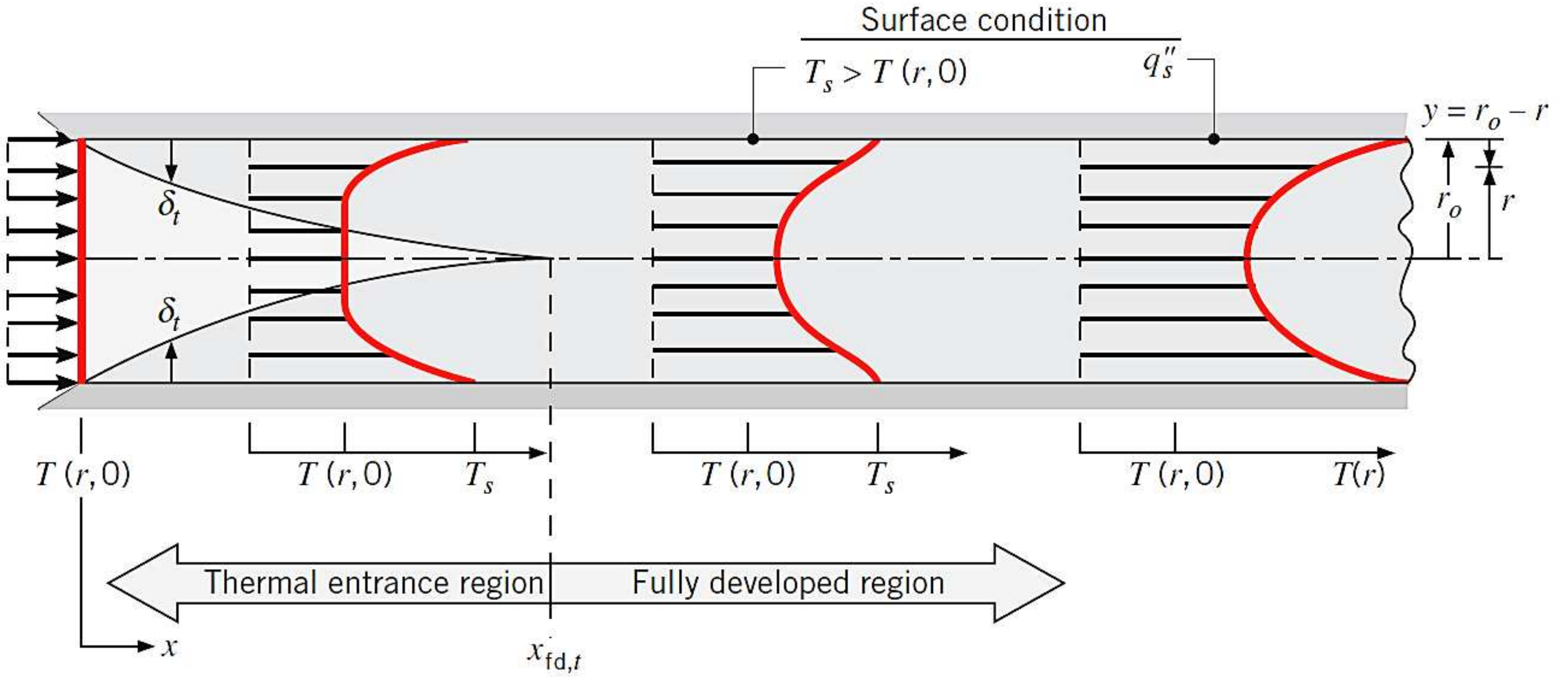
$$\frac{dT_m}{dx} \neq 0$$

Thermally Developed Flow

Velocity Boundary Layer Development



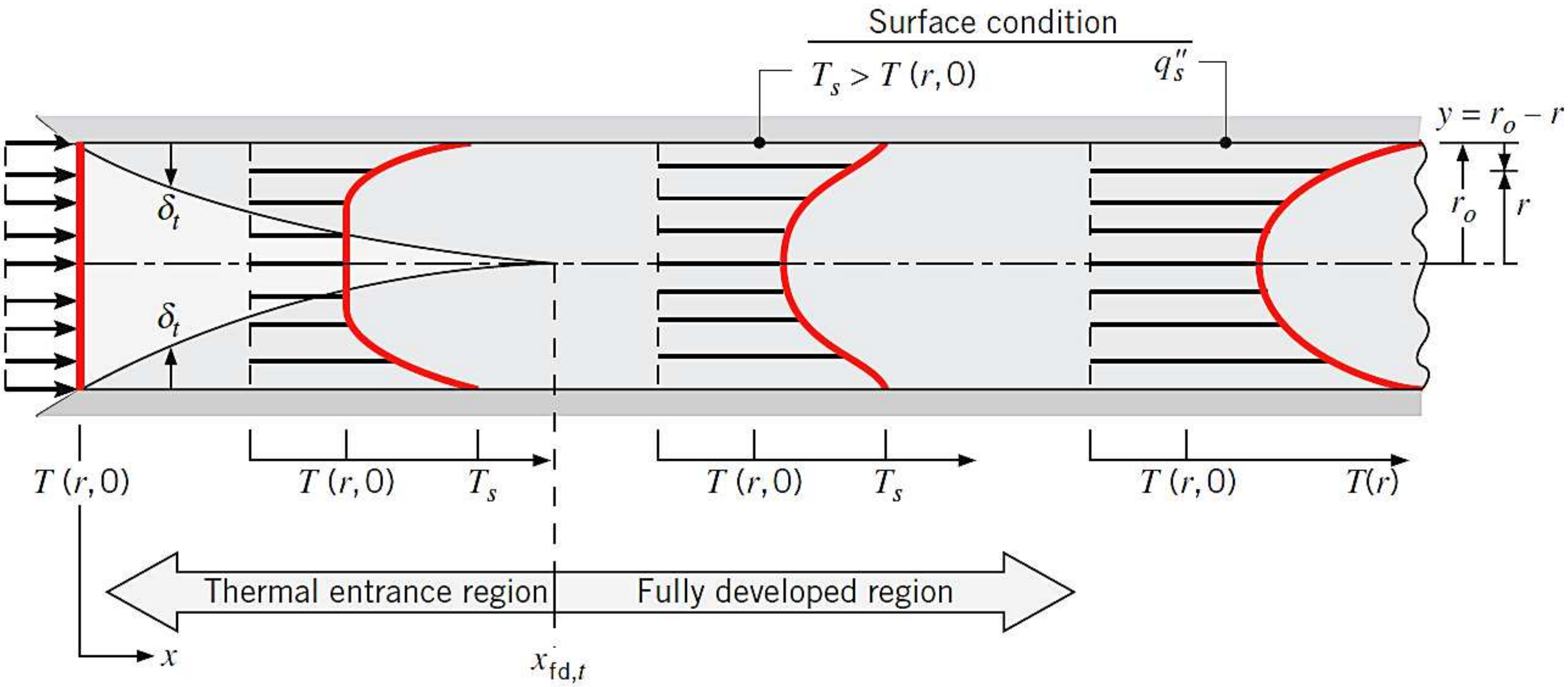
Thermal Boundary Layer Development



Thermally Developed Flow

- As with the fluid dynamic problem, there are the notions of a “thermal entry length” and that of a “thermally developed” flow
- However, in the case of a constant heat flux (or any continuous wall heating) that T_m will never reach a constant value ($dT_m/dx \neq 0$)
- What we really mean for thermally developed flow then, is not constant temperature, but a constant relative shape of the profile (**RECALL SELF-SIMILAR VELOCITY PROFILES**)

Thermal Boundary Layer Development



Thermally Developed Flow

- The requirement for such a constant shape condition for a fully developed thermal BL is formally stated as:

$$\frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right] = 0$$

Eq 8.28

- For convenience, let's define a scaled dimensionless temperature profile:

$$\theta(x,r) = \frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}$$

- If $\partial\theta/\partial x = 0$ for thermally developed flow then it must be that:

$$\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial\theta}{\partial x}\right) = \frac{\partial}{\partial r}(0) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial r}\right) = 0$$

Thermally Developed Flow

- But:

$$\frac{\partial\theta}{\partial r} = -\frac{1}{T_s(x) - T_m(x)}\frac{\partial T}{\partial r} = -\frac{\partial T/\partial r}{T_s - T_m} \neq f(x)$$

NOTE: T_s and T_m are constants insofar as differentiation w.r.t. r is concerned!

- And at the wall:

$$\left.\frac{\partial\theta}{\partial r}\right|_{wall} = -\frac{\left.\partial T/\partial r\right|_{wall}}{T_s - T_m} \neq f(x)$$

- Heat flux at the wall:

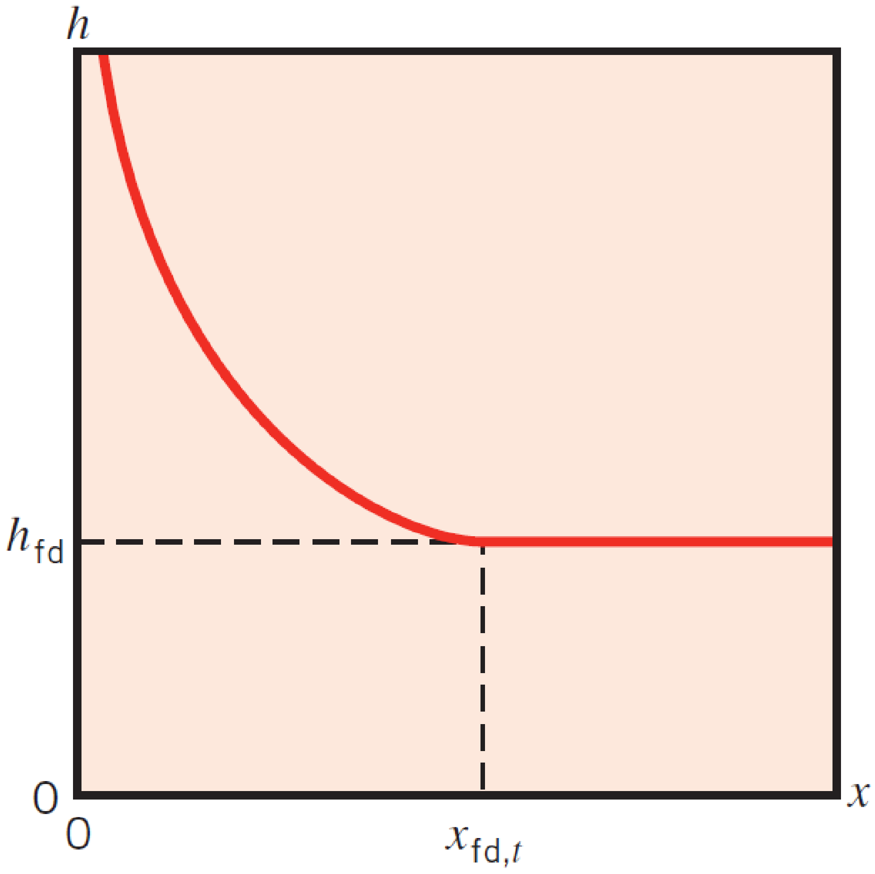
$$q_s" = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} = -k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = h(T_s - T_m) \Rightarrow -\frac{h}{k} = \frac{\left.\partial T/\partial r\right|_{r=r_0}}{T_s - T_m} \neq f(x)$$

$$\boxed{-\frac{h}{k} \neq f(x)}$$

Thermally Developed Flow

- Thus, if k is a constant property, then h will become constant once flow in a pipe becomes thermally developed

- Recall for the velocity boundary layer that the local friction factor (related to the local wall shear stress) became constant once the velocity profile became fully-developed



Thermally Developed Flow

- We can make more interesting points under a simplification of uniform surface heat flux:

By assumption: $q_s" = h(T_s - T_m) = \text{constant}$

For thermally developed flow: $h = \text{constant}$

- Then:

$$\frac{d}{dx}(q_s") = \frac{d}{dx}[h(T_s - T_m)] = 0$$

$$0 = h\left(\frac{dT_s}{dx} - \frac{dT_m}{dx}\right)$$

$$\boxed{\frac{dT_s}{dx} = \frac{dT_m}{dx}}$$

For uniform surface heat flux

Thermally Developed Flow

- Now according to our definition for $\theta(x, r)$

$$\frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right] = 0$$

$$\frac{(T_s - T_m)\left[\frac{dT_s}{dx} - \frac{\partial T}{\partial x}\right] - (T_s - T)\left[\frac{dT_s}{dx} - \frac{dT_m}{dx}\right]}{(T_s - T_m)^2} = 0$$

Quotient rule

$$\frac{dT_s}{dx} - \frac{\partial T}{\partial x} - \left(\frac{1}{T_s - T_m}\right)(T_s - T)\left[\frac{dT_s}{dx} - \frac{dT_m}{dx}\right] = 0$$

$$\left.\frac{\partial T}{\partial x}\right|_{fdt} = \left.\frac{dT_s}{dx}\right|_{fdt} - \left(\frac{T_s - T}{T_s - T_m}\right)\left[\frac{dT_s}{dx} - \frac{dT_m}{dx}\right]_{fdt}$$

Thermally Developed Flow

$$\left.\frac{\partial T}{\partial x}\right|_{fdt} = \left.\frac{dT_s}{dx}\right|_{fdt} - \left(\frac{T_s - T}{T_s - T_m}\right)\left[\frac{dT_s}{dx} - \frac{dT_m}{dx}\right]_{fdt}$$

- Recall that if $q_s" = \text{constant}$: $\frac{dT_s}{dx} = \frac{dT_m}{dx}$

$\left.\frac{\partial T}{\partial x}\right|_{fdt} = \left.\frac{dT_s}{dx}\right|_{fdt}$ ➡ $\frac{\partial T}{\partial x}$ does not depend on r!!!!

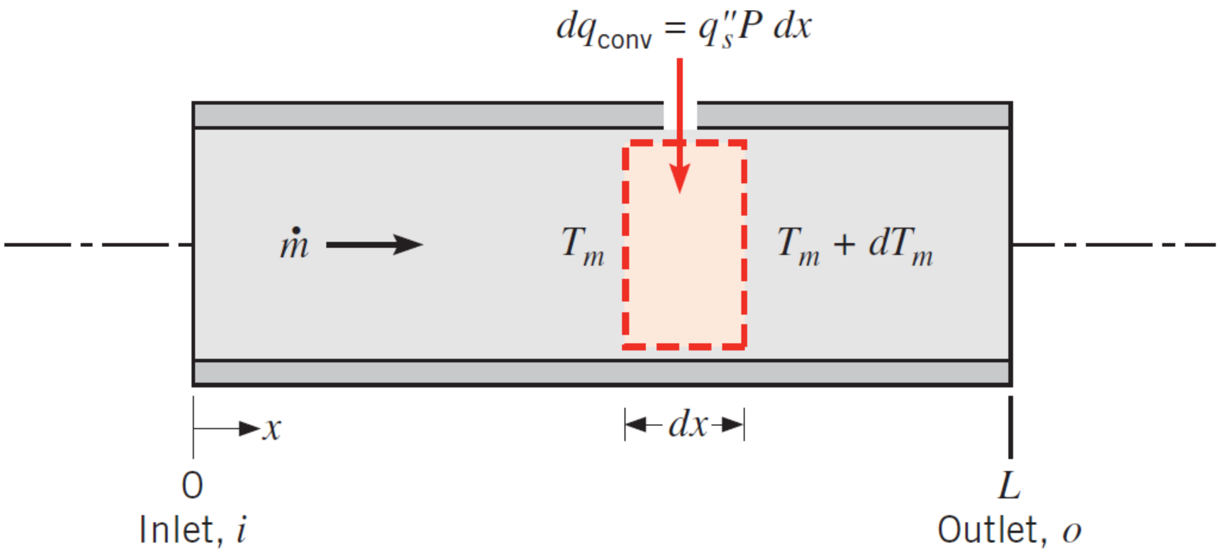
- This is NOT the case for constant surface temperature, here: $\frac{dT_s}{dx} = 0$

$\left.\frac{\partial T}{\partial x}\right|_{fdt} = \left(\frac{T_s - T}{T_s - T_m}\right)\left[\frac{dT_m}{dx}\right]_{fdt}$ ➡ r dependence still exists in T

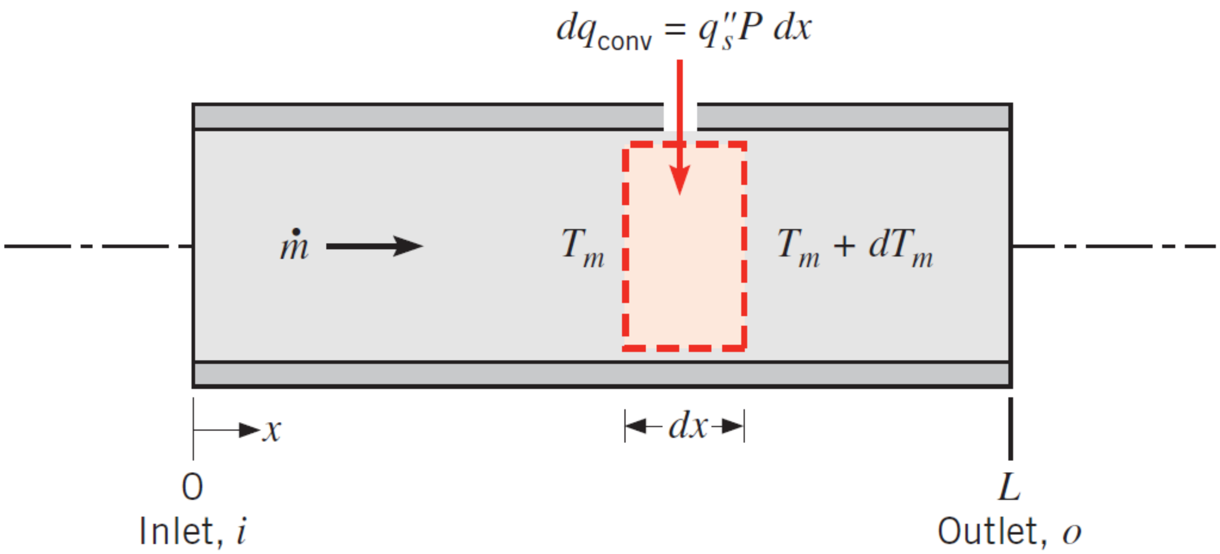
- The uniform flux case is suggestive of Poiseuille flow with $\partial P/\partial x \neq f(r)$ and one may expect a parabolic-type profile

Energy Balance in Pipe Flow

- As flow in a tube is completely enclosed, an energy balance may be applied to determine how the mean temperature varies with position along the tube and how the total convection heat transfer is related to the difference in temperatures at the tube inlet and outlet.



Energy Balance in Pipe Flow



- From a 1D perspective using T_m, a basic energy balance gives:

$$dq_{conv} = d\dot{E} \Rightarrow \underset{\substack{\uparrow \\ \textbf{perimeter}}}{q"_s P} dx = \underset{\substack{\uparrow \\ \textbf{energy storage}}}{\dot{m}C_p} dT_m \Rightarrow \frac{dT_m}{dx} = \frac{q"_s P}{\dot{m}C_p}$$

$$\boxed{\frac{dT_m}{dx} = \frac{hP}{\dot{m}C_p}(T_s - T_m)}$$

Eq 8.37

Energy Balance in Pipe Flow

Constant Heat Flux

- If $q"_s$ is constant, then:

$$\frac{dT_m}{dx} = \frac{q"_s P}{\dot{m} C_p} = \text{constant}$$

- Integrating from $x = 0$:

$$T_m(x) = \underset{\substack{\uparrow \\ \textbf{Inlet Temperature}}}{T_m(x = 0)} + \frac{q"_s P}{\dot{m} C_p} x$$

$$\boxed{T_m(x) = T_{m,i} + \frac{q"_s P}{\dot{m} C_p} x} \Rightarrow \textbf{Linear Variation}$$

Eq 8.40

Energy Balance in Pipe Flow

Constant Surface Temperature

- If T_s is constant, and we let $\Delta T = T_s - T_m$, then:

$$\frac{dT_m}{dx} = -\frac{d(\Delta T)}{dx} = \frac{hP}{\dot{m}C_p}(\Delta T)$$

$$\frac{d(\Delta T)}{dx} + \frac{hP}{\dot{m}C_p}(\Delta T) = 0$$

Here, $h = h(x)$ as the flow is not assumed to be fully developed

- Separating variables:

$$\frac{d(\Delta T)}{(\Delta T)} = -\frac{h(x)P}{\dot{m}C_p}dx$$

Energy Balance in Pipe Flow

Constant Surface Temperature

- Integrate from the inlet to some location x

$$\int_{\Delta T_i}^{\Delta T} \frac{d(\Delta T)}{(\Delta T)} = -\int_0^x \frac{h(x)P}{\dot{m}C_p}dx \quad \Rightarrow \quad \ln\left(\frac{\Delta T}{\Delta T_i}\right) = -\frac{P}{\dot{m}C_p}\int_0^x h(x)dx$$

- Recalling our average heat transfer coefficient definition from Eq. 6.13: $\int_0^x h(x)dx \equiv \bar{h}_x \cdot x$

$$\frac{\Delta T}{\Delta T_i} = \exp\left[-\left(\frac{P\bar{h}_x}{\dot{m}C_p}\right)x\right]$$

- At the outlet where $x = L$

$$\frac{\Delta T_o}{\Delta T_i} = \exp\left[-\left(\frac{P\bar{h}_L}{\dot{m}C_p}\right)L\right]$$

Eq 8.41b

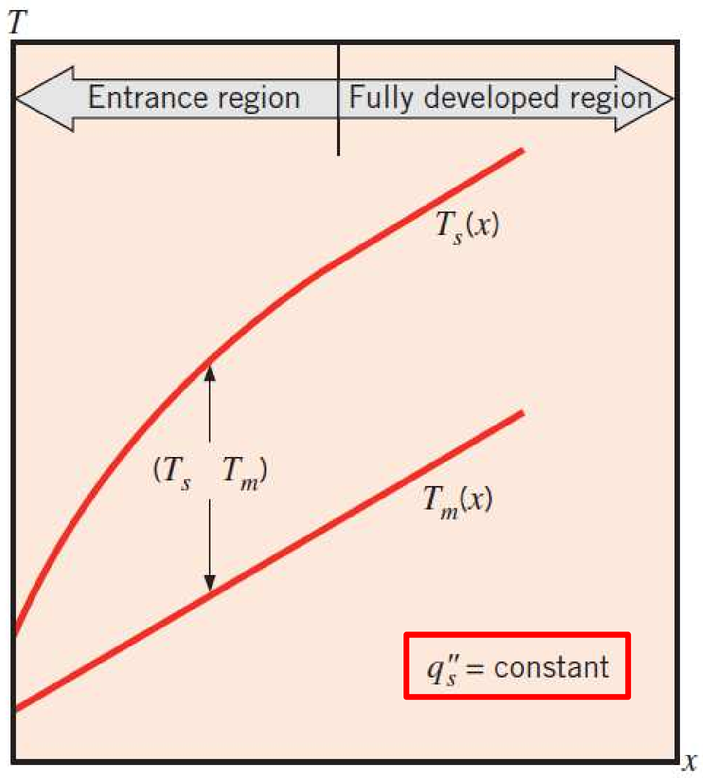
- Or in terms of T_s and T_m:

Exponential Variation

$$\frac{T_s(x) - T_m(x)}{T_s(x=0) - T_m(x=0)} = \boxed{\frac{T_s - T_m(x)}{T_s - T_{m,i}} = \exp\left[-\left(\frac{P\bar{h}_x}{\dot{m}C_p}\right)x\right]}$$

Eq 8.42

Energy Balance in Pipe Flow

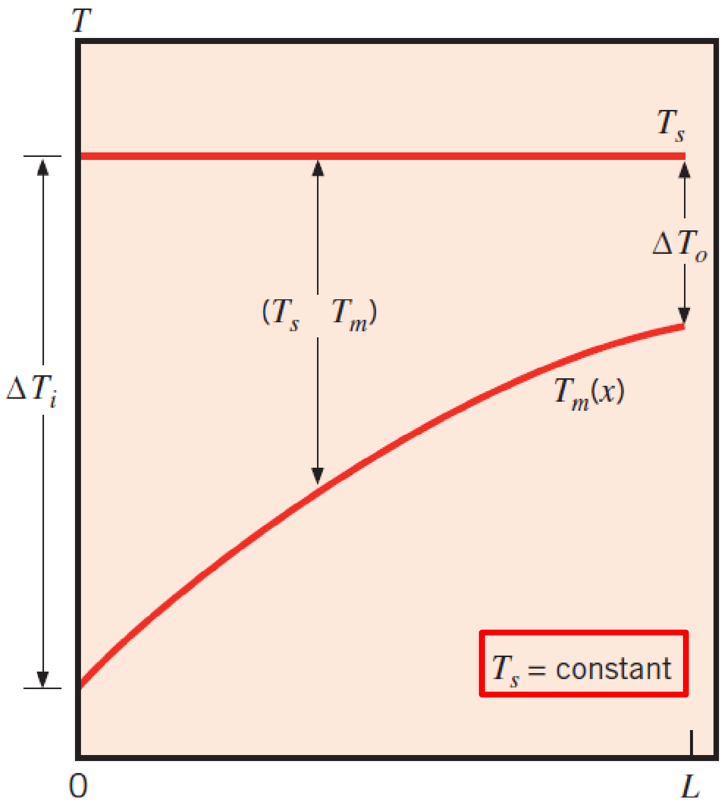


(a)

$$T_m(x) = T_{m,i} + \frac{q''_s P}{\dot{m} C_p} x$$

$$\frac{dT_s}{dx} = \frac{dT_m}{dx}$$

Once fully developed



(b)

$$\frac{T_s - T_m(x)}{T_s - T_{m,i}} = \exp\left[-\left(\frac{P\bar{h}_x}{\dot{m} C_p}\right) x\right]$$

Energy Balance in Pipe Flow

Constant Surface Temperature (Connection to Log Mean Temperature Difference)

- Determining an expression for total heat transfer rate is complicated by the exponential nature of:

$$\frac{T_s - T_m(x)}{T_s - T_{m,i}} = \exp\left[-\left(\frac{P\bar{h}_x}{\dot{m}C_p}\right)x\right]$$

- For a given length L we have:

$$\ln\left(\frac{\Delta T_o}{\Delta T_i}\right) = -\frac{PL}{\dot{m}C_p}\bar{h}_L =- -\frac{A_s}{\dot{m}C_p}\bar{h}_L$$

$$\left\| \dot{m}C_p = -\frac{A_s\bar{h}_L}{\ln\left(\frac{\Delta T_o}{\Delta T_i}\right)} \right\|$$

- Over length L we can also say from a total energy balance that:

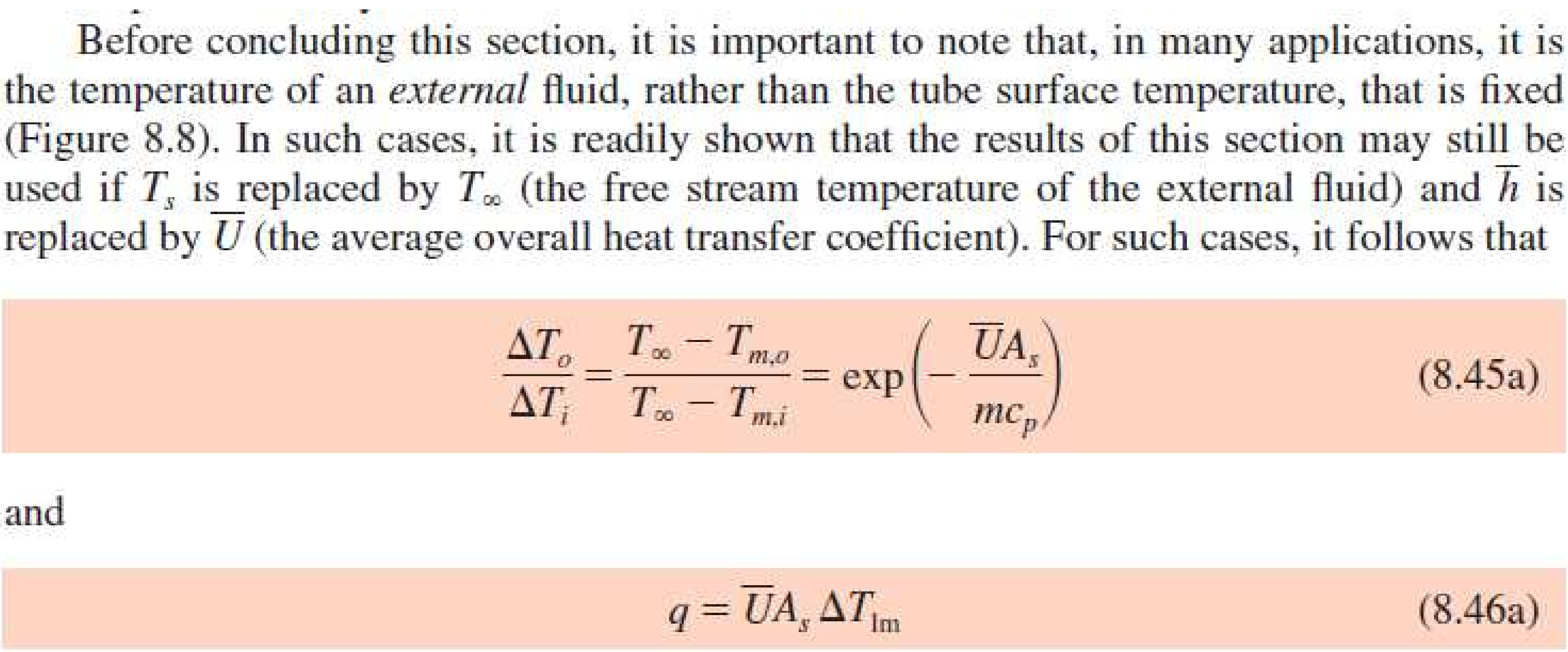
$$q_{conv} = \dot{m}C_p\left(T_{m,o} - T_{m,i}\right) = \dot{m}C_p\left(\left(T_s - T_{m,i}\right) - \left(T_s - T_{m,o}\right)\right) \Rightarrow \left\| q_{conv} = \dot{m}C_p[\Delta T_i - \Delta T_o] \right\|$$

- Combining the two highlighted relations:

$$\boxed{q_{conv} = A_s\bar{h}_L\frac{(\Delta T_o - \Delta T_i)}{\ln(\Delta T_o/\Delta T_i)} = A_s\bar{h}_L\Delta T_{lm}}$$

Eq. 8.43 and 8.44

Energy Balance in Pipe Flow



Before concluding this section, it is important to note that, in many applications, it is the temperature of an *external* fluid, rather than the tube surface temperature, that is fixed (Figure 8.8). In such cases, it is readily shown that the results of this section may still be used if T_s is replaced by T_∞ (the free stream temperature of the external fluid) and $\overline{h}$ is replaced by $\overline{U}$ (the average overall heat transfer coefficient). For such cases, it follows that

$$\frac{\Delta T_o}{\Delta T_i} = \frac{T_\infty - T_{m,o}}{T_\infty - T_{m,i}} = \exp\left(-\frac{\overline{U}A_s}{\dot{m}c_p}\right) \qquad (8.45a)$$

and

$$q = \overline{U}A_s\,\Delta T_{\text{lm}} \qquad (8.46a)$$

Convection Coefficients for Laminar Flow in Pipes

- The preceding results all assume a known value for the convection coefficient h
- Here, we obtain expressions for h for fully developed laminar flow in circular pipes
- In axisymmetric, cylindrical coordinates, the temperature field is governed by:

$$u\frac{\partial T}{\partial x} = \alpha \nabla^2 T = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]$$

- This assumes:

1. Steady flow, $(\partial/\partial t) = 0$
2. Fully developed, $v = 0$
3. Negligible axial conduction $\quad \dfrac{\partial^2 T}{\partial x^2} \ll u\dfrac{\partial T}{\partial x} \qquad \dfrac{\partial^2 T}{\partial x^2} \ll \nabla^2 T$

Convection Coefficients for Laminar Flow in Pipes

$$u\frac{\partial T}{\partial x} = \alpha \nabla^2 T = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]$$

- Now, in the fully developed region

$$u = 2\underset{\substack{\uparrow \\ \text{mean velocity}}}{u_m}\left[1-\left(\frac{r}{R}\right)^2\right]$$

See your fluid dynamics relations for fully developed flow

- Also, for thermally developed flow:

$$\frac{\partial T}{\partial x} = \frac{dT_m}{dx}$$

$$\Rightarrow \qquad 2u_m\left[1-\left(\frac{r}{R}\right)^2\right]\frac{dT_m}{dx} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Convection Coefficients for Laminar Flow in Pipes

- If we are to assume a constant heat flux ($q"_s$ = constant), then:

$$\frac{dT_m}{dx} = \text{constant} \quad \Rightarrow \quad \frac{\partial}{\partial r} = \frac{d}{dr}$$

$$\underbrace{\left(\frac{2u_m}{\alpha}\frac{dT_m}{dx}\right)}_{\textbf{constant}}\underbrace{r\left[1-\left(\frac{r}{R}\right)^2\right]}_{\boldsymbol{f(r)}\textbf{ only}} = \frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

- Integrating twice w.r.t. r:

$$T(r) = \left(\frac{2u_m}{\alpha}\frac{dT_m}{dx}\right)\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + C_1 \ln r + C_2$$

- Apply boundary condition at $r = 0$:

$$\ln(0) = -\infty \quad \Rightarrow \quad C_1 = 0$$

$$\Rightarrow \quad T(r) = \left(\frac{2u_m}{\alpha}\frac{dT_m}{dx}\right)\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + C_2$$

Convection Coefficients for Laminar Flow in Pipes

$$T(r) = \left(\frac{2u_m}{\alpha}\frac{dT_m}{dx}\right)\left[\frac{r^2}{4} - \frac{r^4}{16R^2}\right] + C_2$$

- Apply boundary condition at $r = R, T = T_s$:

$$C_2 = T_s - \left(\frac{2u_m}{\alpha}\frac{dT_m}{dx}\right)\left[\frac{3R^2}{16}\right]$$

- Thus, we arrive at:

$$\boxed{T(r,x) = T_s(x) - \frac{2u_m R^2}{\alpha}\left(\frac{dT_m}{dx}\right)\left[\frac{3}{16} + \frac{1}{16}\left(\frac{r}{R}\right)^4 - \frac{1}{4}\left(\frac{r}{R}\right)^2\right]}$$

Eq. 8.50

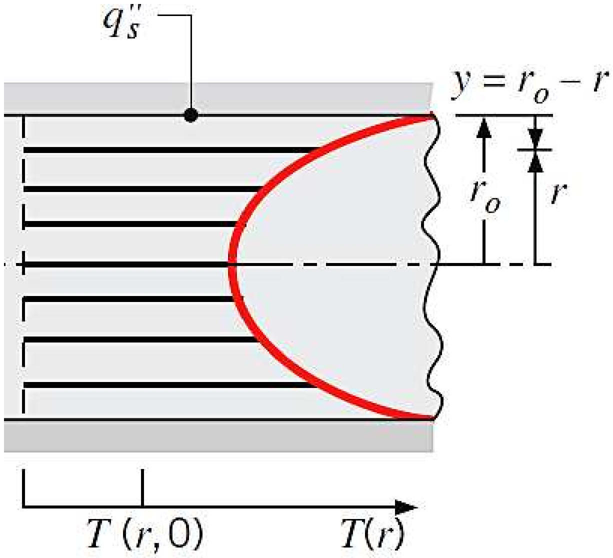
Convection Coefficients for Laminar Flow in Pipes

Nusselt Number

- With the temperature distribution know, one may then compute the wall heat flux
- This may is done by computing $\left.\frac{\partial T}{\partial r}\right|_{r=R}$
- We can also use the following and substitute in for u and T:

$$T_m(x) = \frac{\int_0^R (2\pi r dr)uT}{u_m R^2}$$

$$\boxed{T_m(x) = T_s(x) - \frac{11}{48}\frac{u_m R^2}{\alpha}\left(\frac{dT_m}{dx}\right)}$$



Convection Coefficients for Laminar Flow in Pipes

Nusselt Number (for constant heat flux)

- Next, for thermally developed flow and constant $q"_s$ recall that:

$$\dot{m}C_p\frac{dT_m}{dx} = q"_{conv} = hP(T_s - T_m) \quad \Rightarrow \quad \frac{dT_m}{dx} = \frac{hP}{\dot{m}C_p}(T_s - T_m)$$

- Substituting:

$$T_s(x) - T_m(x) = \frac{11}{48}\frac{u_m R^2}{\alpha}\left(\frac{hP}{\dot{m}C_p}\left(T_s(x) - T_m(x)\right)\right)$$

- Also noting that: $\dot{m} = u_m\frac{\pi}{4}D^2\rho \qquad P = \pi D \qquad \alpha C_p = \frac{k}{\rho C_p}C_p = \frac{k}{\rho}$

- We arrive at:

Eq. 8.53

$$\frac{48}{11} = \frac{u_m(D/2)^2}{\frac{k}{\rho C_p}}\left(\frac{h\pi D}{u_m\frac{\pi}{4}D^2\rho C_p}\right) \quad \Rightarrow \quad \frac{48}{11} = \frac{hD}{k} \equiv \mathrm{Nu}_D \quad \Rightarrow \quad \boxed{\mathrm{Nu}_D = \frac{48}{11} = 4.36}$$

For constant $q"_s$

Convection Coefficients for Laminar Flow in Pipes

Nusselt Number (for constant wall temperature)

- For the case of constant wall temperature (T_s is constant), the governing equation is similar to that for the constant wall flux except that:

$$\frac{\partial T(x,r)}{\partial x} = \frac{T_s - T(x,r)}{T_s - T_m(x)} \frac{dT_m}{dx}(x)$$

- The equation for T then becomes:

$$\frac{2u_m}{\alpha} \frac{dT_m}{dx} \left[1 - \left(\frac{r}{R}\right)^2\right] \left(\frac{T_s - T(x,r)}{T_s - T_m(x)}\right) = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)$$

- The solution to this in NOT trivial… however, with advanced methods it can be shown that :

$$\boxed{\mathrm{Nu}_D = 3.66}$$ **Eq. 8.53**

For constant T_s

- **NOTE**: both the constant heat flux and constant wall temperature cases yield Nu_D independent of Re_D

Turbulent Correlations

- For turbulent flows, the Nusselt number cannot be determined analytically, and one finds empirical correlations again of the form:

$$\mathrm{Nu}_D = (\mathrm{constant})\ {\mathrm{Re}_D}^m \mathrm{Pr}^m$$

- As with external flow analysis, the specific coefficients vary from different flow regimes and experimental conditions and must be used carefully!

- As with external flow analysis, the specific coefficients vary from different flow regimes and experimental conditions and must be used carefully!

Turbulent Correlations

- A typical relation is the Colburn relation: $\mathrm{Nu}_D = 0.023\mathrm{Re}_D{}^{4/5}\mathrm{Pr}^{1/3}$
- A nearly identical relation is the Dittus-Boelter relation:

$$Nu_D = 0.023\, Re_D^{4/5}\, Pr^n \tag{8.60}$$

where $n = 0.4$ for heating ($T_s > T_m$) and 0.3 for cooling ($T_s < T_m$). These equations have been confirmed experimentally for the range of conditions

$$\begin{bmatrix} 0.6 \lesssim Pr \lesssim 160 \\ Re_D \gtrsim 10{,}000 \\ \dfrac{L}{D} \gtrsim 10 \end{bmatrix}$$

- Additional correlations can be found within your text (Section 8.5)

Turbulent Correlations

Special Notes

- In practice, most pipe flows in engineering applications will be turbulent!
- It is essential to estimate critical pipe lengths needed for turbulent transition
- A notable exception is <u>MICROSCALE</u> flows which are inherently laminar

Laminar Flow in Non-circular Tubes

- We have thus far restricted our consideration to internal flows of circular cross section, many engineering applications involve convection transport in *noncircular tubes.*
- Many of the circular tube results may be applied by using *an effective diameter* as the characteristic length as a first order approximation
- This effective diameter is termed the *hydraulic diameter* and is defined as:

$$D_h \equiv \frac{4A_c}{P}$$

- where *Ac* and *P* are the *flow* cross-sectional area and the *wetted perimeter,* respectively.
- It is this diameter that should be used in calculating parameters such as Re_D and Nu_D.

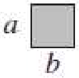
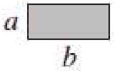
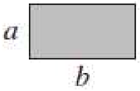
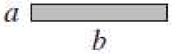
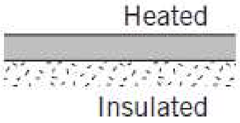
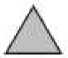
Laminar Flow in Non-circular Tubes

$$D_h \equiv \frac{4A_c}{P}$$

- For turbulent flows ($\mathrm{Re}_D \geq 2300$) we can use the correlations in Section 8.5 with reasonable accuracy provided $\mathrm{Pr} \geq 0.7$.
- However, with sharp corners h changes and the value calculated using 8.5 correlations is an average value over the cross section.
- For laminar flow these sharp corners present a more significant problem.
- For this reason it is prudent to utilize the corrected values in Table 8.1 for different geometric cross-sections
- These values are based on solutions of the differential momentum and energy equations for flow through the different duct cross sections

Laminar Flow in Non-circular Tubes

TABLE 8.1 Nusselt numbers and friction factors for fully developed laminar flow in tubes of differing cross section

Cross Section	$\frac{b}{a}$	$Nu_D \equiv \frac{hD_h}{k}$ (Uniform q_s'')	 (Uniform T_s)	Re_{D_h}
	—	4.36	3.66	64
	1.0	3.61	2.98	57
	1.43	3.73	3.08	59
	2.0	4.12	3.39	62
	3.0	4.79	3.96	69
	4.0	5.33	4.44	73
	8.0	6.49	5.60	82
	∞	8.23	7.54	96
	∞	5.39	4.86	96
	—	3.11	2.49	53

Heat Transfer Enhancement

Generally, heat transfer enhancement may be achieved by using two common means- increasing the convection coefficient and/or by increasing the convection surface area. More specifically:

1) Promoting Turbulence:

- Turbulent transfer can be enhanced through “roughening” of tube walls or the insertion of elements intended to “trip” the flow to a turbulent state (coil-spring inserts)

2) Active Generation of Swirl Vortices:

- By using “vanes” or similar geometries within the tube one can introduce swirl to the flow and enhance convection

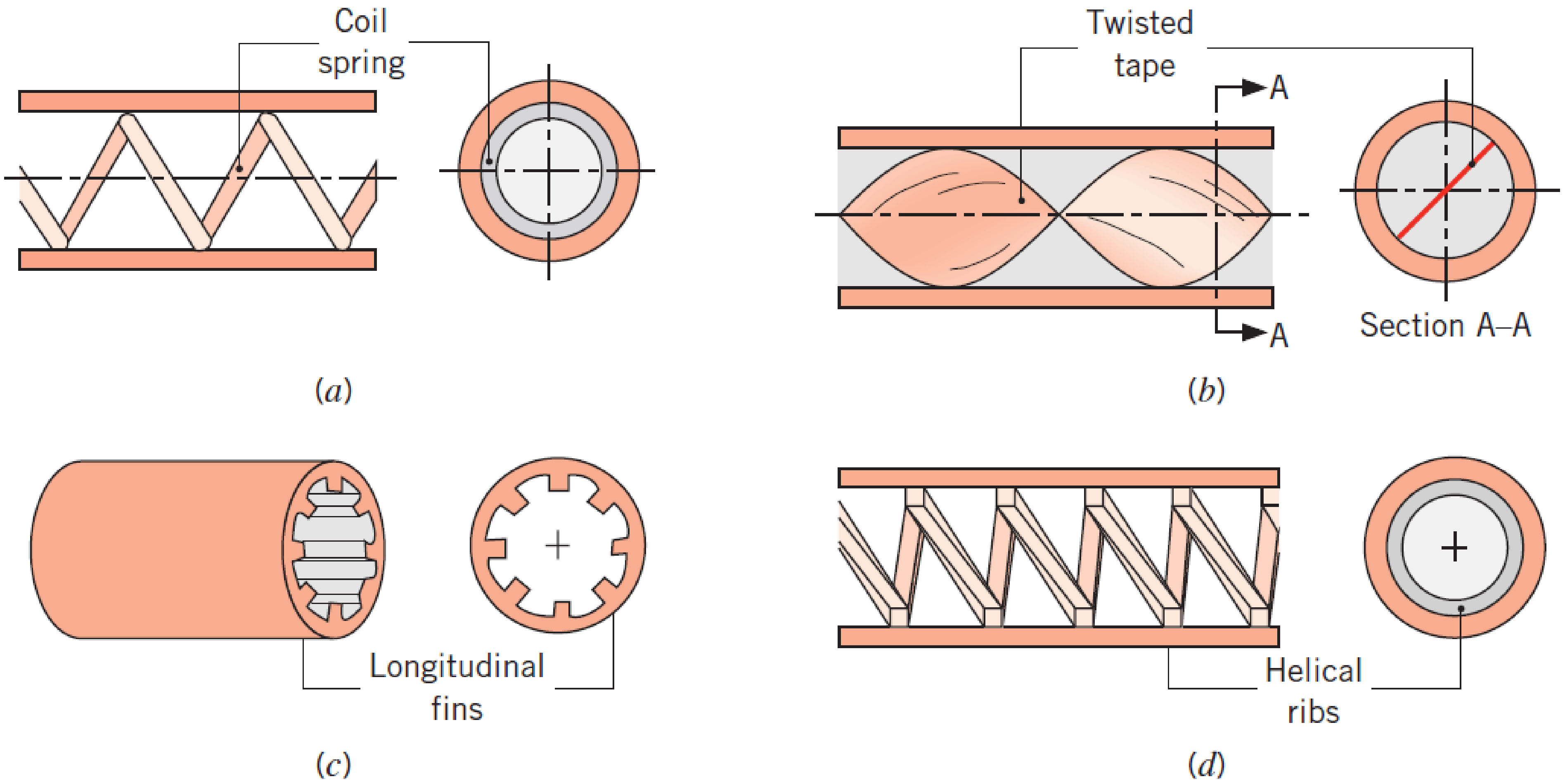
3) Increase of Surface Area via Ribs or Fins

- This is the pipe equivalent to extended surfaces from earlier conduction chapters

4) Passive Generation of Secondary Flows

- (Dean Vortices) For curved pipes, the centripital acceleration can lead to hydrodynamic instabilities giving rise to “secondary-flows”
- These Dean Vortices help provide swirl in the transverse plane which promotes heat transfer (relevant to flow in a coiled pipe)

Heat Transfer Enhancement

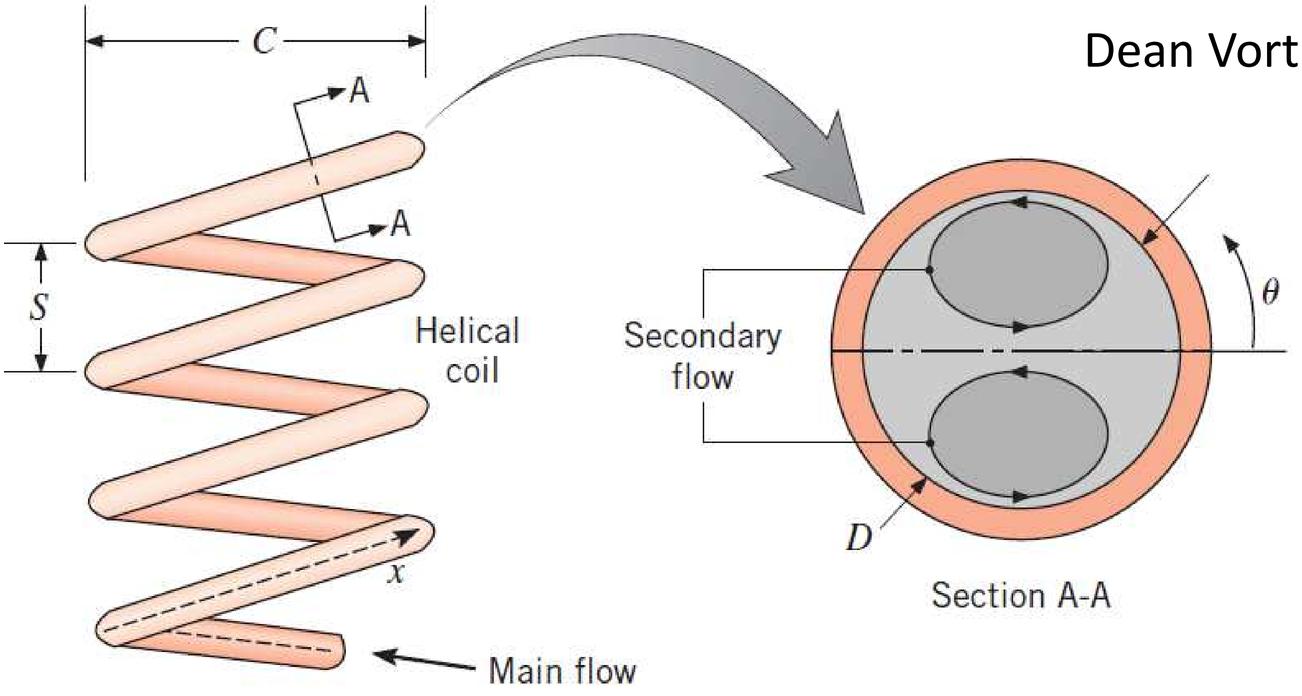


Heat Transfer Enhancement

- By coiling a tube, heat transfer may be enhanced without turbulence or significant additional heat transfer surface area.

- The secondary flow increases heat transfer rates but will also increase friction losses!

- In addition, the secondary flow decreases entrance lengths and reduces the difference between laminar and turbulent heat transfer rates, relative to the straight tube cases

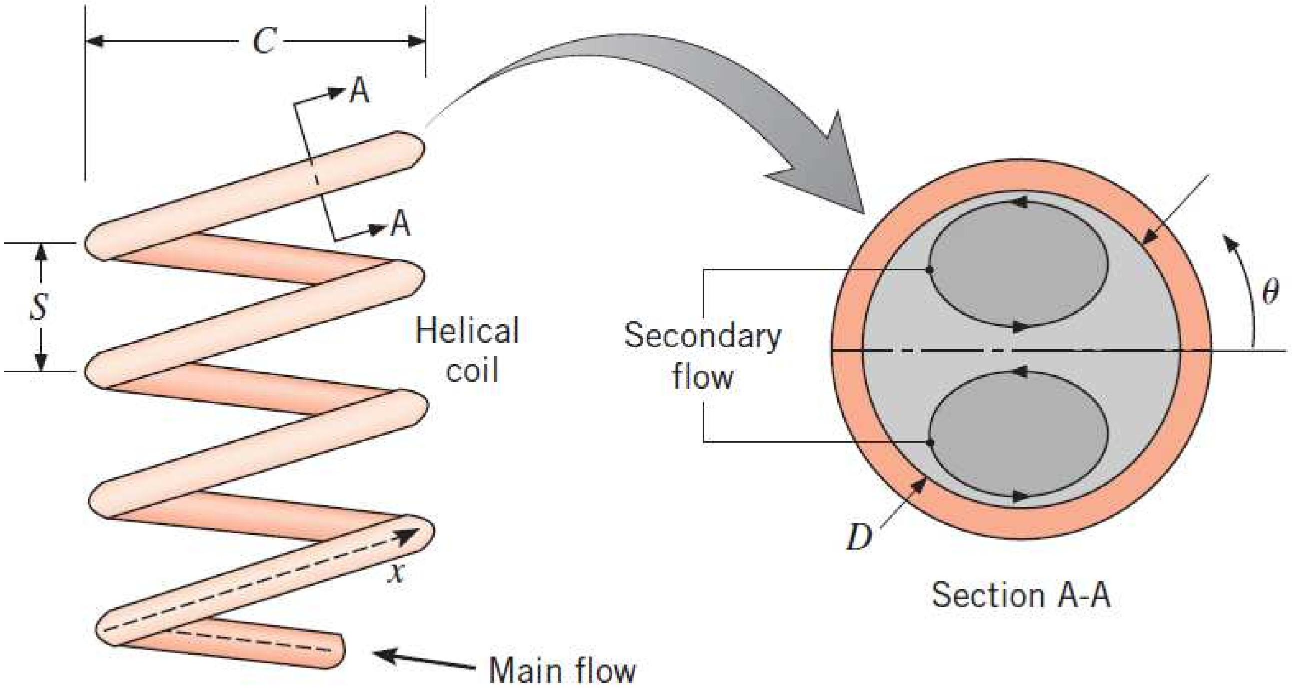


Heat Transfer Enhancement

- The critical Reynolds number corresponding to the onset of turbulence for the helical tube is:

$$Re_{D,c,h} = Re_{D,c}[1 + 12(D/C)^{0.5}]$$

- It is obvious to see that the Reynolds number will indeed increase for the same base geometry implying that a longer transition is needed!



Heat Transfer Enhancement

For fully developed laminar flow with $C/D \gtrsim 3$, the friction factor is

$$f = \frac{64}{Re_D} \qquad Re_D(D/C)^{1/2} \lesssim 30 \tag{8.19}$$

$$f = \frac{27}{Re_D^{0.725}}(D/C)^{0.1375} \qquad 30 \lesssim Re_D(D/C)^{1/2} \lesssim 300 \tag{8.75a}$$

$$f = \frac{7.2}{Re_D^{0.5}}(D/C)^{0.25} \qquad 300 \lesssim Re_D(D/C)^{1/2} \tag{8.75b}$$

Heat Transfer Enhancement

For cases where $C/D \lesssim 3$, recommendations provided in Shah and Joshi [32] should be followed. The heat transfer coefficient for use in Equation 8.27 may be evaluated from a correlation of the form

$$Nu_D = \left[\left(3.66 + \frac{4.343}{a}\right)^3 + 1.158\left(\frac{Re_D(D/C)^{1/2}}{b}\right)^{3/2}\right]^{1/3}\left(\frac{\mu}{\mu_s}\right)^{0.14} \qquad (8.76)$$

where

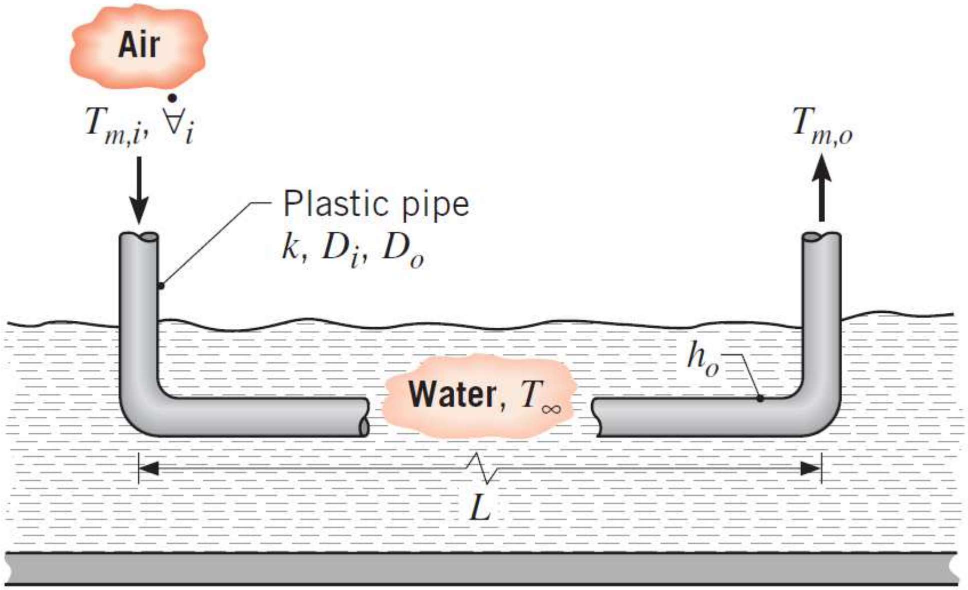
$$a = \left(1 + \frac{927(C/D)}{Re_D^2\,Pr}\right) \qquad \text{and} \qquad b = 1 + \frac{0.477}{Pr} \qquad (8.77\text{a,b})$$

$$\left[\begin{array}{l} 0.005 \lesssim Pr \lesssim 1600 \\ 1 \lesssim Re_D(D/C)^{1/2} \lesssim 1000 \end{array}\right]$$

Example 1 (8.31)

To cool a summer home without using a vapor compression refrigeration cycle, air is routed through a plastic pipe ($k = 0.15$ W/m·K, $D_i = 0.15$ m, $D_o = 0.17$ m) that is submerged in an adjoining body of water. The water temperature is nominally at $T_\infty = 17°\text{C}$, and a convection coefficient of $h_o \approx 1500$ W/m^2·K is maintained at the outer surface of the pipe.

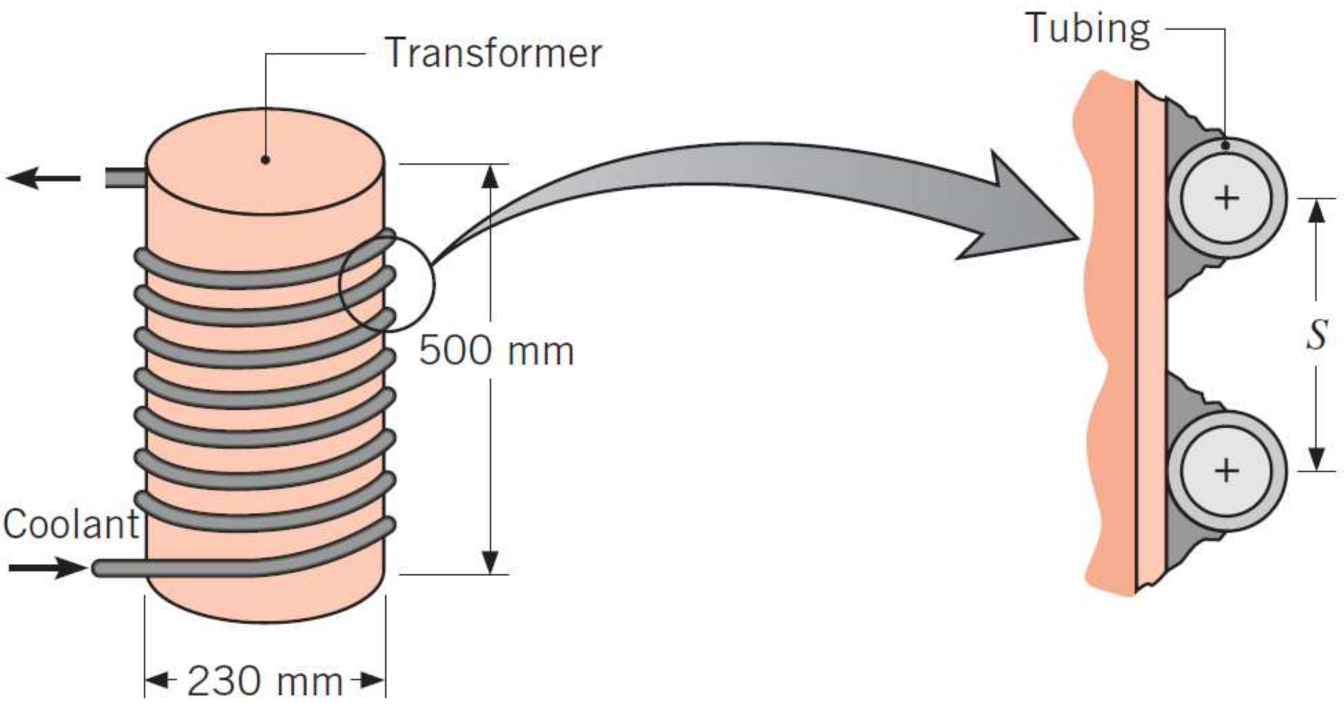
If air from the home enters the pipe at a temperature of $T_{m,i} = 29°\text{C}$ and a volumetric flow rate of $\dot{\forall}_i = 0.025$ m^3/s, what pipe length L is needed to provide a discharge temperature of $T_{m,o} = 21°\text{C}$?



Example 2 (8.103)

An electrical power transformer of diameter 230 mm and height 500 mm dissipates 1000 W. It is desired to maintain its surface temperature at 47°C by supplying ethylene glycol at 24°C through thin-walled tubing of 20-mm diameter welded to the lateral surface of the transformer. All the heat dissipated by the transformer is assumed to be transferred to the ethylene glycol.

Assuming the maximum allowable temperature rise of the coolant to be 6°C, determine the required coolant flow rate, the total length of tubing, and the coil pitch S between turns of the tubing.



References

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ME 144: Heat Transfer
Introduction to Convection

J. M. Meyers

Introductory Remarks

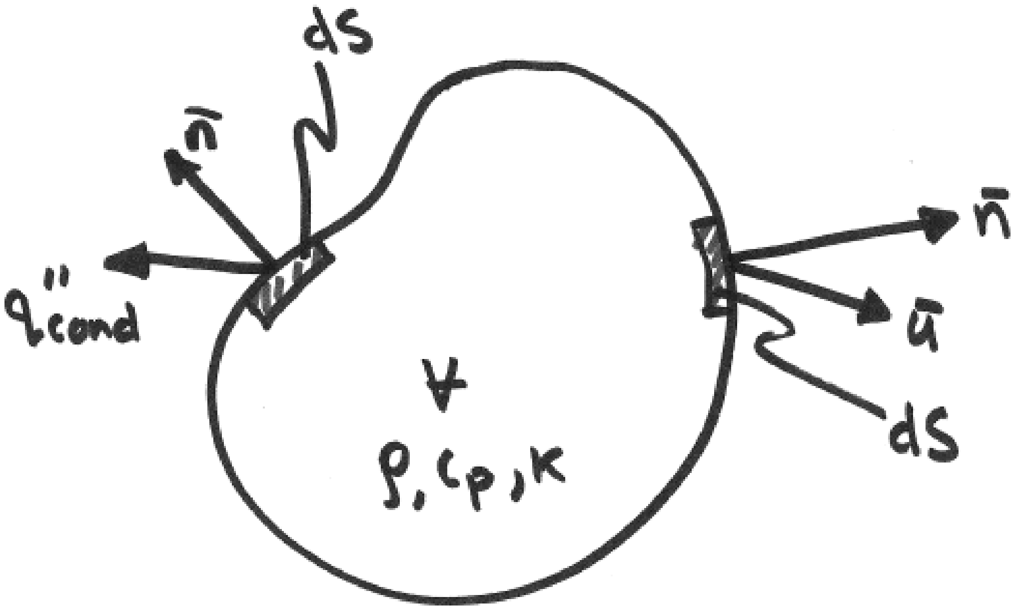
- Convection heat transfer differs from diffusion heat transfer in that a bulk fluid motion is present which augments the overall heat transfer by an advection mechanism:

Convection = Conduction + Advection

- The governing equation field is then no longer a diffusion equation, but rather becomes a convection-diffusion equation

Convection Equation for Incompressible Flow

- Consider a fluid (liquid or gas) body:



- Recall that in our previous derivation for the diffusion equation for conduction (see Chpt. 2 notes) a conservation of energy analysis led us to an integral equation of the form:

$$\frac{\partial}{\partial t}\oiiint \rho C_p T(x,t)dV = -\oiint q"_{cond} \cdot \bar{n}dS$$

This assumes no energy generation ($\dot{q}_S = 0$)

Convection Equation for Incompressible Flow

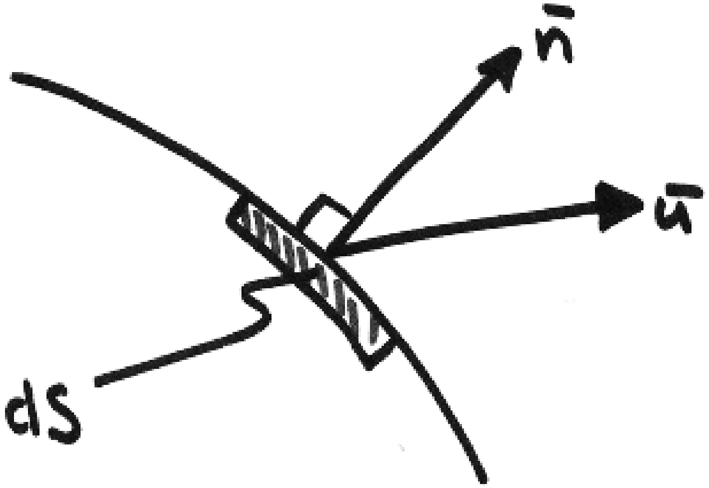
- Here, $q"_{cond} = -k\nabla T$ (conductive heat flux)

- A fluid (liquid or gas) at rest conducts in a manner described by:

$$\frac{\partial}{\partial t}\oiiint \rho C_p T(x,t)dV = -\oiint -k\nabla T\bar{n}dS$$

- If fluid is in motion energy will be convected across boundaries and introduce another term

- Consider a location at the boundary of the domain where the velocity is $\bar{u}$ and the temperature is T



Convection Equation for Incompressible Flow

- A mass crossing the surface in an interval of time Δt can be found to be:

$$dm_\perp = \rho dV = \rho(dS \cdot u_\perp \cdot dt)$$

- More precisely:

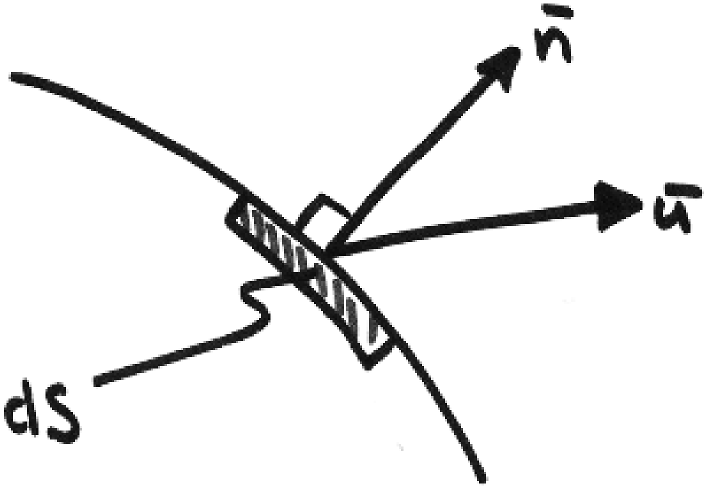
$$d\dot{m} = \frac{dm}{dt} = \rho\bar{u} \cdot \bar{n} dS$$

- There is an associated incremental amount of energy crossing the boundary which can be described mathematically as:

$$d\dot{E}_{conv} = C_p T d\dot{m}$$

Recall that mass specific internal energy is described by $e = C_p T$

$$d\dot{E}_{conv} = \rho C_p(\bar{u} \cdot \bar{n}) T dS$$



- Integrating over the surface, we see that the total heat loss will be:

$$\dot{E}_{conv} = -\oiint \left(\rho C_p \bar{u} T\right) \cdot \bar{n} dS$$

Convection Equation for Incompressible Flow

- This newly derived energy loss term can be used to augment our already derived diffusion equation as follows:

$$\frac{\partial}{\partial t}\oiiint \rho C_p T(x,t)dV = -\oiint\left(\rho C_p \bar{u} T\right)\cdot \bar{n}dS + \oiint k\nabla T\bar{n}dS$$

- Pulling the time derivative inside the left hand side integral and using the divergence theorem (allowing us to convert surface integrals to volume integrals) we find:

$$\oiiint \rho C_p \frac{\partial T}{\partial t}dV = -\oiiint \nabla\left(\rho C_p \bar{u} T\right)dV + \oiiint k\nabla^2 T dV$$

- Integrate over the volume (as it is arbitrary):

$$\rho C_p \frac{\partial T}{\partial t} = -\nabla\left(\rho C_p \bar{u} T\right) + k\nabla^2 T \qquad \Rightarrow \qquad \rho C_p \left[\frac{\partial T}{\partial t} + \nabla(\bar{u}T)\right] = k\nabla^2 T$$

Convection Equation for Incompressible Flow

- Breaking down $\nabla(\bar{u}T)$:

$$\nabla(\bar{u}T) = (\nabla \cdot \bar{u})T + (\bar{u} \cdot \nabla)T$$

- Recall the material derivative for density from fluids:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \nabla\rho \cdot \bar{u}$$

- For incompressible flow:

$$\boxed{\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \rho\nabla \cdot \bar{u} \qquad \frac{D\rho}{Dt} = 0 \qquad \frac{\partial\rho}{\partial t} = 0} \Rightarrow \nabla \cdot \bar{u} = 0$$

- Thus: $\nabla(\bar{u}T) = (\bar{u} \cdot \nabla)T$

- Rearranging our augmented diffusion equation and utilizing thermal diffusivity term ($\alpha = k/\rho C_p$):

$$\boxed{\frac{\partial T}{\partial t} + (\bar{u} \cdot \nabla)T = \alpha\nabla^2 T}$$

Convection-Diffusion Equation

Convection Equation for Incompressible Flow

$$\boxed{\frac{\partial T}{\partial t} + (\bar{u} \cdot \nabla)T = \alpha \nabla^2 T}$$

Convection-Diffusion Equation

- Keep in mind our assumptions used to arrive at the above relation:
    1. Constant properties (k, ρ, C_p)
    2. No internal heat generation within the fluid $(\dot{q}_s = 0)$
    3. Incompressible Flow

- The term $(\bar{u} \cdot \nabla)T$ is difficult to solve as can be seen when showing its full 3D Cartesian coordinates form:

$$(\bar{u} \cdot \nabla)T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

- Here we see strong coupling into an unknown flow field

- The general case requires simultaneous solution of the Navier-Stokes equation for the velocity field coupled with the energy equation to handle heat transfer mechanisms

Convection Equation for Incompressible Flow

$$\boxed{\frac{\partial T}{\partial t} + (\bar{u} \cdot \nabla)T = \alpha \nabla^2 T}$$

Convection-Diffusion Equation

- This chapter (Chapter 6) we derive relations for convective heat flux
- Solving the convection-diffusion equation is difficult
- Normally, numerical solutions (CFD) are utilized but this topic is outside of the scope for this class
- Owing to this, it is normal practice in first treatments of heat transfer to handle our understanding of convective heat flux through an array of approximate solutions as will be discussed in subsequent chapters

Dimensionless Form of the Convection-Diffusion Equation

- As with the Navier-Stokes equation, it is often convenient to non-dimensionalize the convection-diffusion equation

- The main parameters involved (velocity, space, temperature, and time) can be non-dimensionalized as:

Velocity $\bar{u}^* = \frac{\bar{u}}{U} \Rightarrow \bar{u} = U\bar{u}^*$ $U \equiv$ reference velocity (i.e. free stream velocity)

Space $\bar{x}^* = \frac{\bar{x}}{L} \Rightarrow \bar{x} = L\bar{x}^*$ $L \equiv$ characteristic length (i.e. length of a flat plate)

Time $t^* = \frac{t}{L/U} \Rightarrow t = \frac{L}{U}t^*$

Temperature $T^* = \frac{T}{T_0} \Rightarrow T = T_0 T^*$ $T_0 \equiv$ reference temperature (i.e. free stream temperature)

Dimensionless Form of the Convection-Diffusion Equation

- Substituting our dimensionless definitions into the convection-diffusion equation:

$$\frac{\partial T^*}{\partial t^*}\frac{T_0 U}{L} + (\bar{u}^* \cdot \nabla^*)T^* \frac{T_0 U}{L} = \alpha \nabla^{*2} T^* \frac{T_0}{L^2}$$

$$\Downarrow$$

$$\boxed{\frac{\partial T^*}{\partial t^*} + (\bar{u}^* \cdot \nabla^*)T^* = \boxed{\frac{\alpha}{UL}}\nabla^{*2} T^*}$$

Dimensionless Convection-Diffusion Equation

Peclet Number: $\mathrm{Pe} = UL/\alpha$

- The Peclet number is a ratio of advection to diffusion (conduction) transport rates
- Recall that **conduction is a combination of conductive and advective transport mechanisms**
- **Advection:** transport due to bulk fluid motion

6.1 Convection at Fluid-Solid Boundaries

- In heat transfer analyses of solid bodies we are often interested in the role of convection primarily as a surface boundary condition

- We often regard the flow at a finite distance from the body to be essentially "known" and quite possibly even uniform

- Moreover, we often regard the ambient temperature field T_∞ in much the same way

- In order to fully understand the concepts of convection heat transfer the concept of boundary layers must be understood

- Three types of boundary layers are of general interest:

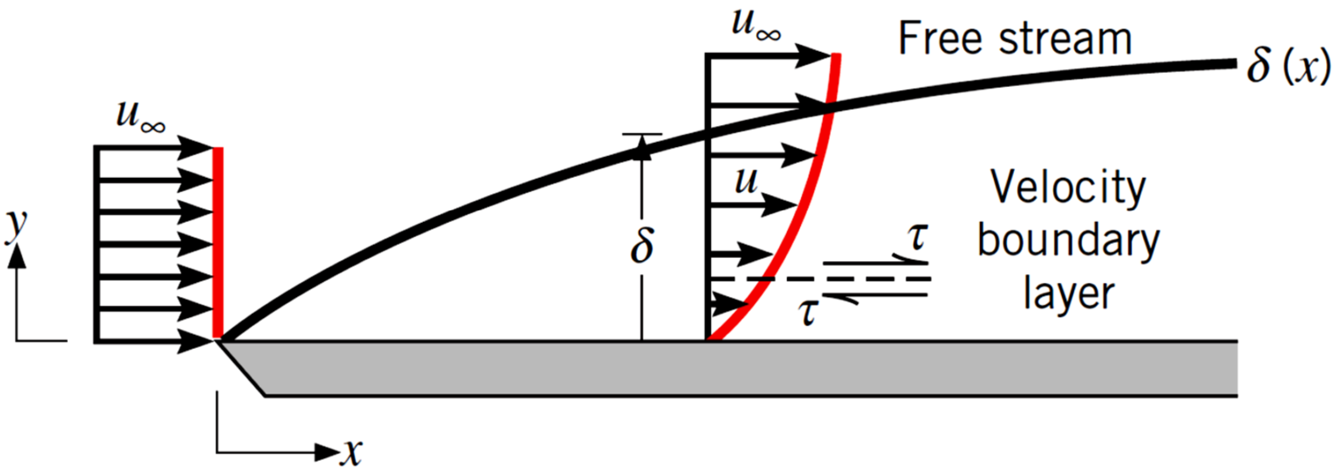
   1. Velocity Boundary Layer
   2. Thermal Boundary Layer
   3. Concentration Boundary Layer (think evaporative mass transport)

- We will focus on the first two for this class

6.1 Convection at Fluid-Solid Boundaries

Velocity Boundary Layer

- We know already from fluid dynamics that there is a thin layer of high velocity gradients adjacent to the solid surface
- This transitions the flow from a no-slip zero velocity at the solid boundary to the free-stream state.

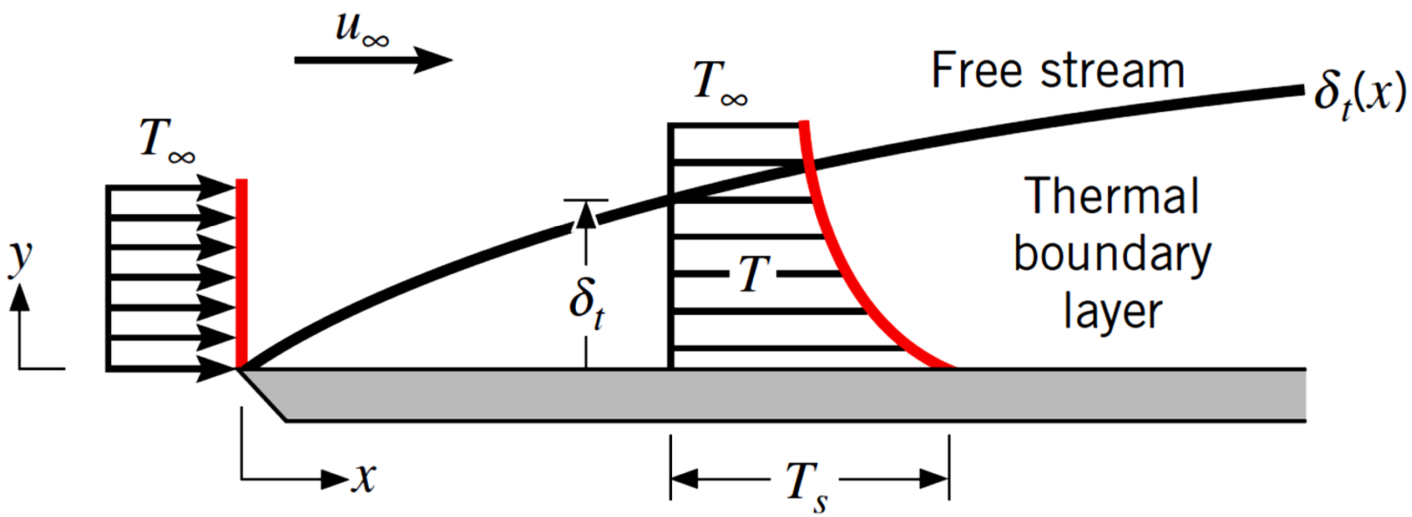


6.1 Convection at Fluid-Solid Boundaries

Thermal Boundary Layer

- Similarly, in heat transfer problems we can speculate that a similar <u>thermal boundary layer</u> can exist

- Here, there is a “sharp transition” from the surface temperature to the free stream value



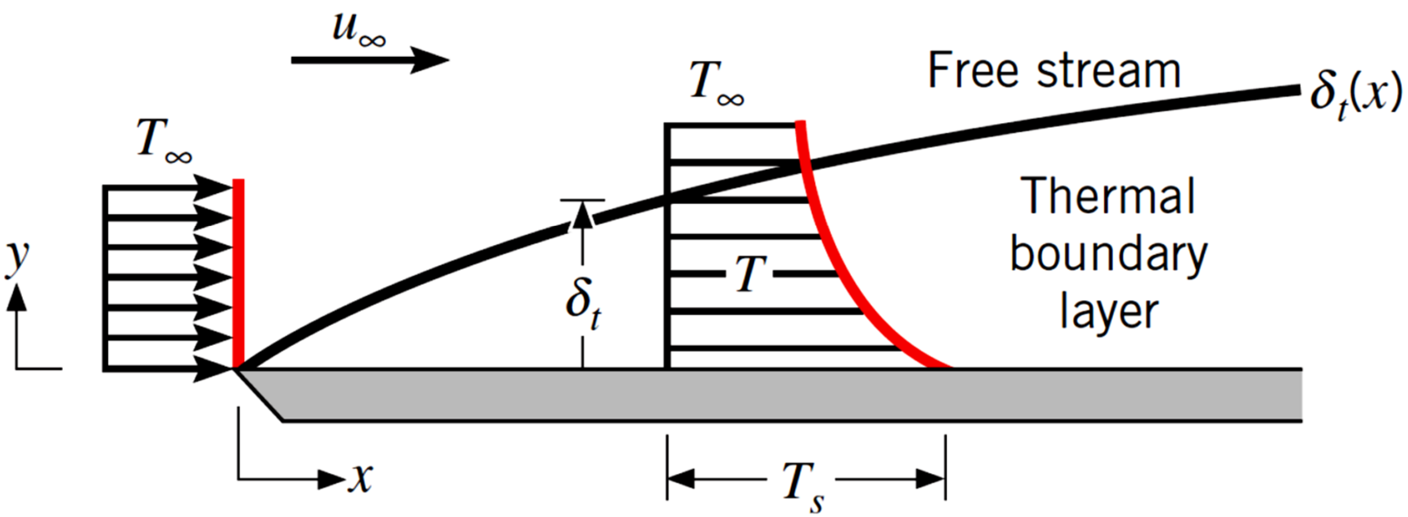
6.1 Convection at Fluid-Solid Boundaries

Thermal Boundary Layer

- At the solid surface ($y = 0$) there will be a normal heat flux:

$$q"_{solid} = q"_{fluid} \quad \Rightarrow \quad -k_s \frac{\partial T}{\partial y}\bigg|_{y=0^-} = -k_f \frac{\partial T}{\partial y}\bigg|_{y=0^+}$$

- **!! NOTE !!** $q"_{fluid}$ at $y = 0$ truly is a conduction flux as the fluid at the plate surface is at rest



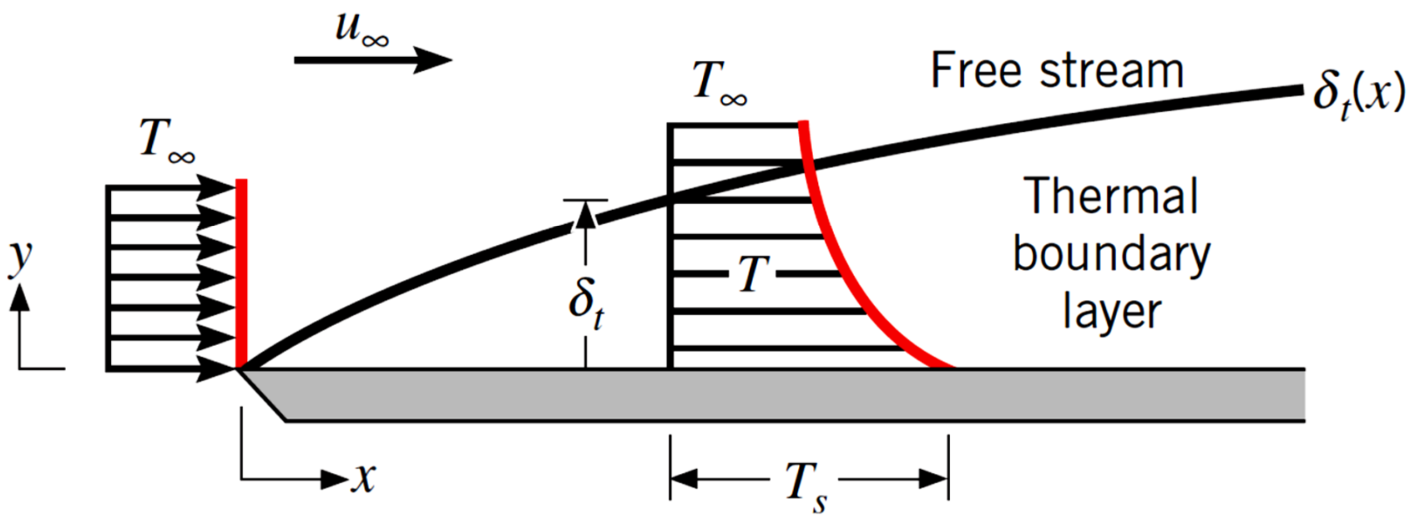
6.1 Convection at Fluid-Solid Boundaries

Thermal Boundary Layer

- Now, the gradient $\left.\frac{\partial T}{\partial y}\right|_{y=0}$ will certainly depend on both the surface temperature (T_s) as well as the free stream temperature (T_∞) through:

$$q"_f = -k_f \left.\frac{\partial T}{\partial y}\right|_{y=0} \propto (T_s - T_\infty)$$

- All we have intuited here is nothing more than Newton's law of cooling which we've had plenty of experience with at this point



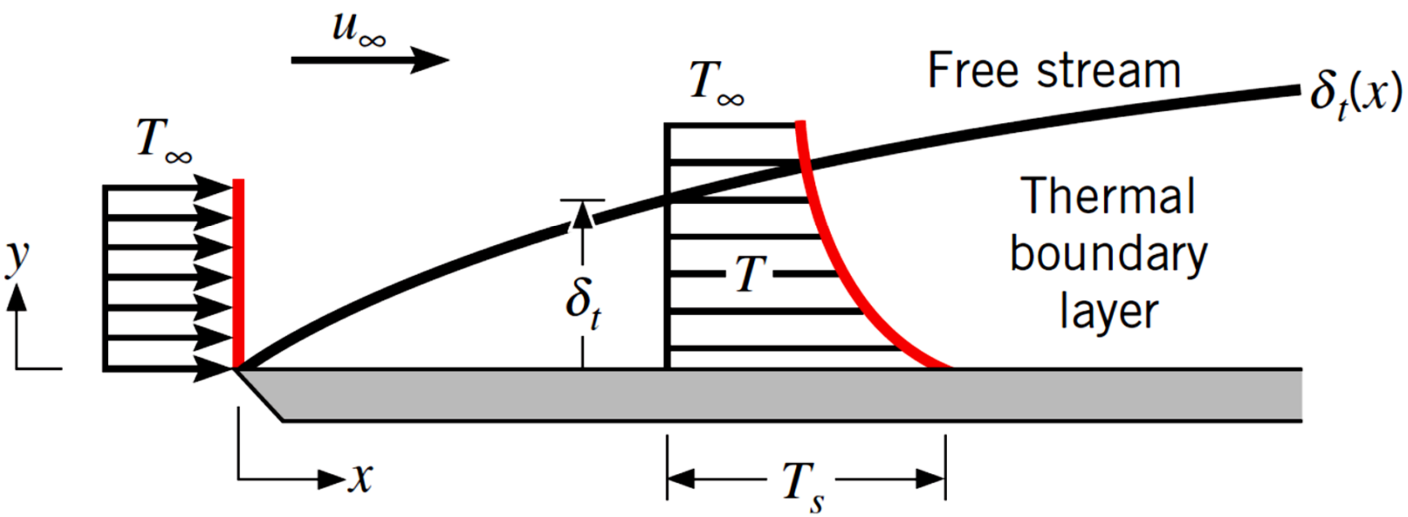
6.1 Convection at Fluid-Solid Boundaries

Thermal Boundary Layer

- Thus, we in fact define the convection coefficient in terms of the surface heat flux as:

$$\boxed{h \equiv \frac{q''_f}{(T_s - T_\infty)} = \frac{-k_f \partial T/\partial y|_{y=0}}{(T_s - T_\infty)}}$$

Equation 6.5 in text



6.2 Local and Average Convection Coefficients

- It is evident that the surface heat flux depends of the temperature gradient $\partial T/\partial y$

- In turn, the gradient magnitude will scale with the thickness of the thermal boundary layer:

$$\frac{\partial T}{\partial y} \sim \frac{(T_s - T_\infty)}{\delta_t} \qquad \text{(scaling)}$$

- Now, even if $(T_s - T_\infty)$ remains constant, we know that $\delta_t = \delta_t(x)$ (that is, the thermal BL thickness varies with location as BL grows)

- Thus, we see that $h = h(x)$ and that $h(x)$ must decrease as the BL increases in thickness

$$h = \frac{-k_f \partial T/\partial y|_{y=0}}{(T_s - T_\infty)}$$

- In other words for constant $T_s - T_\infty$:

$$\delta_t(x) \Uparrow \quad \Rightarrow \quad \frac{\partial T}{\partial y} \Downarrow \quad \Rightarrow \quad h(x) \Downarrow$$

6.2 Local and Average Convection Coefficients

- Since $q"_f$ will vary with location x it is understood that this is a **local** definition
- To define the **average** convection coefficient we write:

$$\bar{h} = \frac{\int q"_f(x) dA}{A} = \frac{\int -k_f \left.\frac{\partial T}{\partial n}\right|_{surf} dA}{A \cdot (T_s - T_\infty)}$$

- Or:

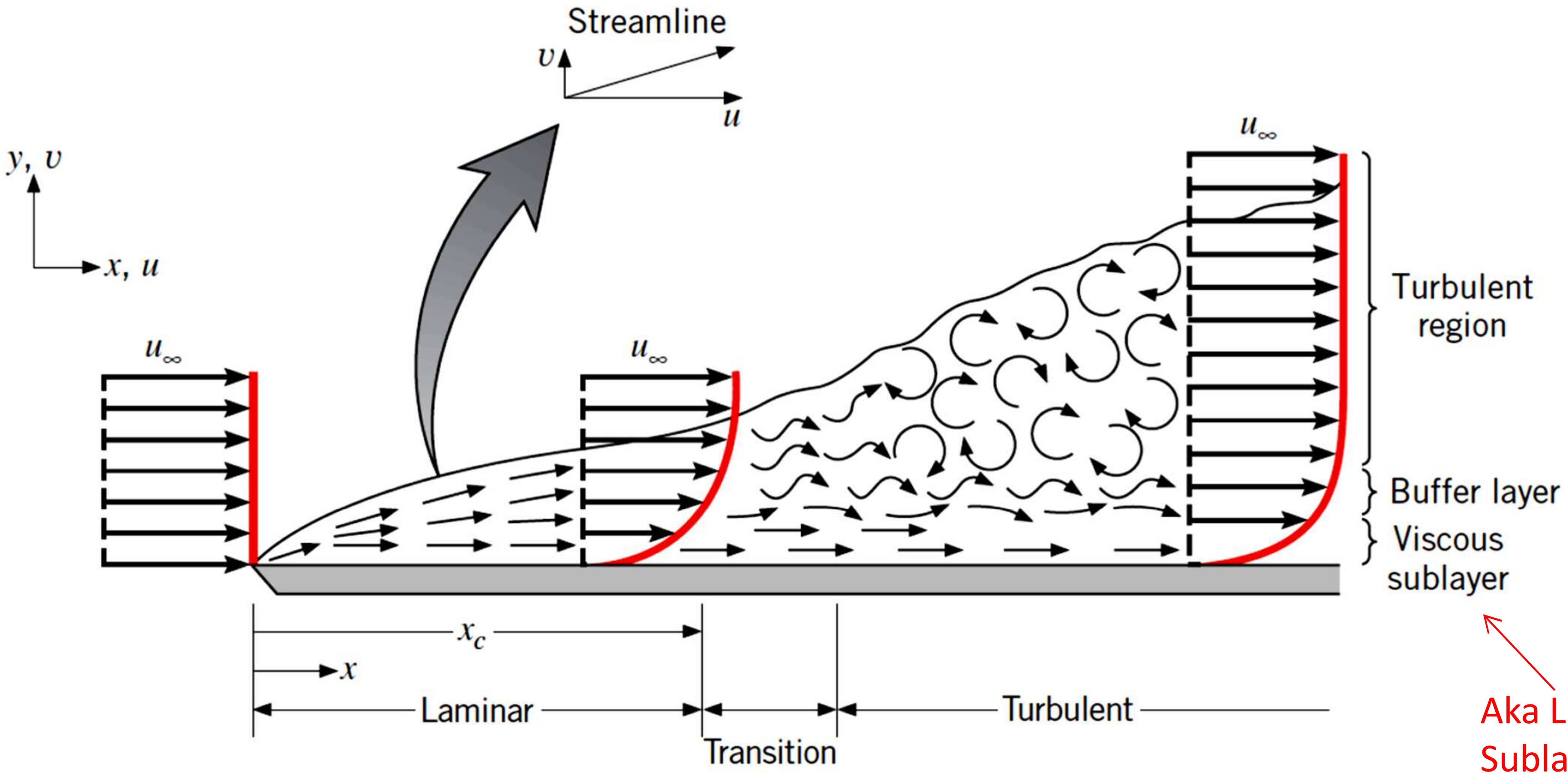
$$\boxed{\bar{h} = \frac{1}{A} \int h(x) dA}$$

Equation 6.13 in text

- Again, why does the gradient $k \, \partial T / \partial y$ scale with δ_t if this problem is convective?
- Because very near the surface the velocity is small (owing to zero-slip condition at the wall) which leads to much if not all of the heat flux owing to conduction in the y-direction

6.3 Laminar vs. Turbulent Boundary Layers

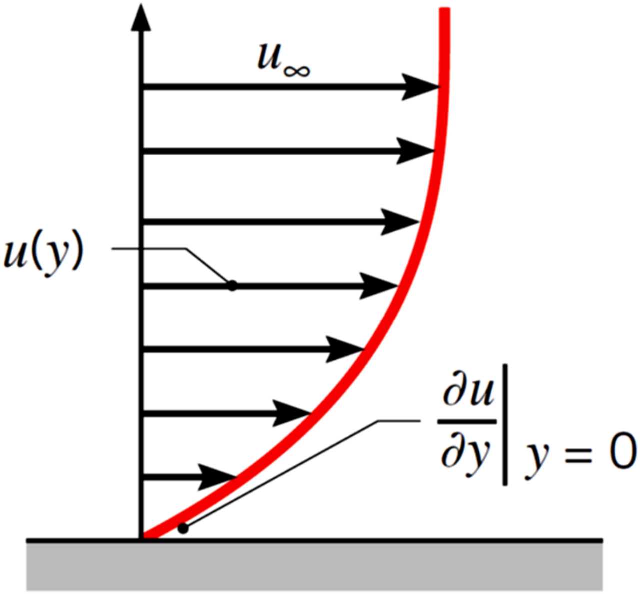
- The heat transfer that occurs within the boundary layer will greatly depend on whether the boundary layer is <u>laminar</u> or <u>turbulent</u>



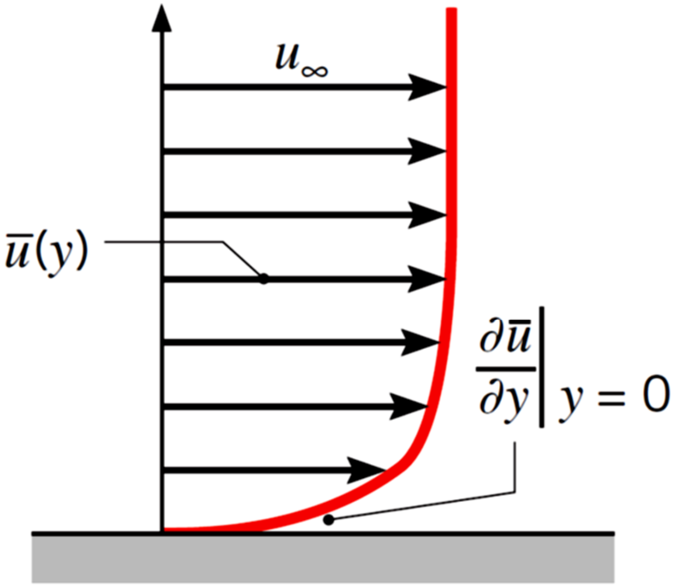
Aka Laminar Sublayer

6.3 Laminar vs. Turbulent Boundary Layers

- The heat transfer that occurs within the boundary layer will greatly depend on whether the boundary layer is laminar or turbulent



Laminar



Turbulent

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} < \left.\frac{\partial \bar{u}}{\partial y}\right|_{y=0}$$

Laminar Turbulent

6.3 Laminar vs. Turbulent Boundary Layers

- Turbulent boundary layers result from an instability in the laminar boundary layer
- The details of the transition mechanisms are quite complicated and remain the subject of active research
- The phrase “boundary layer transition” refers to the transition of a laminar boundary layer to a turbulent one
- For flat plate flow, it is found (experimentally) that transition occurs for a critical Reynolds number of:

$$(\mathrm{Re}_x)_{crit} = \frac{\rho u_\infty x}{\mu} \cong 500{,}000$$

Transition Reynolds number for a flat plate boundary layer

6.3 Laminar vs. Turbulent Boundary Layers

- For example, let's assume we have a slow water flow over a flat plate:

$$\rho = 1000\ \mathrm{kg/m^3}$$

$$u_\infty = 1\ \mathrm{m/sec}$$

$$\mu = 10^{-3}\ \mathrm{Pa{-}sec}$$

$$(\mathrm{Re}_x)_{crit} = \frac{\rho u_\infty x}{\mu} \quad \Rightarrow \quad x \cong \frac{(500{,}000)(10^{-3})}{(1000)(1)} \cong 0.5\ \mathrm{m}$$

- That's a transition length of 50 cm for a flow of only 1 m/sec!

- Transition can occur earlier (lower value of x) if one introduces roughness to promote earlier onset of flow instabilities

6.3 Laminar vs. Turbulent Boundary Layers

- For example, let's assume we have a slow water flow over a flat plate:

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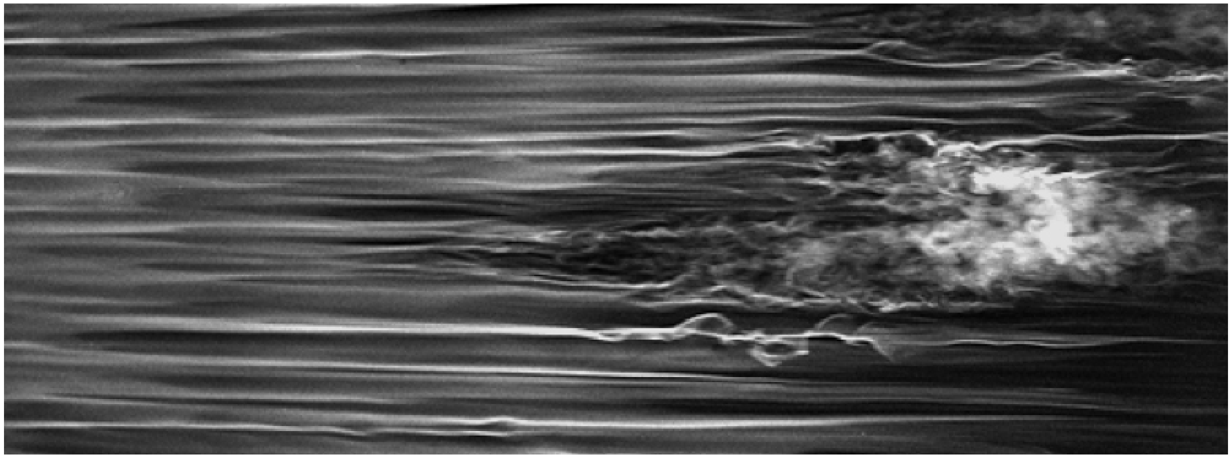
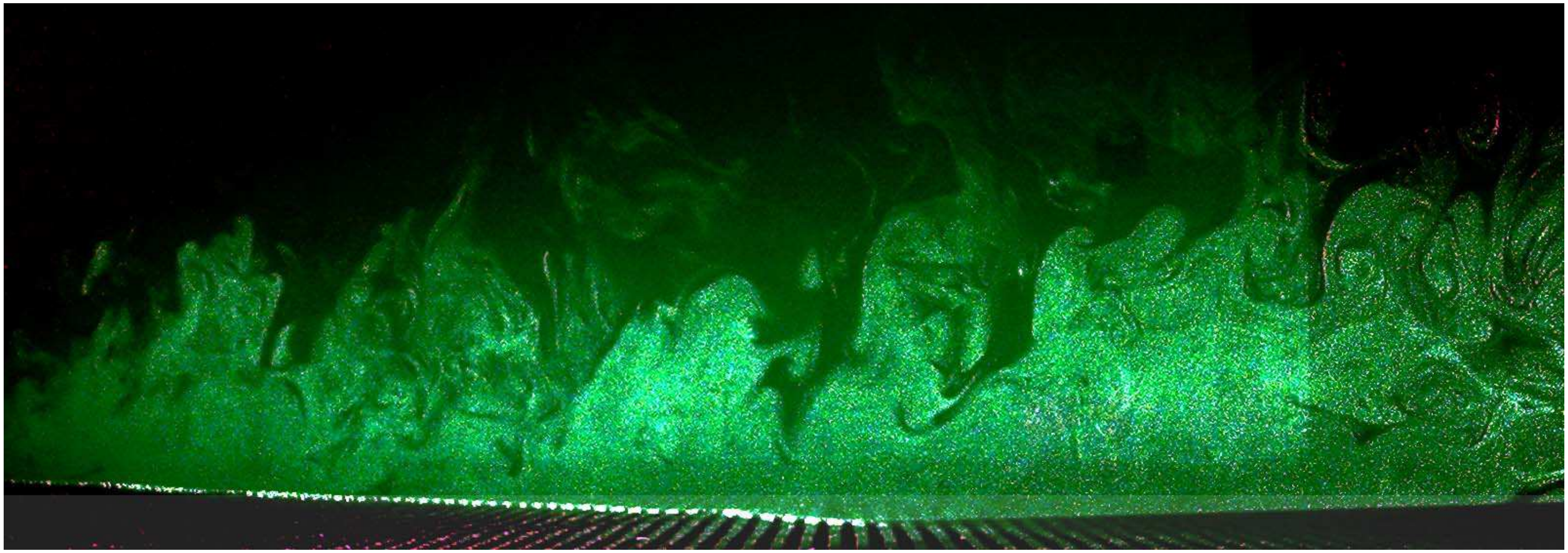
- That's a transition length of 50 cm for a flow of only 1 m/sec!
- Transition can occur earlier (lower value of x) if one introduces roughness to promote earlier onset of flow instabilities

6.3 Laminar vs. Turbulent Boundary Layers

- Transition can be characterized by a progression of visual turbulent “spots” and “streaks” progressing ultimately to full turbulence





6.3 Laminar vs. Turbulent Boundary Layers

- What are the consequences of turbulence?

$$(\delta_t)_{lam} < (\delta_t)_{turb}$$

$$\left(\left. \frac{\partial u}{\partial y} \right|_{y=0} \right)_{lam} < \left(\left. \frac{\partial \bar{u}}{\partial y} \right|_{y=0} \right)_{turb} \Rightarrow (\tau_{wall})_{lam} < (\tau_{wall})_{turb}$$

- Recalling:

$$h \equiv \frac{q"_f}{(T_s - T_\infty)} = \frac{-k_f \partial T / \partial y|_{y=0}}{(T_s - T_\infty)}$$

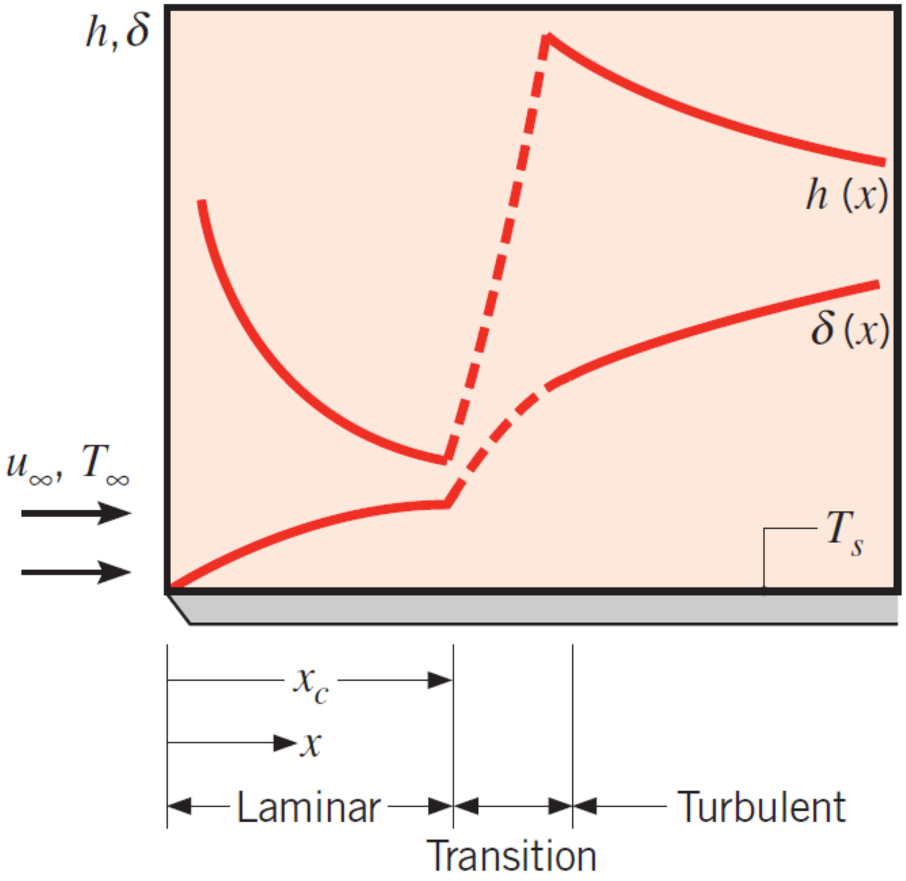
- Consequently:

!!!! $\boxed{h_{lam} < h_{turb}}$!!!!

- Why if the BL thickness is greater? It owes to the fact that the turbulent fluctuations tend to promote a greater transfer of heat.

6.3 Laminar vs. Turbulent Boundary Layers

- What are the consequences of turbulence?



6.4 Boundary Layer Equations

- Let’s assume a laminar (no turbulence), steady flow ($\partial/\partial t = 0$).
- Within the boundary layer, the flow and heat transfer are governed by:

Mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \qquad x\text{-dir}$$

$\nu = {}^{\mu}/_{\rho}$ ≡kinematic viscosity

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \qquad y\text{-dir}$$

Energy*

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

*Neglecting viscous dissipation which is an adequate assumption for low speed incompressible flows

6.4 Boundary Layer Equations

Boundary Layer Scaling and Approximations

- Within the boundary layer, $u \gg v$ and the flow is nearly 1-D.
- We can see this from mass conservation:

$$\text{If} \quad \frac{\partial u}{\partial x} \sim \frac{u}{L} \quad \text{and} \quad \frac{\partial v}{\partial y} \sim \frac{v}{\delta} \quad \Rightarrow \quad \frac{u}{L} \sim \frac{v}{\delta} \quad \Rightarrow \quad v \sim \delta\left(\frac{u}{L}\right) \ll 1$$

- Also within the boundary layer:

$$\frac{\partial P}{\partial y} \approx 0$$

6.5 Boundary Layer Similarity

- Let’s now non-dimensionalize the equations as follows:

Assumes $T_\infty > T_s$

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{\delta} \qquad u^* = \frac{u}{U_\infty} \qquad v^* = \frac{v}{\frac{\delta}{L} U_\infty} \qquad T^* = \frac{T - T_s}{T_\infty - T_s} \qquad P^* = \frac{P}{\rho {U_\infty}^2}$$

$$\Downarrow$$

$$x = Lx^* \qquad y = \delta y^* \qquad u = U_\infty u^* \qquad v = \frac{\delta}{L} U_\infty v^* \qquad T - T_s = (T_\infty - T_s) T^* \qquad P = \rho {U_\infty}^2 P^*$$

Mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \frac{U_\infty}{L} \frac{\partial u^*}{\partial x^*} + \frac{\delta U_\infty}{L \delta} \frac{\partial v^*}{\partial y^*} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0}$$

6.5 Boundary Layer Similarity

Momentum

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{{U_\infty}^2}{L}u^*\frac{\partial u^*}{\partial x^*}+\frac{{U_\infty}^2\delta}{L\delta}v^*\frac{\partial u^*}{\partial y^*}=-\frac{\rho {U_\infty}^2}{\rho L}\frac{\partial P^*}{\partial x^*}+\nu\left[\frac{U_\infty}{L^2}\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{U_\infty}{\delta^2}\frac{\partial^2 u^*}{\partial {y^*}^2}\right]$$

Multiply through by $\dfrac{L}{{U_\infty}^2}$

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial P^*}{\partial x^*}+\frac{\nu}{U_\infty L}\left[\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial {y^*}^2}\right]$$

$$\boxed{u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial P^*}{\partial x^*}+\frac{1}{\mathrm{Re}_L}\left[\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial {y^*}^2}\right]}$$

6.5 Boundary Layer Similarity

Momentum

Consider the viscous term; we can say 2 things:

1) Clearly $\frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}} \gg \frac{\partial^2 u^*}{\partial x^{*2}}$ so the latter is negligible here

2) All other terms are of $O(1)$ and the viscous term is of the order:

$$\frac{1}{\mathrm{Re}_L}\frac{L^2}{\delta^2}$$

For this term to be of $O(1)$ we need to have: $\frac{\delta}{L} \sim \frac{1}{\sqrt{\mathrm{Re}_L}}$

The working form of the momentum boundary layer equation is therefore:

$$\boxed{u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{1}{\mathrm{Re}_L}\frac{\partial^2 u^*}{\partial y^{*2}}}$$

6.5 Boundary Layer Similarity

Energy

The procedure is identical to the momentum case except that the variable being convected is temperature

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\cancelto{0}{-\frac{1}{\rho}\frac{\partial P}{\partial y}}+\alpha\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]$$

⇨ Assumes $T_\infty > T_s$

$$(T_\infty - T_s)\frac{U_\infty}{L}\left[u^*\frac{\partial T^*}{\partial x^*}+\frac{\delta}{\delta}v^*\frac{\partial T^*}{\partial y^*}\right]=(T_\infty - T_s)\frac{\alpha}{L^2}\left[\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{1}{\delta_t{}^2}\frac{\partial^2 T^*}{\partial y^{*2}}\right]$$

$\delta_t \equiv$ thermal BL thickness

$$u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{\alpha}{U_\infty L}\left[\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{L^2}{\delta_t{}^2}\frac{\partial^2 T^*}{\partial y^{*2}}\right]$$

$$u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{1}{\mathrm{Pe}_L}\left[\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{L^2}{\delta_t{}^2}\frac{\partial^2 T^*}{\partial y^{*2}}\right]$$

Recall:

$$\mathrm{Pe}_L=\frac{U_\infty L}{\alpha}=\mathrm{Re}_L\mathrm{Pr}$$

6.5 Boundary Layer Similarity

Energy

1) Again

$$\frac{\partial^2 T^*}{\partial y^{*2}} \gg \frac{\partial^2 T^*}{\partial x^{*2}}$$

2) The diffusion term is of the order:

$$\frac{1}{\mathrm{Pe}_L} \frac{L^2}{{\delta_t}^2}$$

For this term to be of $O(1)$ we need to have: $\frac{\delta_t}{L} \sim \frac{1}{\sqrt{\mathrm{Pe}_L}}$

The final working form for the energy boundary layer equation is then:

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Pe}_L} \frac{\partial^2 T^*}{\partial y^{*2}}$$

6.5.2 Functional Forms of Solutions

- Based on the BL equations, we can infer the key dimensionless variables involved.

Velocity Field

$$u^* = f_1\left(x^*, y^*, \frac{\partial P^*}{\partial x^*}, \mathrm{Re}_L\right)$$

$x^*, y^* \Rightarrow$ position dependence

$\frac{\partial P^*}{\partial x^*} \Rightarrow$ related to variations in U_∞ due to changes in geometry

- think of Bernoulli-like effects
- **Note**: $\partial P^*/_{\partial x^*} = 0$ for a flat plate!

$\mathrm{Re}_L \Rightarrow$ ratio of viscosity to inertia

6.5.2 Functional Forms of Solutions

- Based on the BL equations, we can infer the key dimensionless variables involved.

Temperature Field

$$T^* = f_2\left(x^*, y^*, \frac{\partial P^*}{\partial x^*}, \mathrm{Re}_L, \mathrm{Pe}_L\right)$$

$$x^*, y^* \Rightarrow \text{position dependence}$$

$$\frac{\partial P^*}{\partial x^*} \Rightarrow \text{related to variations in } U_\infty \text{ due to changes in geometry}$$

- think of Bernoulli-like effects
- **Note**: $\partial P^* / \partial x^* = 0$ for a flat plate!

$$\mathrm{Re}_L \Rightarrow \text{ratio of viscosity to inertia}$$

$$\mathrm{Pe}_L \Rightarrow \text{ratio of advection to diffusion}$$

6.5.2 Functional Forms of Solutions

- Recall now that our wall heat flux is:

$$q"_{wall} = -k_f \left.\frac{\partial T}{\partial y}\right|_{y=0} = h(T_s - T_\infty)$$

$$h = \frac{-k_f \partial T/\partial y|_{y=0}}{(T_s - T_\infty)}$$

- If we substitute our scaling terms:

$$h = -k_f \frac{(T_\infty - T_s)\frac{1}{\delta}\frac{\partial T^*}{\partial y^*}}{(T_s - T_\infty)} \Rightarrow \frac{\partial T^*}{\partial y^*} = \frac{h\delta}{k_f} \equiv \mathrm{Nu}_\delta$$

6.5.2 Functional Forms of Solutions

- Our Nusselt number here is, by definition, based on the BL thickness δ

$$\mathrm{Nu}_\delta = \frac{h\delta}{k_f} = \frac{\partial T^*}{\partial y^*}$$

- We recognize that δ is a function of x or L, so we can alternatively use the definition:

$$\mathrm{Nu}_x = \frac{hx}{k_f} \qquad \mathrm{Nu}_L = \frac{hL}{k_f}$$

- The Nusselt number is a non-dimensional gradient at the wall which is a measure of convection. In fact it is interpreted as

$$\mathrm{Nu} = \frac{\text{convection heat flux}}{\text{conduction heat flux}}$$

- Functionally it follows that Nu_x

$$\mathrm{Nu}_x = f_3(x^*, \mathrm{Re}_x, \mathrm{Pr})$$

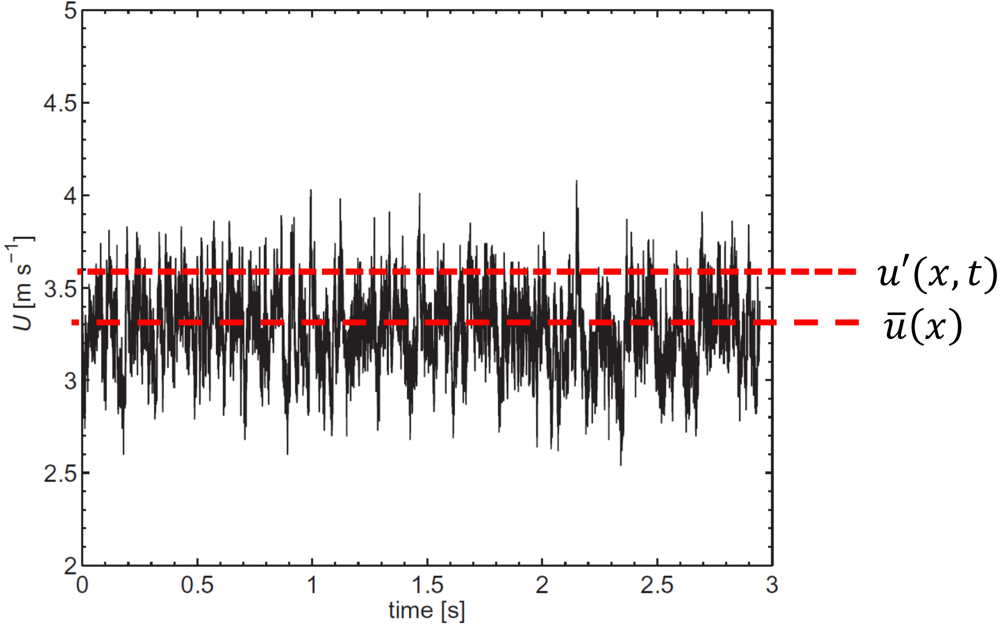
- And the averaged quantity over a surface will be:

$$\overline{\mathrm{Nu}} = f_4(\mathrm{Re}, \mathrm{Pr})$$

Role of Turbulence in the Boundary Layer Equations

- All of the preceding equations have assumed a steady, laminar flow
- The situation becomes much more complicated with the introduction of turbulence
- Here the flow is characterized by random fluctuations on top of a mean, base flow

$$\underset{\substack{\uparrow\\ \textbf{absolute}\\ \textbf{velocity}}}{u(x,t)} = \underset{\substack{\uparrow\\ \textbf{mean}\\ \textbf{velocity}}}{\bar{u}(x)} + \underset{\substack{\uparrow\\ \textbf{velocity}\\ \textbf{fluctuation}}}{u'(x,t)}$$



- The study of turbulence is inherently statistical in nature leading to the development of time averaged equations of motion approach
- This is known as the Reynolds Averaged Navier Stokes ("RANS") method and can be shown to lead to the following boundary layer equations

Role of Turbulence in the Boundary Layer Equations

Momentum

Take our momentum relation (x-direction) and rearrange and realizing that $\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2}$:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad \Rightarrow \quad \rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

A mean form of this relation with a turbulent Reynolds stress correction can be found to be:

$$\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right] = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y}\left(\overline{\rho u'v'}\right)$$

This new term is the so-called Reynolds stress:

$$\overline{\rho u'v'} \qquad\qquad {}' \Rightarrow \text{fluctuating velocity}$$

And it is interpreted as the flux of momentum ($\rho u'$) being transported in the y-direction by v'

Role of Turbulence in the Boundary Layer Equations

Energy

Here the boundary layer equation is modified by a similar fluctuating term ($\alpha = k/\rho C_p$):

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \cancelto{0}{-\frac{1}{\rho}\frac{\partial P}{\partial y}} + \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] \quad \Rightarrow \quad \rho C_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

Assuming $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$:

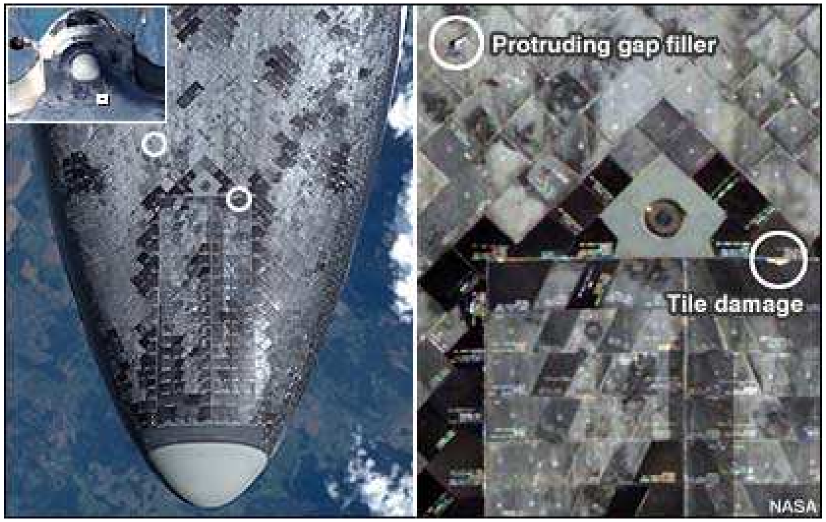
$$\rho C_p\left[\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right] = k\frac{\partial^2 \bar{T}}{\partial y^2} - \underbrace{\frac{\partial}{\partial y}\left(\overline{\rho C_p T'}\right)}_{\text{Turbulent mixing term}}$$

This new term represents enhanced thermal mixing within the boundary layers due to turbulent fluctuations and eddies

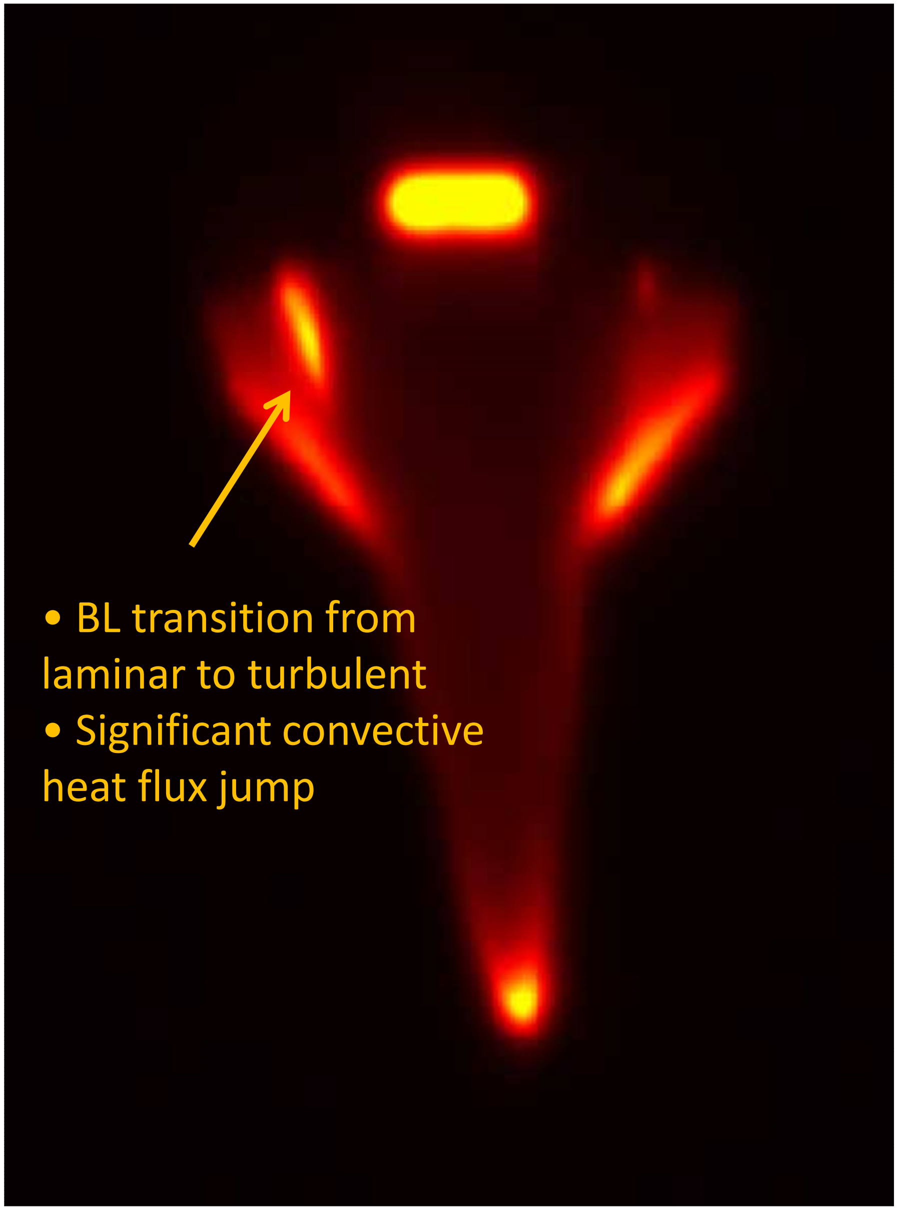
This is why $h_{turb} > h_{lam}$

Role of Turbulence in the Boundary Layer Equations

Shuttle in orbit



Shuttle during reentry



- BL transition from laminar to turbulent
- Significant convective heat flux jump

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