BEAMS: SHEAR FORCE & BENDING MOMENT

Introduction

The term *beam* refers to a slender bar that carries transverse loading; that is, the applied forces are perpendicular to the bar. In a beam, the internal force system consists of a shear force and a bending moment acting on the cross section of the bar. The study of beams, however, is complicated by the fact that the shear force and the bending moment usually vary continuously along the length of the beam.

Supports, Types and Loads

Beams are classified according to their supports and may be summarized as:

1. A *simply supported beam*, Figure 1-*a*, has a pin support at one end and a roller support at the other end. The pin support prevents displacement of the end of the beam, but not its rotation. The term *roller support* refers to a pin connection that is free to move parallel to the axis of the beam; hence, this type of support suppresses only the transverse displacement.
2. A *cantilever beam* is built into a rigid support at one end, with the other end being free, Figure 1-*b*. The built-in support prevents displacements as well as rotations of the end of the beam.

3. An *overhanging beam*, illustrated in Figure 1-*c*, is supported by a pin and a roller support, with one or both ends of the beam extending beyond the supports.

The above three types of beams are ***statically determinate*** because the support reactions can be found from the equilibrium equations.

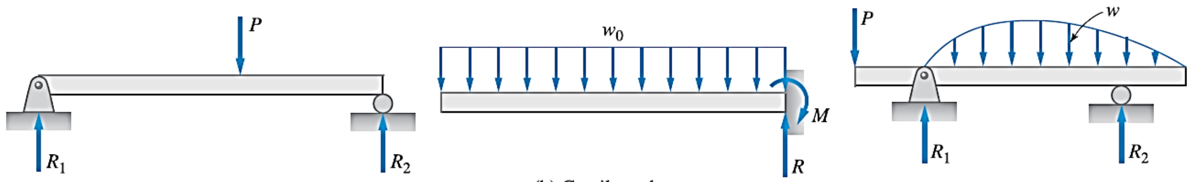


Figure 1: Statically determinate beams.

Figure 2 shows other types of beams. These beams are over-supported in the sense that each beam has at least one more reaction than is necessary for support. Such beams are ***statically indeterminate***; the presence of these redundant supports requires the use of additional equations obtained by considering the deformation of the beam. These types of beams may be summarized as:

4. A *propped cantilever beams* (Figure 1-*a*); is a beam with a built in support at one side, and a point support at the other.
5. *Fixed or built-in beams*; Figure 1-*b*, is a beam with a built in supports at both sides.
6. *Continuous beams* (Figure 1-*c*) is a multi-span beam on hinged support. The end spans may be cantilever, may be freely supported or fixed supported. At least one of the supports of a continuous beam must be able to develop a reaction along the beam axis.

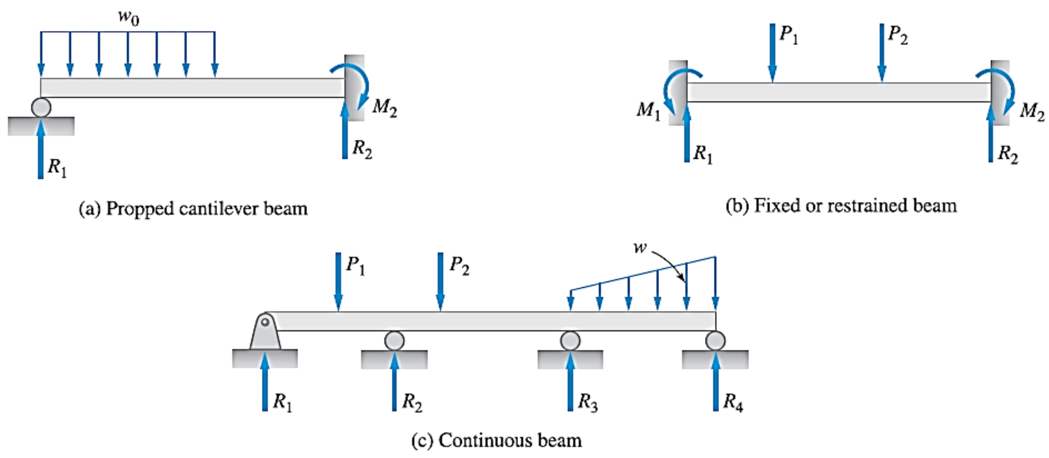


Figure 2: Statically indeterminate beams.

A *concentrated load*, such as *P* in Figure 1-*a*, is an approximation of a force that acts over a very small area. In contrast, a *distributed load* is applied over a finite area. If the distributed load acts on a very narrow area, the load may be approximated by a line load. The intensity *w* of this loading is expressed as force per unit length (N/m). The load distribution may be uniform, as shown in Figure 1-*b*, or it may vary with distance along the beam, as in Figure 1-*c*.

Shear-Moment Equations and Shear-Moment Diagrams

Consider the cantilever beam shown in Figure 3-*a*, which is subjected to a concentrated load *P* at the free end. If a cutting plane at *C* is drawn, a free-body diagram through this section (Figure 3-*b*) shows a shear forces *V* and bending moment *M* at the cutting section. It is the objective in this section to determine the shear force *V* and the bending moment *M* at every cross section of the beam. To accomplish this task, we must derive the expressions for *V* and *M* in terms of the distance *x* measured along the beam. By plotting these expressions to scale, we obtain the *shear force and bending moment diagrams* for the beam.

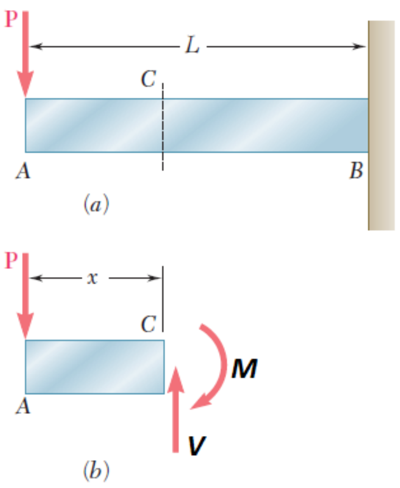


Figure 3: (*a*) Cantilever beam subjected to a concentrated load, (*b*) Section through *C*.

Sign Conventions

It is necessary to adopt sign conventions for applied loading, shear forces, and bending moments. We will use the conventions shown in Figure 4, which assume the following to be *positive*.

- *Shear forces* that tend to rotate a beam element *clockwise*.
- *Bending moments* that tend to bend a beam element *concave upward*.

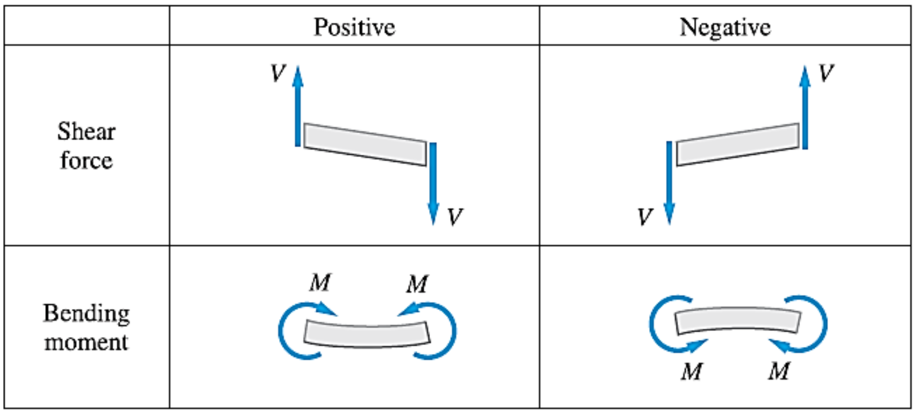


Figure 4: Sign Conventions.

Procedure for Determining Shear force and Bending Moment Diagrams

The following is a general procedure for obtaining shear force and bending moment diagrams of a statically determinate beam:

- Compute the support reactions from the *FBD* of the entire beam.
- Divide the beam into segments so that the loading within each segment is continuous. Thus, the end-points of the segments are discontinuities of loading, including concentrated loads and couples.

Perform the following steps for each segment of the beam:

- Introduce an imaginary cutting plane within the segment, located at a distance *x* from the left end of the beam, that cuts the beam into two parts.
- Draw a *FBD* for the part of the beam lying either to the left or to the right of the cutting plane, whichever is more convenient. At the cut section, show *V* and *M* acting in their positive directions.

- Determine the expressions for V and M from the equilibrium equations obtainable from the FBD. These expressions, which are usually functions of x, are the shear force and bending moment equations for the segment. These equation can be obtained using the following:

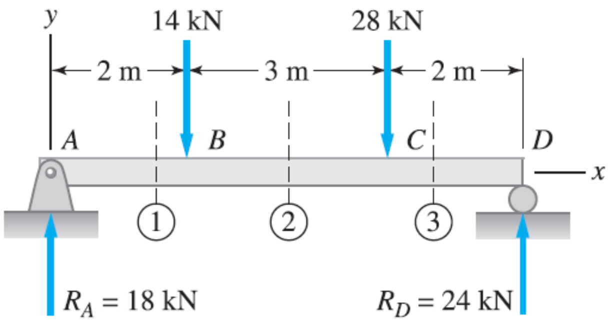
$$V = (\sum F_y)_L = (\sum F_y)_R \qquad \text{(for shear force)}$$

$$M = (\sum M)_L = (\sum M)_R \qquad \text{(for bending moment)}$$

- Plot the expressions for V and M for the segment. It is visually desirable to draw the V-diagram (SFD) below the FBD of the entire beam, and then draw the M-diagram (BMD) below the V-diagram.

Example 1:

The simply supported beam in Fig. (a) carries two concentrated loads. (1) Derive the expressions for the shear force and the bending moment for each segment of the beam. (2) Draw the shear force and bending moment diagrams. Neglect the weight of the beam. Note that the support reactions at A and D have been computed and are shown in Fig. (a).



(a)

Solution:

Part 1

The determination of the expressions for V and M for each of the three beam segments (AB, BC, and CD) is explained below.

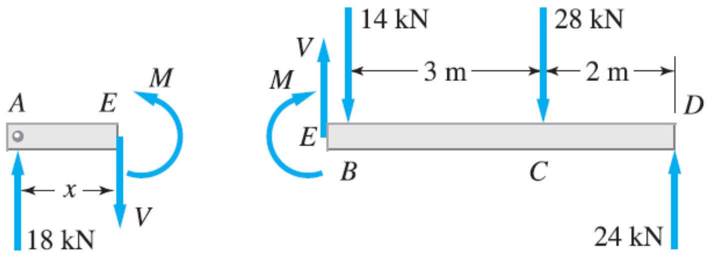
Segment AB **($0 < x < 2$ m)** Figure (b) shows the FBDs for the two parts of the beam that are separated by section ①, located within segment AB. Note that we show V and M acting in their positive directions according to the sign conventions in Fig. 4. Because V and M are equal in magnitude and oppositely directed on the two FBDs, they can be computed using either FBD. The analysis of the FBD of the part to the left of section ① yields

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - V = 0$$

$$V = +18 \text{ kN}$$ *Answer*

$$\Sigma M_E = 0 \quad +\circlearrowleft \quad -18x + M = 0$$

$$M = +18x \text{ kN} \cdot \text{m}$$ *Answer*



(b) FBDs

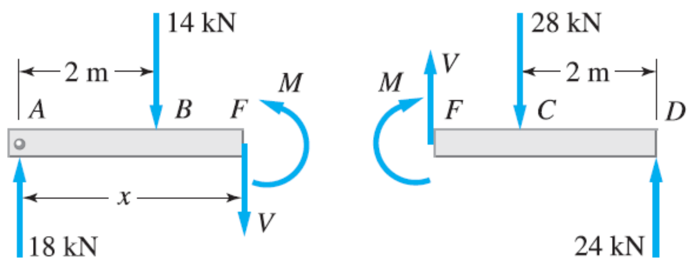
***Segment BC* (2 m < *x* < 5 m)** Figure (c) shows the FBDs for the two parts of the beam that are separated by section (2), an arbitrary section within segment *BC*. Once again, *V* and *M* are assumed to be positive according to the sign conventions in Fig. 4. The analysis of the part to the left of section (2) gives

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - 14 - V = 0$$

$$V = +18 - 14 = +4 \text{ kN}$$ *Answer*

$$\Sigma M_F = 0 \quad +\circlearrowleft \quad -18x + 14(x - 2) + M = 0$$

$$M = +18x - 14(x - 2) = 4x + 28 \text{ kN} \cdot \text{m}$$ *Answer*



(c) FBDs

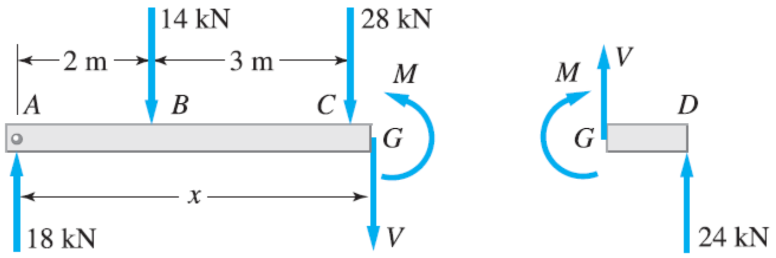
***Segment CD* (5 m < *x* < 7 m)** Section (3) is used to find the shear force and bending moment in segment *CD*. The FBDs in Fig. (d) again show *V* and *M* acting in their positive directions. Analyzing the portion of the beam to the left of section (3), we obtain

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - 14 - 28 - V = 0$$

$$V = +18 - 14 - 28 = -24 \text{ kN}$$ *Answer*

$$\Sigma M_G = 0 \quad +\circlearrowleft \quad -18x + 14(x-2) + 28(x-5) + M = 0$$

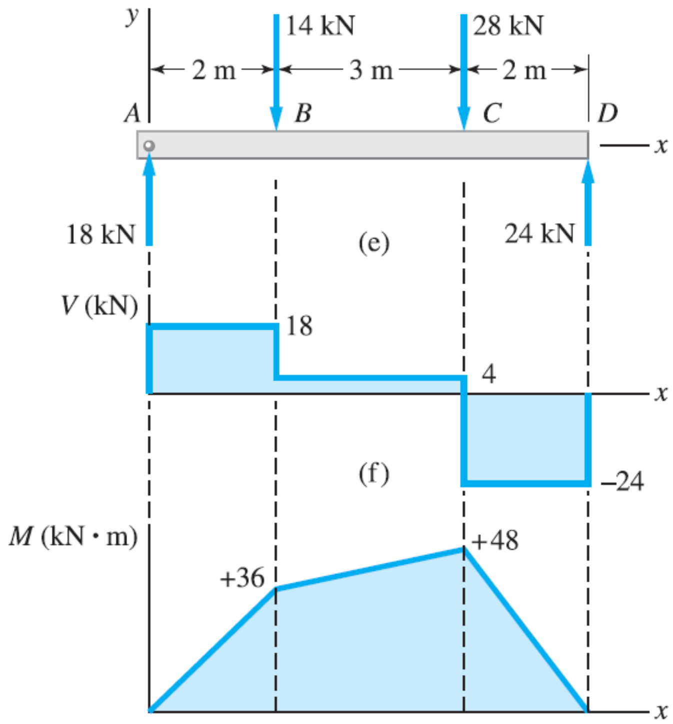
$$M = +18x - 14(x-2) - 28(x-5) = -24x + 168 \text{ kN} \cdot \text{m}$$ *Answer*



(d) FBDs

Part 2

The shear force and bending moment diagrams in Figs. (f) and (g) are the plots of the expressions for V and M derived in Part 1. By placing these plots directly below the sketch of the beam in Fig. (e), we establish a clear visual relationship between the diagrams and locations on the beam.



Shear force and bending moment diagrams

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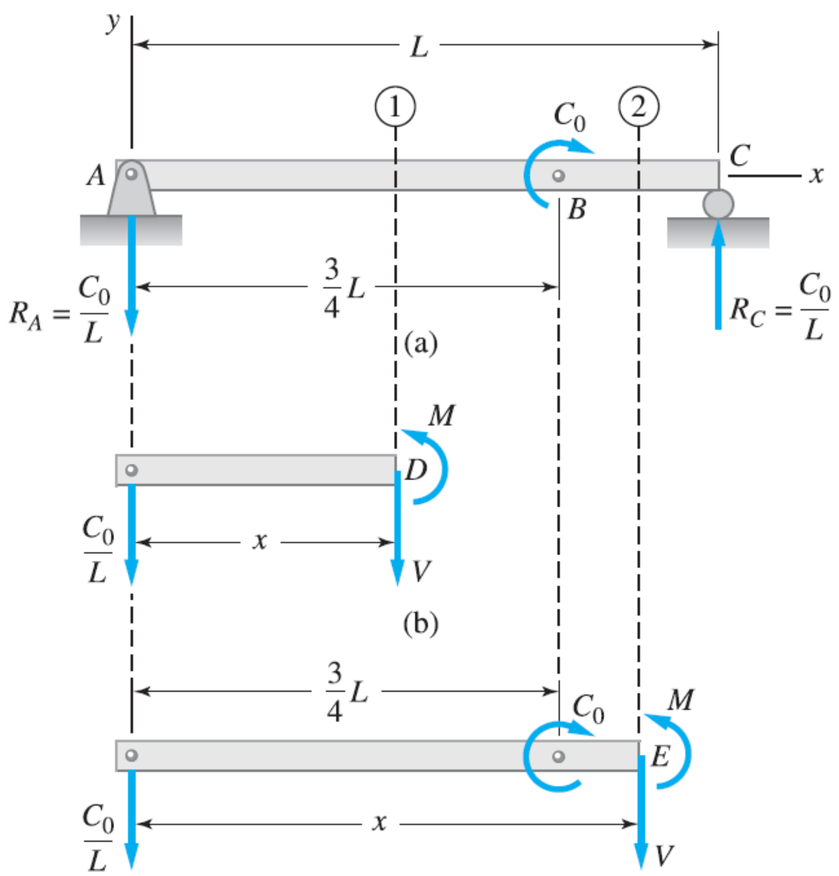
An inspection of the V-diagram reveals that the largest shear force in the beam is -24 kN and that it occurs at every cross section of the beam in segment CD. From the M-diagram we see that the maximum bending moment is $+48$ kN·m, which occurs under the 28-kN load at C. Note that at each concentrated force the V-diagram "jumps" by an amount equal to the force. Furthermore, there is a discontinuity in the slope of the M-diagram at each concentrated force.

Example 1:

The simply supported beam in Fig. (a) is loaded by the clockwise couple C_0 at B. (1) Derive the shear force and bending moment equations, and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam. The support reactions A and C have been computed, and their values are shown in Fig. (a).

Solution:



Part 1

Due to the presence of the couple C_0, we must analyze segments AB and BC separately.

Segment AB **($0 < x < 3L/4$)** Figure (b) shows the FBD of the part of the beam to the left of section ① (we could also use the part to the right). Note that V and M are assumed to act in their positive directions according to the sign conventions in Fig. 4.3. The equilibrium equations for this portion of the beam yield

$$\Sigma F_y = 0 \quad +\uparrow \quad -\frac{C_0}{L} - V = 0 \qquad V = -\frac{C_0}{L} \qquad \text{Answer}$$

$$\Sigma M_D = 0 \quad +\circlearrowleft \quad \frac{C_0}{L}x + M = 0 \qquad M = -\frac{C_0}{L}x \qquad \text{Answer}$$

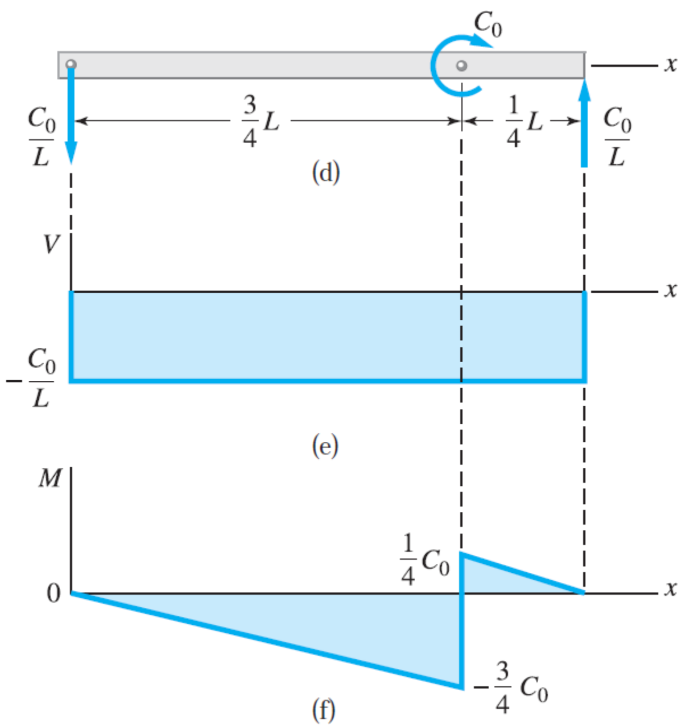
Segment BC **($3L/4 < x < L$)** Figure (c) shows the FBD of the portion of the beam to the left of section ② (the right portion could also be used). Once again, V and M are assumed to act in their positive directions. Applying the equilibrium equations to the beam segment, we obtain

$$\Sigma F_y = 0 \quad +\uparrow \quad -\frac{C_0}{L} - V = 0 \qquad V = -\frac{C_0}{L} \qquad \text{Answer}$$

$$\Sigma M_E = 0 \quad +\circlearrowleft \quad \frac{C_0}{L}x - C_0 + M = 0 \qquad M = -\frac{C_0}{L}x + C_0 \qquad \text{Answer}$$

Part 2

The sketch of the beam is repeated in Fig. (d). The shear force and bending moment diagrams shown in Figs. (e) and (f) are obtained by plotting the expressions for V and M found in Part 1. From the V-diagram, we see that the shear force is the same for all cross sections of the beam. The M-diagram shows a jump of magnitude C_0 at the point of application of the couple.



Shear force and bending moment diagrams

Area Method for Drawing Shear-Moment Diagrams

Useful relationships between the loadings, shear force, and bending moment can be derived from the equilibrium equations. These relationships enable us to plot the shear force diagram directly from the load diagram, and then construct the bending moment diagram from the shear force diagram. This technique, called the *area method*, allows us to draw the shear force and bending moment diagrams without having to derive the equations for V and M. We first consider beams subjected to distributed loading and then discuss concentrated forces and couples.

- **Distributed Loading**

Consider the beam in Figure 5-*a* that is subjected to a distributed load of intensity $w(x)$. The free-body diagram of an infinitesimal element of the beam, located at the distance x from the left end, is shown in Figure 5-*b*. The segment will carry a shear force and a bending moment at each end, which are denoted by V and M at the left end and by $V + dV$ and $M + dM$ at the right end. The infinitesimal differences dV and dM represent the changes that occur over the differential length dx of the element.

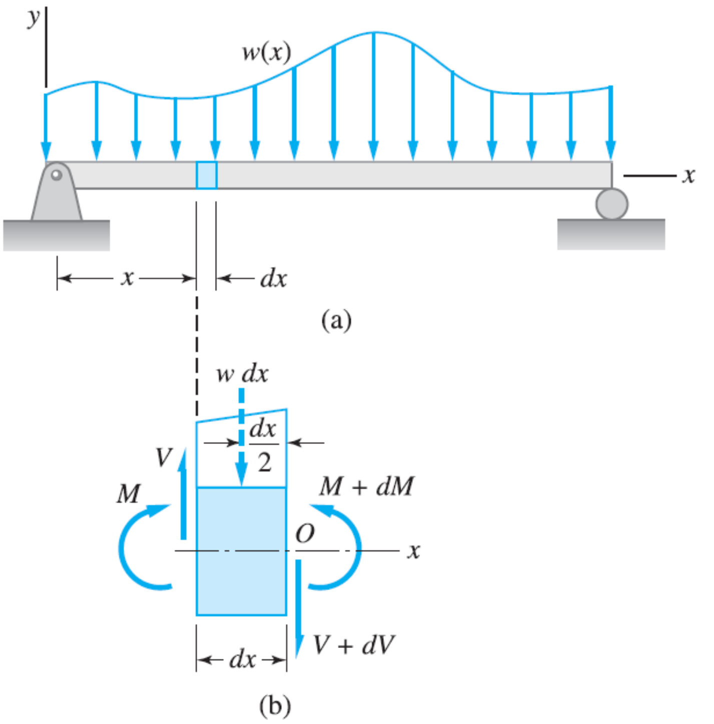


Figure 5: (*a*) Simply supported beam carrying distributed loading; (*b*) free-body diagram of an infinitesimal beam segment.

The force equation of equilibrium for the element is

$$\Sigma F_y = 0 \quad +\uparrow \quad V - w\,dx - (V + dV) = 0$$

from which we get,

$$w = -\frac{dV}{dx} \qquad (1)$$

The moment equation of equilibrium yields

$$\Sigma M_O = 0 \quad +\circlearrowleft \quad -M - V\,dx + (M + dM) + w\,dx\,\frac{dx}{2} = 0$$

Because dx is infinitesimal, the last term can be dropped, yielding

$$V = \frac{dM}{dx} \qquad (2)$$

It can be deduced from equations (1) that

$$\int_{x_A}^{x_B} \frac{dV}{dx}\,dx = V_B - V_A = -\int_{x_A}^{x_B} w\,dx$$

Recognizing that the integral on the right-hand side of this equation represents the area under the load diagram between A and B (Figure 6), we get

$$V_B - V_A = -Area\ of\ w - diagram]_A^B \qquad (3)$$

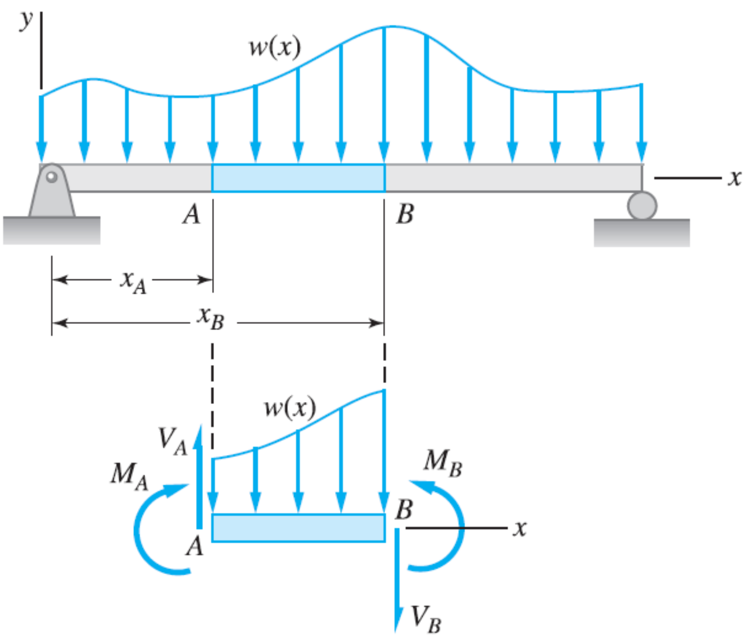


Figure 6: (a) Simply supported beam carrying distributed loading; (b) free-body diagram of a finite beam segment.

Similarly, from equation (2),

$$\int_{x_A}^{x_B} \frac{dM}{dx}\, dx = M_B - M_A = \int_{x_A}^{x_B} V\, dx$$

Because the right-hand side of this equation is the area of the shear force diagram between A and B, we obtain

$$M_B - M_A = Area\ of\ V - diagram]_A^B \qquad (4)$$

- **Concentrated Load and Couples**

The area method for constructing shear force and bending moment diagrams described above for distributed loads can be extended to beams that are loaded by concentrated forces and/or couples.

Following the same procedure, described for the distributed load (above), the shear force may be expressed as

$$V_A^+ = V_A^- \mp P_A \qquad (5)$$

Equation (5) indicates that a positive concentrated force causes a negative jump discontinuity in the shear force diagram at *A*.

Similarly, for the concentrated moment

$$M_A^+ = M_A^- \mp C_A \qquad (6)$$

Procedure for the Area Method

The following steps outline the procedure for constructing shear force and bending moment diagrams by the area method:

- Compute the support reactions from the FBD of the entire beam.
- Draw the load diagram of the beam (which is essentially a FBD) showing the values of the loads, including the support reactions.
- Working from left to right, construct the *V*- and *M*-diagrams for each segment of the beam using Equations. (1)–(6).

At first glance, using the area method may appear to be more cumbersome than plotting the shear force and bending moment equations. However, with practice you will find that the area method is not only much faster but also less susceptible to numerical errors because of the self-checking nature of the computations.

BENDING STRESS IN BEAMS

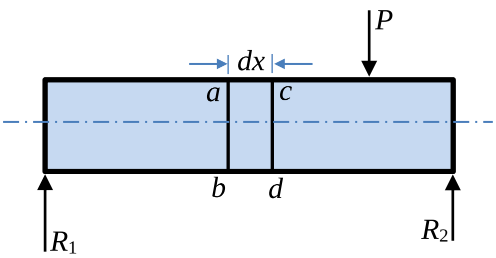
Introduction

In deriving the relations between the bending moment and the bending stress, the following assumptions are made:

1- Plane section of the beam, originally plane remains plane.
2- The material of the beam is homogenous and obeys Hooke's law.
3- The beams are initially straight of constant cross-section.

Derivation

The stress caused by bending moment are known as bending or *flexure* stresses, and the relation between theses stresses and the bending moment is expressed by the *flexure formula*.



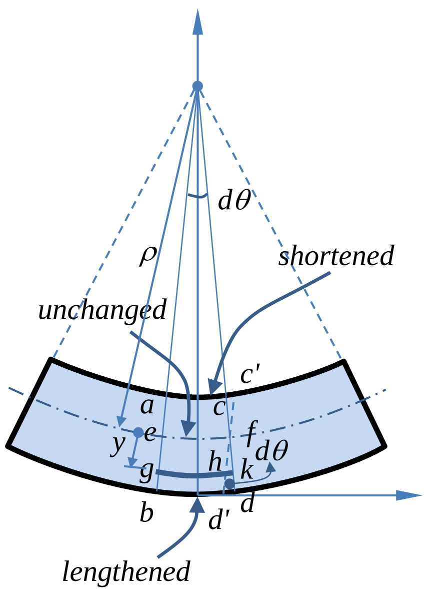


Figure 1:

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Note: the plane that fiber *ef* is called the *neutral surface* because such fiber remains unchanged in length and hence carries **NO STRESS**.

Consider now the deformation of a typical fiber *gh* located *y* units from the neutral surface. Its elongation is the arc of a circle of radius *y* with angle $d\theta$ and is given by:

$$\delta = hk = yd\theta$$

and the strain

$$\varepsilon = \frac{\delta}{L} = \frac{yd\theta}{ef}$$

If ρ is the radius of curvature of the neutral surface, then

$$\varepsilon = \frac{y\,d\theta}{\rho\,d\theta} = \frac{y}{\rho}$$

Applying Hooke's law:

$$\sigma = E\varepsilon = \left(\frac{E}{\rho}\right) y \qquad (1)$$

Now consider equilibrium of forces along *x*-axis;

$$\sum F_x = 0: \qquad \int \sigma dA = 0$$

Using eq. (1), the above integral becomes,

$$\int \left(\frac{E}{\rho}\right) ydA = 0 \quad \rightarrow \quad \left(\frac{E}{\rho}\right) \int ydA = 0$$

Since *y dA* is the moment of the differential area *dA* or the *first moment of area* about the neutral axis, hence

$$\left(\frac{E}{\rho}\right) A\bar{y} = 0$$

However, since only $\bar{y}$ in this relation can be zero. We conclude that the distance from the neutral axis to the centroid of the cross-sectional area must be *zero*, i.e., the neutral axis must contain the centroid of the cross-sectional area.

Consider now the bending moment *M* induced in the beam, as shown in Figure 2, which can be expressed as;

$$M = \int y(\sigma dA) \qquad (2)$$

Substitute eq. (1) into eq. (2), we get

$$M = \left(\frac{E}{\rho}\right)\int y^2\, dA = \left(\frac{E}{\rho}\right) I$$

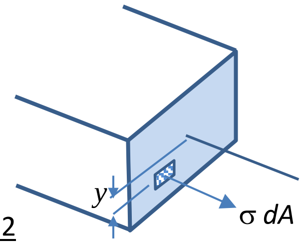


Figure 2

where I is the *moment of inertia* or the 2nd *moment of area* of the beam section about the neutral axis.

$$M = \left(\frac{E}{\rho}\right) I \qquad (3)$$

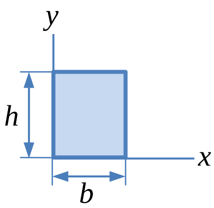
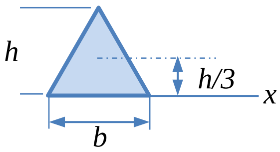
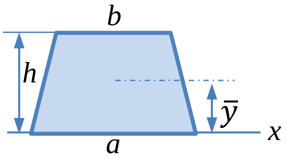
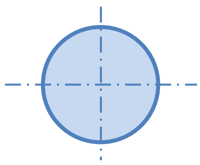
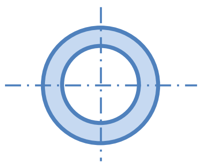
Using eq. (1) and eq. (3)

$$\frac{E}{\rho} = \frac{M}{I} = \frac{\sigma}{y}$$

This leads directly to the flexure formula.

$$\sigma = \frac{My}{I}$$

Moment of Inertia or 2nd Moment of Area

Shape	2nd Moment of Area
	$I_{N.A.} = \frac{bh^3}{12}$
	$I_{N.A.} = \frac{bh^3}{36}$
	$I_{N.A.} = \left(\frac{h^3}{36}\right)\left(\frac{a^2+4ab+b^2}{a+b}\right)$ where $\bar{y} = \left(\frac{h}{3}\right)\left(\frac{a+2b}{a+b}\right)$
	$I_{N.A.} = \frac{\pi D^4}{64}$
	$I_{N.A.} = \frac{\pi}{64}(D_o^4 - D_i^4)$

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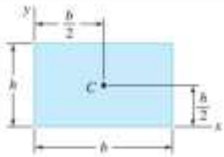
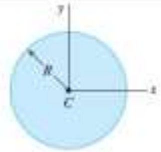
Rectangle



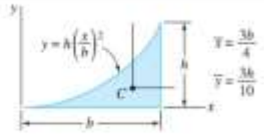
$$\bar{I}_x = \frac{bh^3}{12} \quad \bar{I}_y = \frac{b^3h}{12} \quad \bar{I}_{xy} = 0$$
$$I_x = \frac{bh^3}{3} \quad I_y = \frac{b^3h}{3} \quad I_{xy} = \frac{b^2h^2}{4}$$

Circle



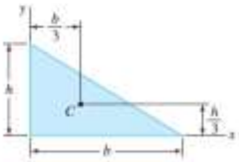
$$I_x = I_y = \frac{\pi R^4}{4} \quad I_{xy} = 0$$

Half parabolic complement



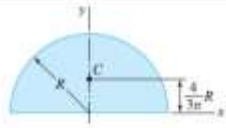
$$\bar{I}_x = \frac{37bh^3}{2100} \quad I_x = \frac{bh^3}{21}$$
$$\bar{I}_y = \frac{b^3h}{80} \quad I_y = \frac{b^3h}{5}$$
$$\bar{I}_{xy} = \frac{b^2h^2}{120} \quad I_{xy} = \frac{b^2h^2}{12}$$

Right triangle



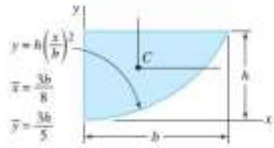
$$\bar{I}_x = \frac{bh^3}{36} \quad \bar{I}_y = \frac{b^3h}{36} \quad \bar{I}_{xy} = -\frac{b^2h^2}{72}$$
$$I_x = \frac{bh^3}{12} \quad I_y = \frac{b^3h}{12} \quad I_{xy} = \frac{b^2h^2}{24}$$

Semicircle



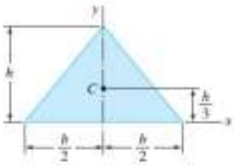
$$\bar{I}_x = 0.1098R^4 \quad \bar{I}_{xy} = 0$$
$$I_x = I_y = \frac{\pi R^4}{8} \quad I_{xy} = 0$$

Half parabola



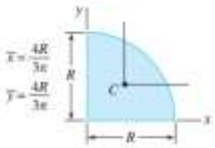
$$\bar{I}_x = \frac{8bh^3}{175} \quad I_x = \frac{2bh^3}{7}$$
$$\bar{I}_y = \frac{19b^3h}{480} \quad I_y = \frac{2b^3h}{15}$$
$$\bar{I}_{xy} = \frac{b^2h^2}{60} \quad I_{xy} = \frac{b^2h^2}{6}$$

Isosceles triangle



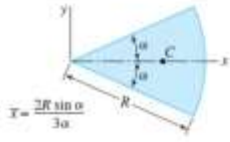
$$\bar{I}_x = \frac{bh^3}{36} \quad \bar{I}_y = \frac{b^3h}{48} \quad \bar{I}_{xy} = 0$$
$$I_x = \frac{bh^3}{12} \quad I_{xy} = 0$$

Quarter circle



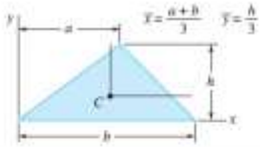
$$\bar{I}_x = \bar{I}_y = 0.05488R^4 \quad I_x = I_y = \frac{\pi R^4}{16}$$
$$\bar{I}_{xy} = -0.01647R^4 \quad I_{xy} = \frac{R^4}{8}$$

Circular sector



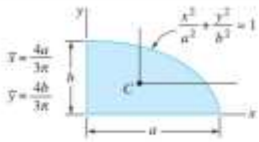
$$I_x = \frac{R^4}{8}(2\alpha - \sin 2\alpha)$$
$$I_y = \frac{R^4}{8}(2\alpha + \sin 2\alpha)$$
$$I_{xy} = 0$$

Triangle



$$\bar{I}_x = \frac{bh^3}{36} \quad I_x = \frac{bh^3}{12}$$
$$\bar{I}_y = \frac{bh}{36}(a^2 - ab + b^2) \quad I_y = \frac{bh}{12}(a^2 + ab + b^2)$$
$$\bar{I}_{xy} = \frac{bh^2}{72}(2a - b) \quad I_{xy} = \frac{bh^2}{24}(2a + b)$$

Quarter ellipse



$$\bar{I}_x = 0.05488ab^3 \quad I_x = \frac{\pi ab^3}{16}$$
$$\bar{I}_y = 0.05488a^3b \quad I_y = \frac{\pi a^3b}{16}$$
$$\bar{I}_{xy} = -0.01647a^2b^2 \quad I_{xy} = \frac{a^2b^2}{8}$$

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Parallel-Axis Theorems

For the section shown in Figure 3, the axis X_o-X_o is passing through the centroid, while X-X axis is distanced d from it. Consider now calculating the 2nd moment of area through X-X axis,

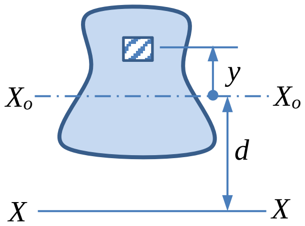


Figure 3

$$I_{X-X} = \int (y+d)^2\, dA \quad = \int y^2 dA + 2d \int y dA + d^2 \int dA$$

Since $\int ydA$ is the 1st moment of area (centroid), then $\int ydA = A\bar{y}$, since the axis X_o-X_o passes through the centroid, hence $\bar{y}$ has a value of zero. Therefore,

$$I_{X-X} = \bar{I} + A\, d^2$$

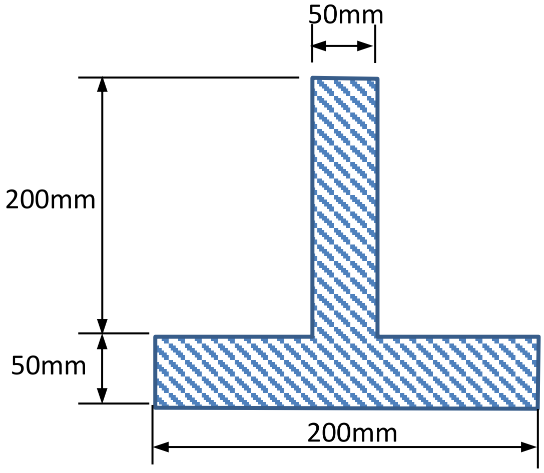
Example 1:

$$A\bar{Y} = \int ydA = \sum_{i=1}^{n} a_i\, \bar{y}_i$$

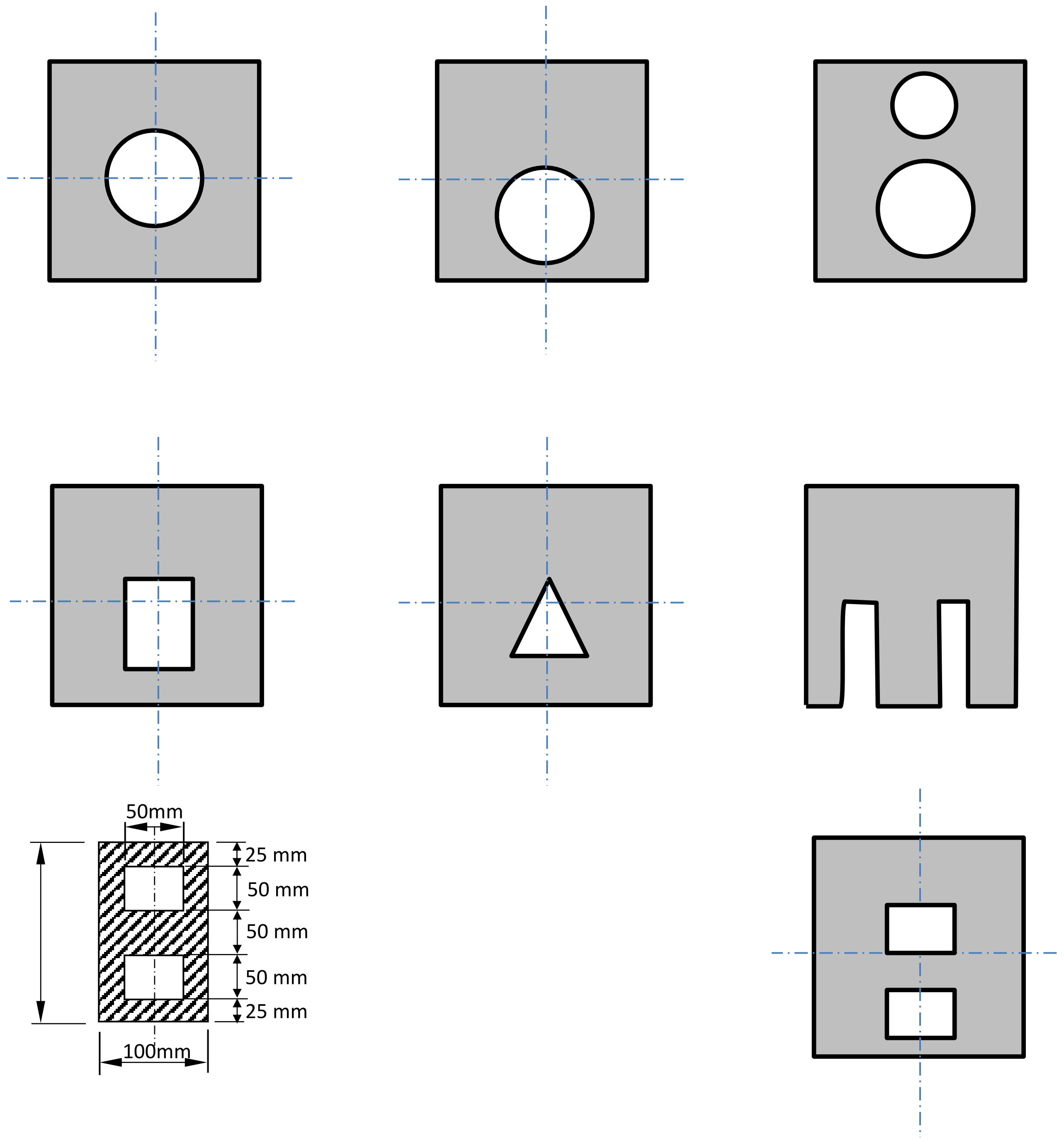
$$(200 \times 50 + 200 \times 50)\bar{Y} = (200 \times 50) \times 25 + (200 * 50) * 150$$

$$\bar{Y} = \frac{175}{2} = 87.5\, mm$$

$$I_{NA} = \left(\frac{50 \times 200^3}{12} + (50 \times 200) \times (150 - 87.5)^2\right) + \left(\frac{200 \times 50^3}{12} + (50 \times 200) \times (87.5 - 25)^2\right) = 113.5 \times 10^6$$ (Answer)



Some Interested Beam Sections



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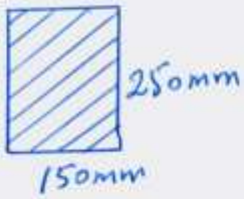
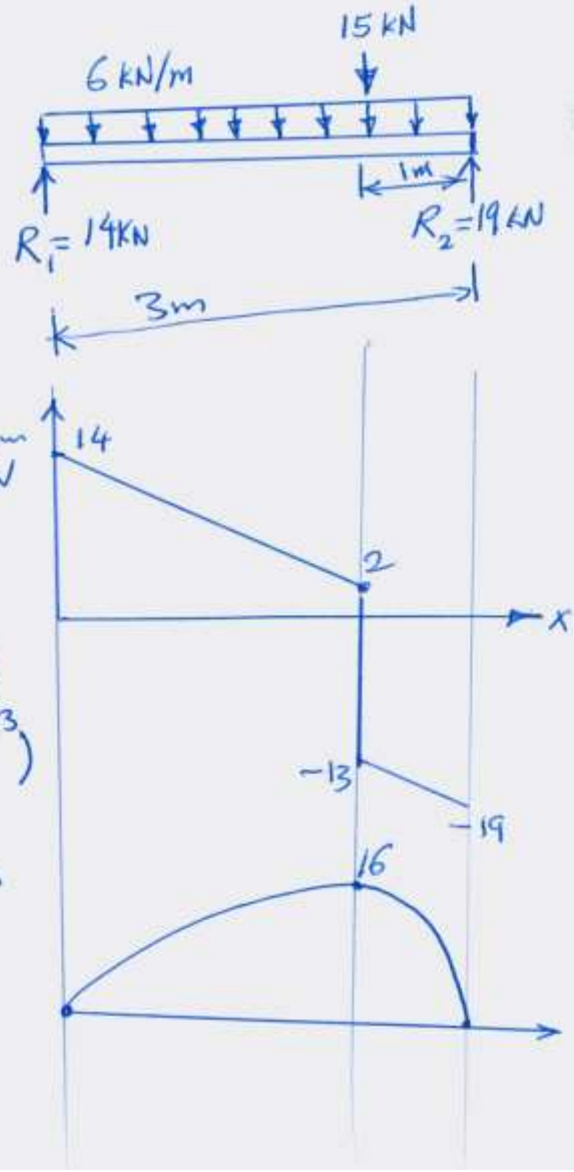
Ex. A beam 150 mm wide by 250 mm deep, supports the loads shown in the figure. Determine the max. bending stress.





Sol.

Max. Bending moment = 16 kN.m

$$\sigma_{max} = \sigma\Big|_{y=\frac{h}{2}} = \frac{M_{max}\, y_{max}}{I}$$

$$= \frac{16\times10^{3} * \left(\frac{0.25}{2}\right)}{\left(\frac{0.15 * 0.25^{3}}{12}\right)}$$

$$= 10.24\ MPa$$

Ans.

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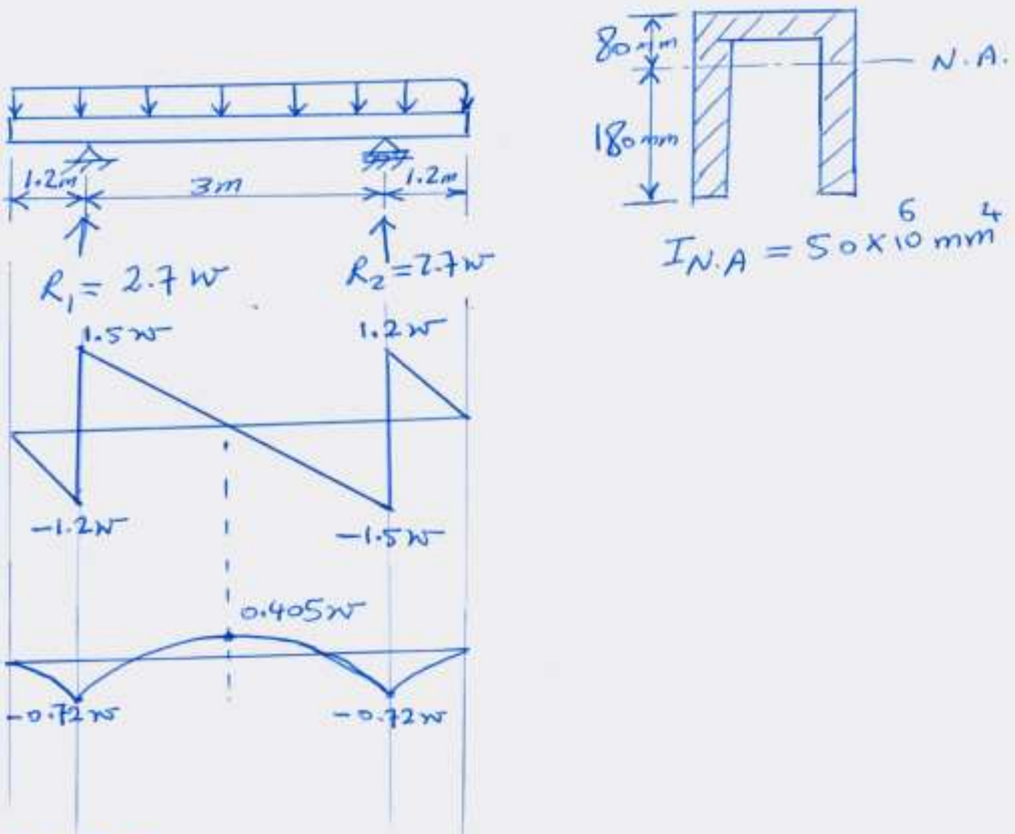
Example 1:

The overhanging beam shown in the figure is made of cast iron for which the allowable stresses are $\sigma_t = 40$ MPa and $\sigma_c = 100$ MPa. Determine the maximum uniformly distributed load that can be supported.



at $x = 1.2$ m

$M = -0.72w$

$M = \frac{\sigma I}{y}$

$\therefore M_t = \frac{40\times10^6 * (50\times10^{-6})}{0.08} = 25.0$ kN.m

$M_c = \frac{100\times10^6 * (50\times10^{-6})}{0.18} = 27.8$ kN.m

u.d.l.

$\therefore 0.72w = 25\times10^3 \longrightarrow w = 34.7$ kN/m

at $x = 2.7$ m (zero shear)

$M = \frac{\sigma I}{y}$

$M_c = \frac{100\times10^6 * 50\times10^{-6}}{0.08} = 62.5$ kN.m

$M_t = \frac{40\times10^6 * 50\times10^{-6}}{0.18} = 11.1$ kN.m

u.d.l. $0.405w = 11.1\times10^3 \longrightarrow w = 27.4$ kN/m ← Ans.

H.W (for position ⊔) Ans 15.4

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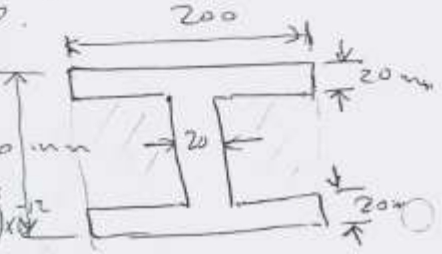
2

A Simply Supported beam over a span of 7m have the section shown. The beam carries a distributed load of 5 kN/m and a concentrated load of 20 kN at mid-span. Determine:

a) The second moment of area of the cross-section

b) The maximum stress set-up.

Sol.-



$$I = I_{rectangle} - I_{shaded\ portions}$$

$$= \left(\frac{200 \times 300^3}{12}\right) 10^{-12} - 2\left(\frac{90 \times 260^3}{12}\right) \times 10^{-12}$$

$$= (4.5 - 2.64) \times 10^{-4} = 1.86 \times 10^{-4}\ m^4$$

b) $$\sigma_{max} = \frac{M_{max}\, y_{max}}{I}$$

$$M_{max} = \frac{WL}{4} + \frac{wL^2}{8} = \left[\frac{20 \times 10^3 * 7}{4}\right] + \left(\frac{5 \times 10^3 * 7^2}{8}\right)$$

$$= (35. + 30.63) \times 10^3 = 65.63\ kN \cdot m$$

$$\sigma_{max} = \frac{65.63 * \times 10^3 * 150 \times 10^{-3}}{1.86 \times 10^{-4}}$$

$$= 51.8\ MN/m^2$$

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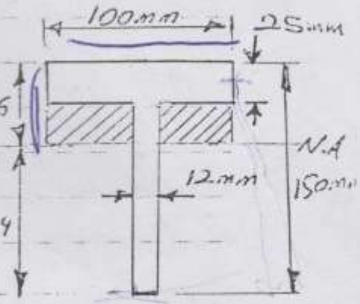
Ex.3 A uniform T-section beam is 100mm wide and 150mm deep with a flange thickness of 25mm and a web thickness of 12mm. If the limiting bending stresses for the material of the beam are 80 MN/m² in compression and 160 MN/m² in tension, find the maximum uniformaly distributed load that the beam can carry over a simply supported span of 5m.



Solution

To determine the position of the neutral axis we have to find the position of the centroid of the section

$$A\bar{Y} = \sum a\bar{y}$$

$$[(100*25)+(125*12)]\bar{Y} = (100*25)*(150-\frac{25}{2}) + (125*12)*(\frac{125}{2})$$

$$\therefore \bar{Y} = 109.4\ mm$$

Moment of inertia

$$I_{NA} = \frac{1}{3}[(100*40.6^3)-(88*15.6^3)] + \frac{1}{3}(12*109.4^3)$$

$$= 22.07\times10^{6}\ mm^4$$

$$= 22.07\times10^{-6}\ m^4 \quad 72.995$$

Maximum Compressive Stress

$$\sigma_c = \frac{M_c y}{I} \Rightarrow M_c = \frac{\sigma I}{y} = \frac{80\times10^6 * 22.07\times10^{-6}}{40.6\times10^{-3}}$$

$$= 43.5\ kNm$$

Maximum Tensile Stress

$$\sigma_t = \frac{M_t y}{I} \Rightarrow M_t = \frac{\sigma_t I}{y} = \frac{160\times10^6 * 22.07\times10^{-6}}{109.4\times10^{-3}}$$

$$= 32.3\ kNm$$

Therefor the maximum moment is 32.3 kN.m

or simply supported beam with uniformly distributed load

$$M_{max} = \frac{wL^2}{8} \Rightarrow w = \frac{8M}{L^2} = \frac{8*32.3\times10^3}{5^2}$$

$$= 10.3\ kN/m$$

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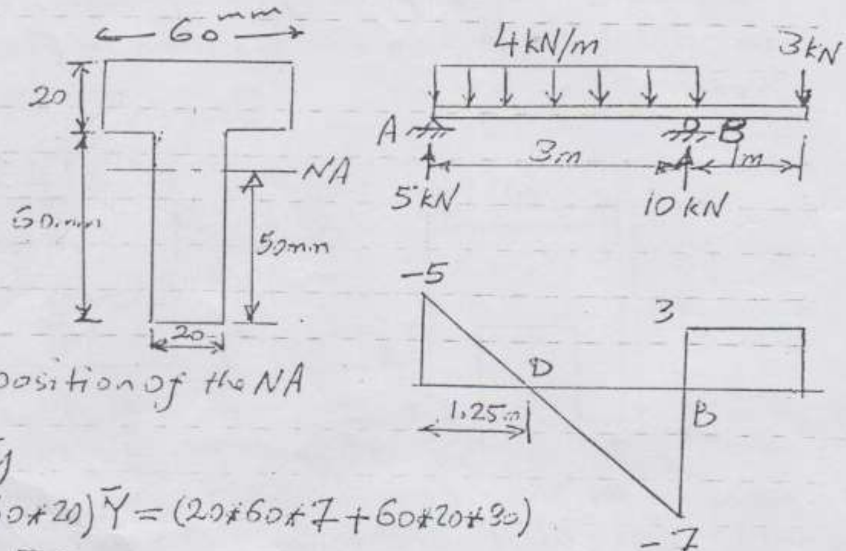
Ex 4 An overhanging beam of T-shaped section is loaded as shown in the figure. Determine the maximum tensile and compressive bending stresses.

Solution

STATICS

$R_A = 5 kN$

$R_B = 10 kN$



To find the position of the NA

$A\bar{Y} = \sum a\bar{y}$

$(20*60 + 60*20)\bar{Y} = (20*60*7 + 60*20*30)$

$\therefore \bar{Y} = 50 mm$

Moment of inertia

$$I_{NA} = \left[\frac{60*20^3}{12} + 20*60*(20)^2\right] + \left[\frac{20*60^3}{12} + 20*60*(20)^2\right]$$

$(y_{NA} - y)^2$

$= 136 \times 10^4 mm^4$

DRAW THE SHEAR FORCE DIAGRAM & LOCATE THE SECTIONS OF ZERO SHEAR OR GREATEST MOMENT.

1. at $x = 1.25$ (zero shear) $M_D = 3.125 kN.m$
2. AT $x = 3$, $M_B = -3 kN.m$

AT $x = 1.25 m$

$$\sigma_c = \frac{M_D\, y}{I} = \frac{3.125 \times 10^3 * 0.03}{136 \times 10^{-8}} = 68.9 MPa \text{ Comp.}$$

$$\sigma_t = \frac{3.125 \times 10^3 * 0.05}{136 \times 10^{-8}} = 114.9 MPa \text{ Tension}$$

AT $x = 3 m$

$$\sigma_c = \frac{3 \times 10^3 * 0.05}{136 \times 10^{-8}} = 110.3 MPa \text{ Compression}$$

$$\sigma_t = \frac{3 \times 10^3 * 0.03}{136 \times 10^{-8}} = 66.2 MPa \text{ tension}$$

$\therefore$ MAX TENSILE STRESS = 114.9 MPa AT bottom of beam at D **Ans.**

MAX. COMPRESSIVE STRESS = 110.3 MPa AT TOP of beam at B **Ans.**

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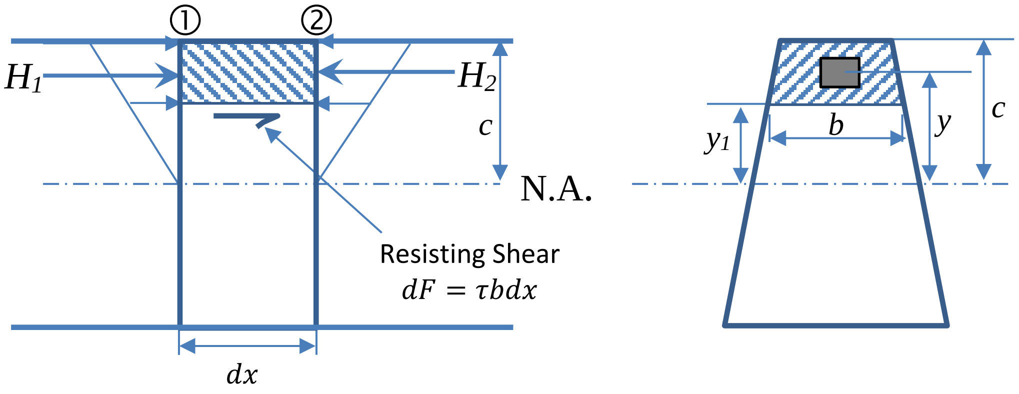
SHEAR STRESS IN BEAMS

Horizontal Shearing Stress in Beams

Consider two sections, (1) and (2), in a beam separated by the distance dx, as shown in Figure (1)



$\sum F_x = 0$: $\quad dF = H_2 - H_1 \quad = \int_{y_1}^{c} \sigma_2 dA - \int_{y_1}^{c} \sigma_1 dA$

Since $\sigma = \frac{My}{I}$, then

$$dF = \frac{M_2}{I}\int_{y_1}^{c} y dA - \frac{M_1}{I}\int_{y_1}^{c} y dA = \frac{M_2 - M_1}{I}\int_{y_1}^{c} y dA = \frac{dM}{I}\int_{y_1}^{c} y dA$$

From the above figure, $dF = \tau b dx$;

$$\tau = \frac{1}{b}\frac{dF}{dx} = \frac{1}{bI}\frac{dM}{dx}\int_{y_1}^{c} y dA$$

But $\frac{dM}{dx} = V$, then

$$\tau = \frac{V}{bI}\int_{y_1}^{c} ydA$$

The integration $\int_{y_1}^{c} ydA$ is the *first moment of area* of the shaded area A' about the neutral axis, then

$$\tau = \frac{V}{bI}A'\,\bar{y}'$$

Note: The above equation sometimes written as $\tau = \frac{V}{bI}Q$, where $Q = A'\,\bar{y}'$ is called shear flow.

Example 1:

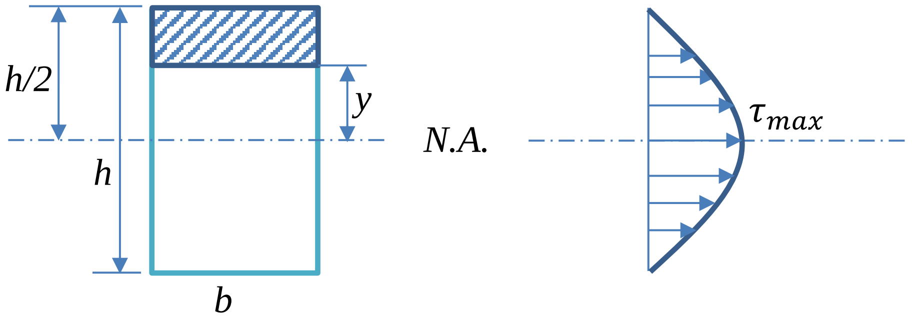
Draw the shear stress due to bending for the rectangular cross section shown.

$$\tau = \frac{V}{bI}A'\,\bar{y}' = \frac{V}{Ib}\left[b\left(\frac{h}{2} - y\right)\left(y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right)\right]$$

which reduced to

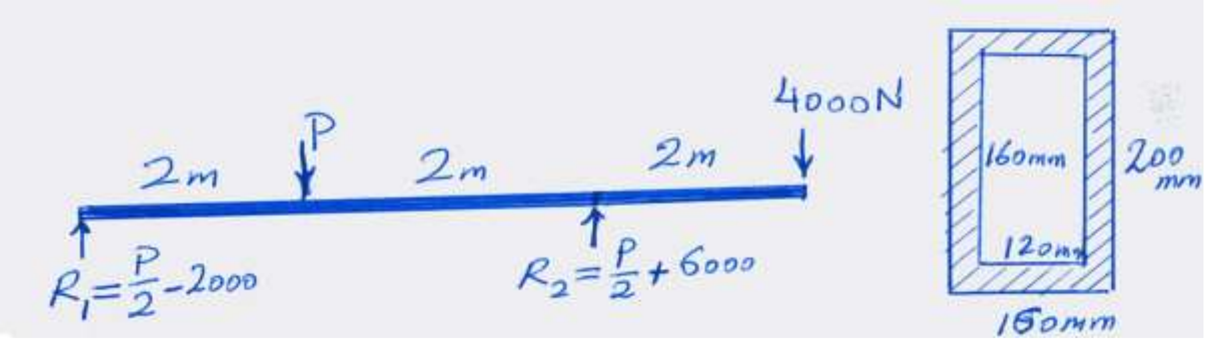
$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right)$$

$$\tau_{max} = \frac{V}{2\left(\frac{bh^3}{12}\right)}\left(\frac{h^2}{4}\right) = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A}$$

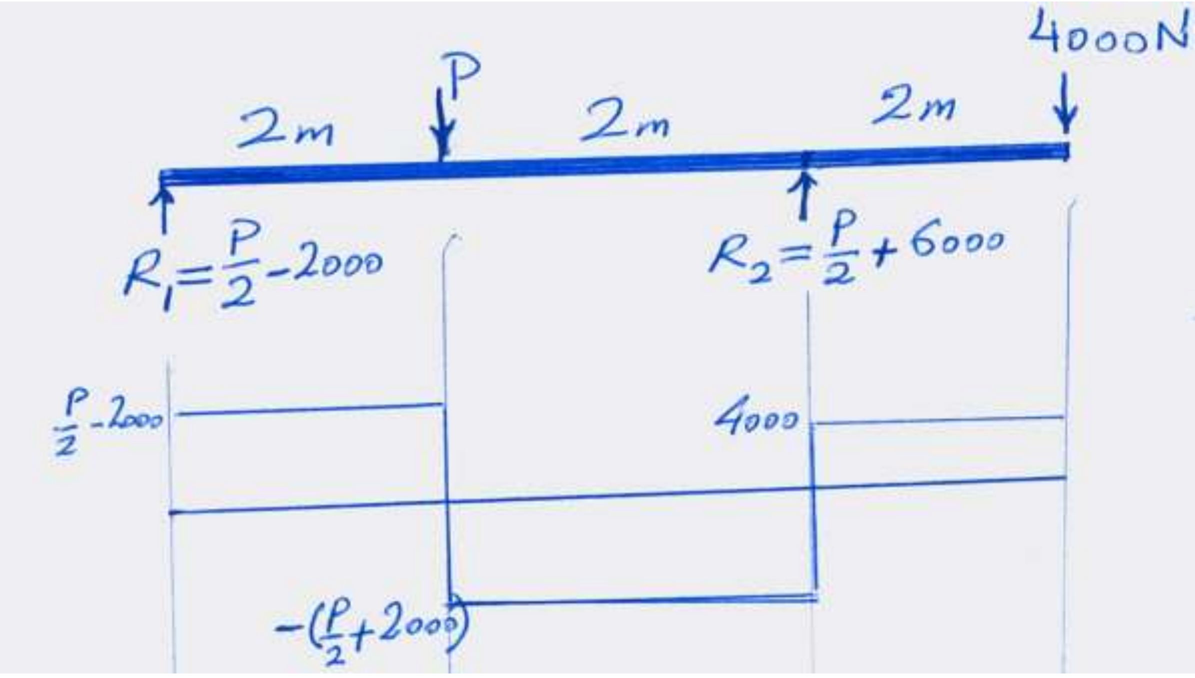


Example 2:

A box beam supports the loads shown in the figure. Compute the maximum value of P that will not exceed a flexural (or bending) stress $\sigma = 8$ MPa or a shearing stress $\tau = 1.2$ MPa for the section between the supports.



Solution:



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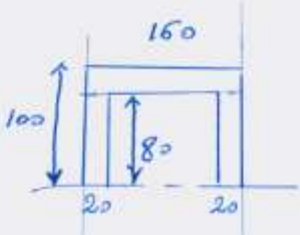
$$I = \Sigma \frac{bh^3}{12} = \frac{160*200^3}{12} - \frac{120*160^3}{12}$$
$$= 65.7\times10^{-6}\ m^4$$

from the shear force diagram the maximum shear force V is $-(\frac{P}{2}+2000)$



$$\tau = \frac{V}{bI} Q$$

$$Q = \Sigma a\bar{y} = (160*20)*90 + 2(80*20)*40$$
$$= 416\times10^3\ mm^3$$

oR $Q = (160*100)*(50) - (120*80)*40$
$$= 416\times10^3\ mm^3$$

Shearing

$$\tau = \frac{V}{bI} Q \Rightarrow 1.2\times10^6 = \frac{\frac{1}{2}P+2000}{(0.04)*65.7\times10^{-6}} \Rightarrow P = 11.2\ kN$$

Bending max. moment between the supports in terms of P is at $x=2m$

$$\therefore M = (\tfrac{1}{2}P - 2000)*2 = P - 4000$$

$$M = \frac{\sigma I}{y} \Rightarrow P - 4000 = \frac{8\times10^6 * 65.7\times10^{-6}}{0.1} \Rightarrow P = 9.26\ kN$$ (Max. P Ans.)

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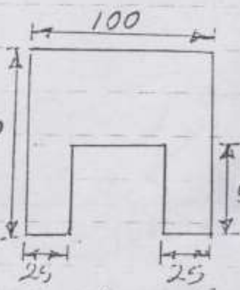
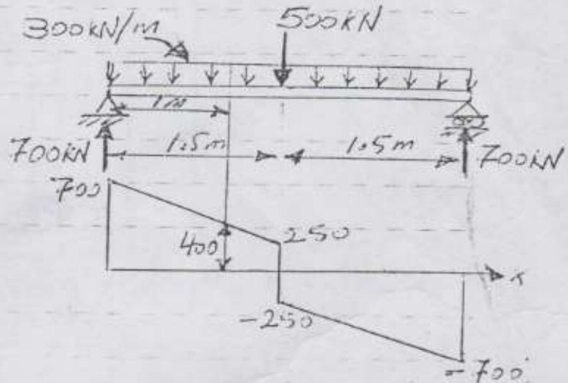
Ex. 5 At a certain section a beam has the cross-section shown in the figure. The beam is simply supported at its ends and carries a central concentrated load of 500kN together with 300 kN/m uniformly distributed load across the complete span of 3m. Draw the shear stress distribution diagram for a section 1m from the left-hand support.

Solution





- First draw the shear force diagram (S.F.D.), as shown above
- The shear force at the section 1m from the left-hand support is 400 kN

The position of N.A. of the beam section:

$$A\bar{Y} = \Sigma a\bar{y}$$

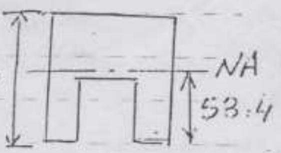
$$(100\times100 - 50\times50)\,\bar{Y} = (100\times100)*50 - (50\times50)*25$$

$$\therefore \bar{Y} = 58.4\,mm$$

Moment of Inertia about N.A.

$$I_{N.A} = \left[\frac{100*41.6^3}{3} + 2\left(\frac{25*58.4^3}{3}\right) + \frac{50*8.4^3}{3}\right]$$

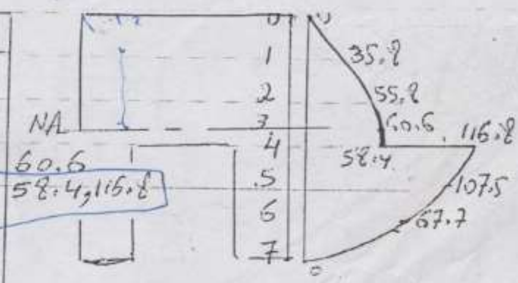
$$= 5.72\times10^6\,mm^4 = 5.72\times10^{-6}\,m^4$$



$$\tau = \frac{VA\bar{y}}{Ib} = \frac{400\times10^3 * A\bar{y}}{5.72\times10^{-6}\,b}$$

$$= 7\times10^{10}\frac{A\bar{y}}{b}$$

section	$A\times10^{-6}$ (m^2)	$\bar{y}\times10^{-3}$ m	$b\times10^{-3}$ (m)	$\tau = 7\times10^{10}\frac{A\bar{y}}{b}$
0	0	—	—	0
1	1500	34.1	100	35.8
2	3000	26.6	100	55.8
3	4160	20.8	100	60.6
4	2500, 2500	33.4	100, 50	58.4, 116.8
5	2000	38.4	50	107.5
6	1000	48.4	50	67.7
7	0	—	—	0



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The following example from:

- F. P. Beer, E. R. Johnston, J. T. Dewolf, D. F. Mazurek, *Mechanics of Materials*, 6th Edition, McGraw-Hill, New York, 2012.

EXAMPLE 6.01

A beam is made of three planks, 20 by 100 mm in cross section, nailed together (Fig. 6.8). Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is $V = 500$ N, determine the shearing force in each nail.

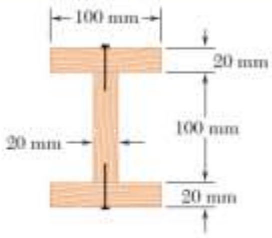

Fig. 6.8

We first determine the horizontal force per unit length, q, exerted on the lower face of the upper plank. We use Eq. (6.5), where Q represents the first moment with respect to the neutral axis of the shaded area A shown in Fig. 6.9*a*, and where I is the moment of inertia about the same axis of the entire cross-sectional area (Fig. 6.9*b*). Recalling that the first moment of an area with respect to a given axis is equal to the product of the area and of the distance from its centroid to the axis,† we have

$$Q = A\bar{y} = (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m}) = 120 \times 10^{-6} \text{ m}^3$$

$$I = \tfrac{1}{12}(0.020 \text{ m})(0.100 \text{ m})^3 + 2[\tfrac{1}{12}(0.100 \text{ m})(0.020 \text{ m})^3 + (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m})^2]$$
$$= 1.667 \times 10^{-6} + 2(0.0667 + 7.2)10^{-6} = 16.20 \times 10^{-6} \text{ m}^4$$

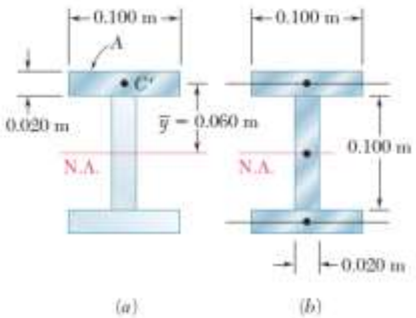

Fig. 6.9

Substituting into Eq. (6.5), we write

$$q = \frac{VQ}{I} = \frac{(500 \text{ N})(120 \times 10^{-6} \text{ m}^3)}{16.20 \times 10^{-6} \text{ m}^4} = 3704 \text{ N/m}$$

Since the spacing between the nails is 25 mm, the shearing force in each nail is

$$F = (0.025 \text{ m})q = (0.025 \text{ m})(3704 \text{ N/m}) = 92.6 \text{ N}$$

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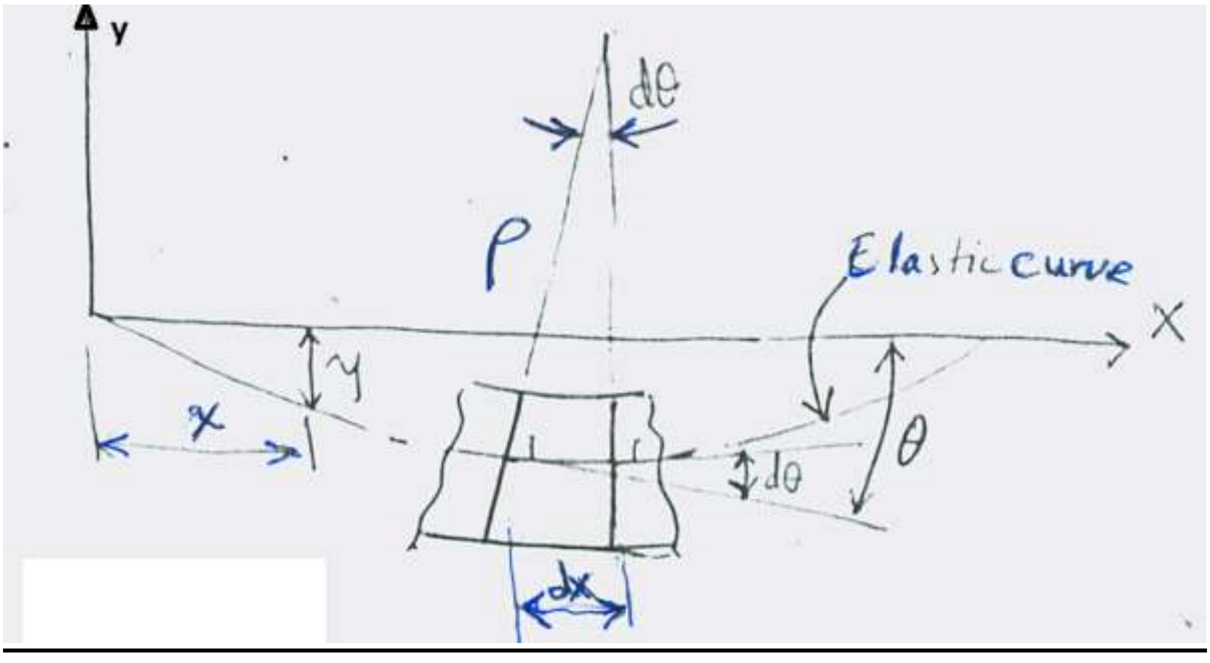
BEAM DEFLECTION

Introduction

When we consider designing beams based on rigidity consideration the deflection of the beam must be known at specific or critical location.

Several methods are avaible for determining beam deflection. Although based on the same principles, they differ in technique and in their immediate objective.

<u>Double- Integration Method</u>



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Slope =tan $\theta = \frac{dy}{dx}$

Since θ is small, then tan $\theta \approx \theta$

$\theta = \frac{dy}{dx}$ and $\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$ **(1)**

The defferential length *ds* can be expressed in terms of radius of curvature ρ as:-

$$ds = \rho \, d\theta$$

Or $\frac{1}{\rho} = \frac{d\theta}{ds} \approx \frac{d\theta}{dx}$ **(2)**

Subtitube eq.(1) and eq.(2)

$\frac{1}{\rho} = \frac{d^2y}{dx^2}$ **(3)**

In deriving the flexure formula (Bending Stress Lecture), we obtained the relation:

$\frac{1}{\rho} = \frac{M}{EI}$ **(4)**

Equating eq.(3) and (4), we have

$$EI\frac{d^2y}{dx^2} = M \qquad \textbf{(5)}$$

This is known as the differential equation of the elastic curve of the beam.

Integrating equation (5), assuming *EI* constant, we obtain

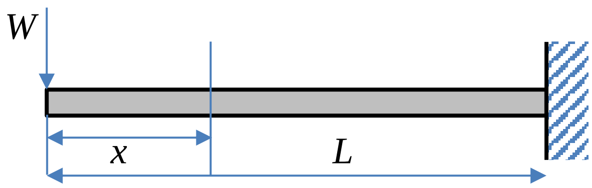
$$EI\frac{dy}{dx} = \int M dx + c_1 \qquad (slope\ equation)$$

And

$$EIy = \int(\int M dx)dx + c_1x + c_2 \qquad (deflection\ equation)$$

Note: The double integration method can be used if the moment (*M*) has a single expression for the whole beam.

Example 1:



$$M_{xx} = EI\frac{d^2y}{dx^2} = -Wx$$

Assuming *EI* constant, integrating the above equation,

$$EI\frac{dy}{dx} = -W\frac{x^2}{2} + A \qquad (1)$$

And,

$$EIy = -W\frac{x^3}{6} + Ax + B \qquad (2)$$

The above constants (*A* and *B*) can be found using the available boundary conditions.

Boundary Conditions (*BCs*)

1) when $x = L$, $\frac{dy}{dx} = 0$, using eq. (1) leads to $A = \frac{WL^2}{2}$
2) when $x = L$, $y = 0$, using eq. (2) leads to $B = -\frac{WL^3}{3}$

Therefore, the equation of the elastic curve is,

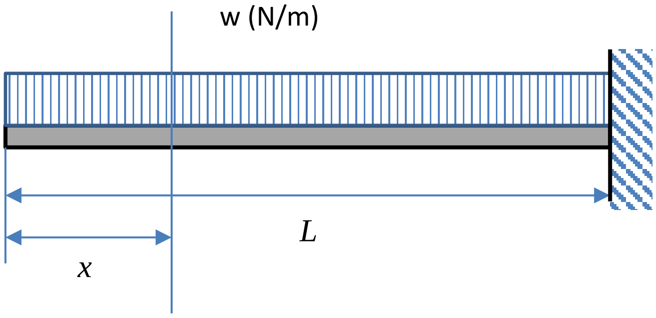
$$y = \frac{W}{EI}\left[-\frac{x^3}{6} + \frac{L^2x}{2} - \frac{L^3}{3}\right]$$

The maximum deflection occur at the free end, i.e. at *x*=0, then

At $x = 0$, $y = y_{max}$

$$y_{max} = -\frac{WL^3}{3EI}$$

Example 2:



$$EI\frac{d^2y}{dx^2} = M = -\frac{wx^2}{2}$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + A \qquad (1)$$

And,

$$EIy = -\frac{wx^4}{24} + Ax + B \qquad (2)$$

Boundary Conditions (*BCs*)

1) when $x = L$, $\frac{dy}{dx} = 0$, using eq. (1) leads to $A = \frac{wL^3}{6}$

2) when $x = L$, $y = 0$, using eq. (2) leads to $B = \frac{wL^4}{24} - \frac{wL^4}{6} = -\frac{wL^4}{8}$

Therefore, the equation of the elastic curve is,

$$y = \frac{w}{EI}\left[-\frac{x^4}{24} + \frac{L^3}{6}x - \frac{L^4}{8}\right]$$

The maximum deflection occur at the free end, i.e. at *x*=0, then

$$At\ x = 0\ , \qquad y_{max} = \frac{wL^4}{8EI}$$

The following example is from:

F.P. Beer, and E.R. Johnston, *Mechanics Of Materials*, Sixth Edtion, McGraw-Hill, 2012.

EXAMPLE 9.02

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 9.12). Determine the equation of the elastic curve and the maximum deflection of the beam.

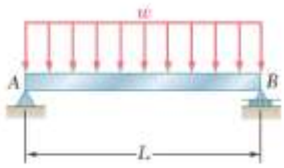


Fig. 9.12

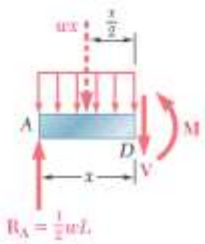


Fig. 9.13

Drawing the free-body diagram of the portion AD of the beam (Fig. 9.13) and taking moments about D, we find that

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2 \qquad (9.12)$$

Substituting for M into Eq. (9.4) and multiplying both members of this equation by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx \qquad (9.13)$$

Integrating twice in x, we have

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1 \qquad (9.14)$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2 \qquad (9.15)$$

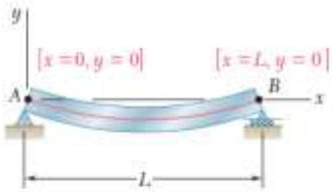


Fig. 9.14

Observing that $y = 0$ at both ends of the beam (Fig. 9.14), we first let $x = 0$ and $y = 0$ in Eq. (9.15) and obtain $C_2 = 0$. We then make $x = L$ and $y = 0$ in the same equation and write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_1L$$
$$C_1 = -\tfrac{1}{24}wL^3$$

Carrying the values of C_1 and C_2 back into Eq. (9.15), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wLx^3 - \tfrac{1}{24}wL^3x$$

or

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \qquad (9.16)$$

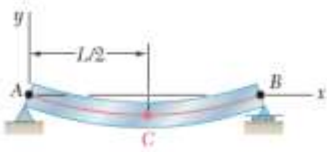


Fig. 9.15

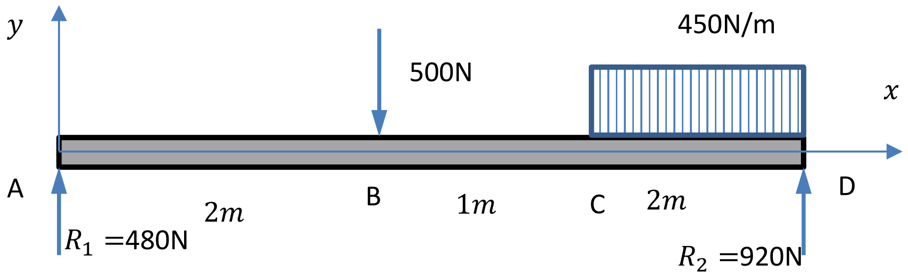
Substituting into Eq. (9.14) the value obtained for C_1, we check that the slope of the beam is zero for $x = L/2$ and that the elastic curve has a minimum at the midpoint C of the beam (Fig. 9.15). Letting $x = L/2$ in Eq. (9.16), we have

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{max} = \frac{5wL^4}{384EI}$$

Discussion (Macaulay's Method)



$$M_{AB} = 480x$$

$$M_{BC} = 480x - 500(x-2)$$

$$M_{CD} = 480x - 500(x-2) - \frac{450}{2}(x-3)^2$$

Macaulay's Method ?

$$M(x) = 480x - 500\langle x-2\rangle - \frac{450}{2}\langle x-3\rangle^2$$

Note: the above equation is for the beam only when the term inside the two brackets $\langle x-a\rangle$ has the *positive value*, i.e, $x \geq a$, and if $x < a$ (negative value) the value of the term is zero.

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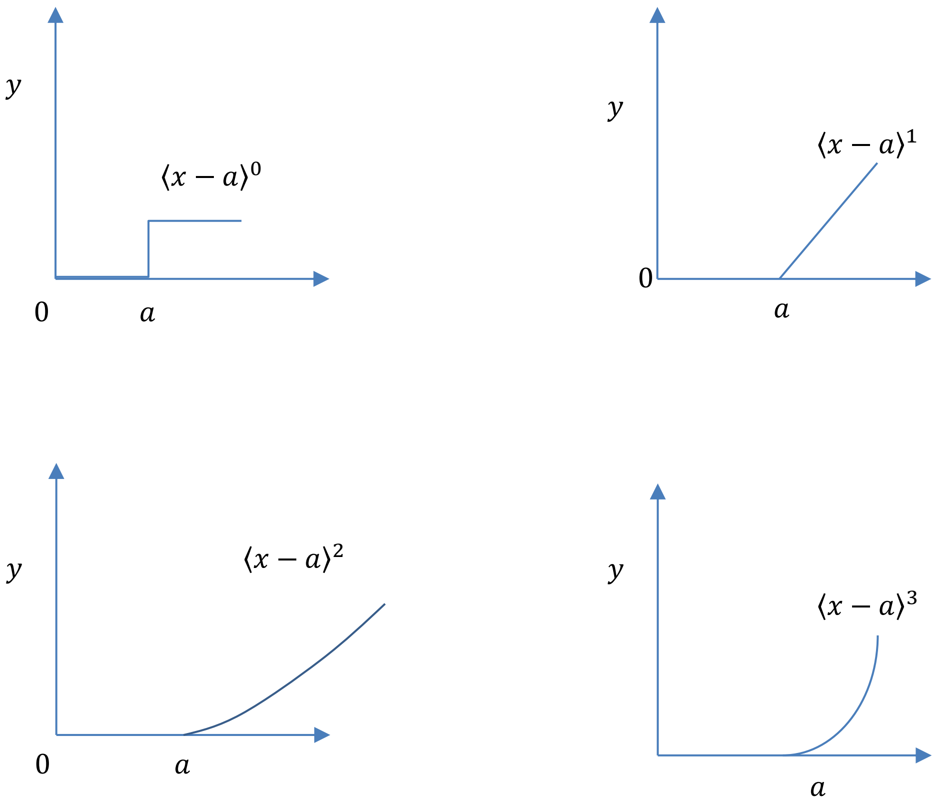
Macaulay's Method

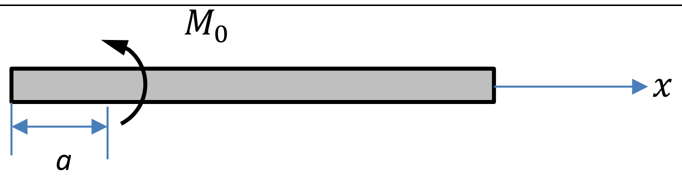
The simple integration method used in the previous examples can only be used when a single expression for bending moment applies along the complete length of the beam. In general this is not the case, and the method has to be adapted to cover all loading conditions. Following the procedure adapted by Macaulay, the step fuctions which have the following properties, must be used



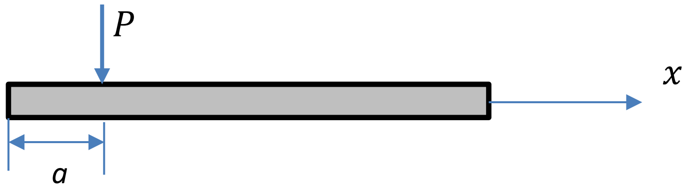
Generally $$\langle x-a\rangle^n = \begin{cases} (x-a)^n & when \quad x \geq a \\ 0 & when \quad x < a \end{cases}$$

Integral of $\langle x-a\rangle^n$ is $\int \langle x-a\rangle^n dx = \frac{1}{(n+1)} \langle x-a\rangle^{n+1}$

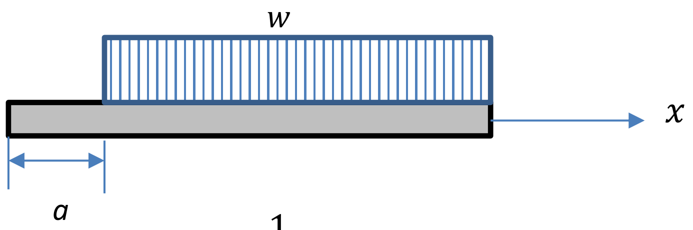
Different types of loading in Macaulay's notations



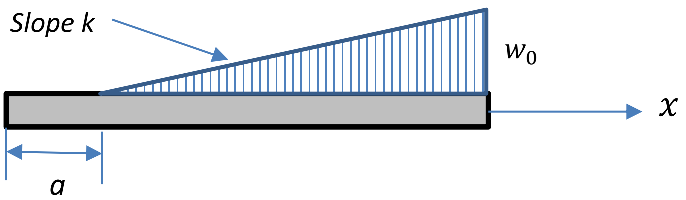
$$M(x) = M_0\langle x-a\rangle^0$$



$$M(x) = -P\langle x-a\rangle^1$$



$$M(x) = -\frac{1}{2}w\langle x-a\rangle^2$$



$$M(x) = -\frac{k}{6}\langle x-a\rangle^3$$

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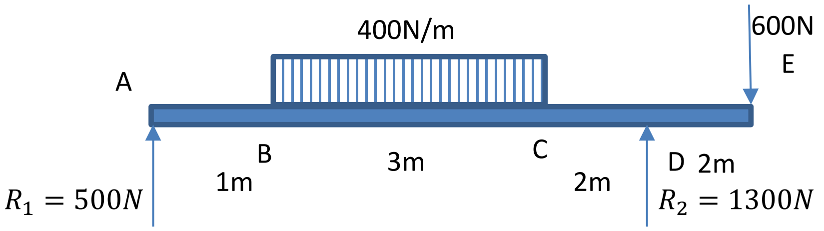
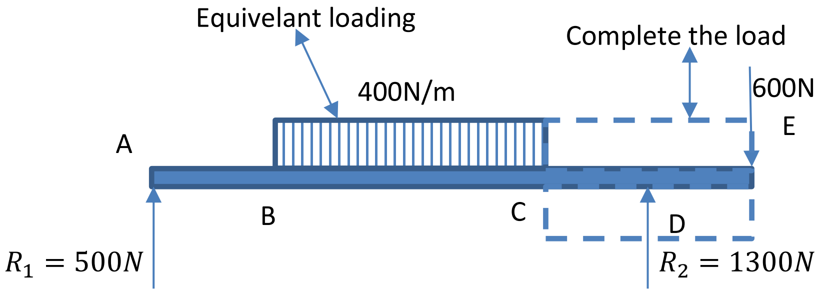
Example 3:

write the complete moment equation for the beam shown



solution:



$$M = 500x - \frac{400}{2}\langle x-1\rangle^2 + \frac{400}{2}\langle x-4\rangle^2 + 1300\langle x-6\rangle$$

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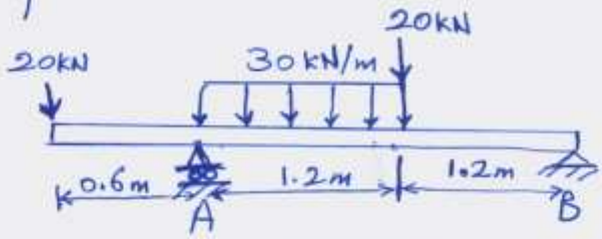
Example 4:

Determine the deflection at a point 1m from the left-hand end of the beam loaded as shown in the figure using Macaulay's method.

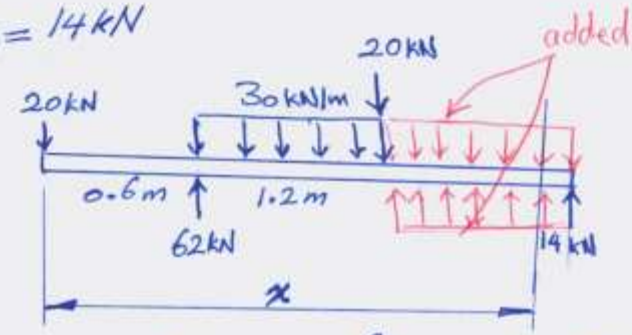
$EI = 0.65\ MNm^2$



Sol.

From Statics

$R_A = 62\ kN$ & $R_B = 14\ kN$



$$M(x) = \left[-20x + 62\langle x-0.6\rangle - \frac{30}{2}\langle x-0.6\rangle^2 - 20\langle x-1.8\rangle + \frac{30}{2}\langle x-1.8\rangle^2\right] \times 10^3$$

$$EI\frac{d^2y}{dx^2} = M(x)$$

$$\therefore \frac{EI}{10^3}\frac{d^2y}{dx^2} = M(x) = -20x + 62\langle x-0.6\rangle - \frac{30}{2}\langle x-0.6\rangle^2 - 20\langle x-1.8\rangle + \frac{30}{2}\langle x-1.8\rangle^2$$

integrating

$$\frac{EI}{10^3}\frac{dy}{dx} = -\frac{20}{2}x^2 + \frac{62}{2}\langle x-0.6\rangle^2 - \frac{30}{6}\langle x-0.6\rangle^3 - \frac{20}{2}\langle x-1.8\rangle^2 + \frac{30}{6}\langle x-1.8\rangle^3 + C_1$$

and

$$\frac{EI}{10^3}y = \frac{-20}{6}x^3 + \frac{62}{6}\langle x-0.6\rangle^3 - \frac{30}{24}\langle x-0.6\rangle^4 - \frac{20}{6}\langle x-1.8\rangle^3 + \frac{30}{24}\langle x-1.8\rangle^4 + C_1x + C_2$$

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BCs

1) at $x=0.6$, $y=0$

$$\therefore 0 = -\frac{20}{6} * 0.6^3 + 0 - 0 - 0 + 0 + 0.6C_1 + C_2$$

$$\therefore C_2 + 0.6C_1 = 0.72 \quad \text{——— (1)}$$

2) at $x=3$, $y=0$ from which

$$0 = -\frac{20}{6}3^3 + \frac{62}{6} * 2.4^3 - \frac{30}{24} * 2.4^4 - \frac{20}{6} * 1.2^3 + \frac{30}{24} * 1.2^4 + 3C_1 + C_2$$

$$\therefore 3C_1 + C_2 = -8.208 \quad \text{——— (2)}$$

from eq. (1) & (2)

$C_1 = -3.72$ & $C_2 = 2.952$

∴ The equation of elastic curve,

$$\frac{EI}{10^3} y = -\frac{20}{6}x^3 + \frac{62}{6}\langle x-0.6\rangle^3 - \frac{30}{24}\langle x-0.6\rangle^4 + \frac{30}{24}\langle x-1.8\rangle^4 - \frac{20}{6}\langle x-1.8\rangle^3 - 3.72x + 2.952$$

Deflection at $x = 1m$

$$y = \frac{10^3}{EI}\left[-\frac{20}{6} + \frac{62}{6} * 0.4^3 - \frac{30}{24} * 0.4^4 + 0 - 0 - 3.72 + 2.952\right]$$

Since $EI = 0.65 \times 10^6 \; Nm^2$

$$\therefore y_{x=1} = -0.00534\,m = -5.34\,mm \quad \text{Ans.}$$

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The following example is from:

F.P. Beer, and E.R. Johnston, *Mechanics of Materials*, Sixth Edtion, McGraw-Hill, 2012.

EXAMPLE 9.06

For the beam and loading shown (Fig. 9.29*a*) and using singularity functions, (*a*) express the slope and deflection as functions of the distance x from the support at A, (*b*) determine the deflection at the midpoint D. Use $E = 200$ GPa and $I = 6.87 \times 10^{-6}$ m^4.

(*a*) We note that the beam is loaded and supported in the same manner as the beam of Example 5.05. Referring to that example, we recall that the given distributed loading was replaced by the two equivalent open-ended loadings shown in Fig. 9.29*b* and that the following expressions were obtained for the shear and bending moment:

$$V(x) = -1.5\langle x-0.6\rangle^1 + 1.5\langle x-1.8\rangle^1 + 2.6 - 1.2\langle x-0.6\rangle^0$$

$$M(x) = -0.75\langle x-0.6\rangle^2 + 0.75\langle x-1.8\rangle^2 + 2.6x - 1.2\langle x-0.6\rangle^1 - 1.44\langle x-2.6\rangle^0$$

Integrating the last expression twice, we obtain

$$EI\theta = -0.25\langle x-0.6\rangle^3 + 0.25\langle x-1.8\rangle^3 + 1.3x^2 - 0.6\langle x-0.6\rangle^2 - 1.44\langle x-2.6\rangle^1 + C_1 \qquad (9.48)$$

$$EIy = -0.0625\langle x-0.6\rangle^4 + 0.0625\langle x-1.8\rangle^4 + 0.4333x^3 - 0.2\langle x-0.6\rangle^3 - 0.72\langle x-2.6\rangle^2 + C_1x + C_2 \qquad (9.49)$$

The constants C_1 and C_2 can be determined from the boundary conditions shown in Fig. 9.30. Letting $x = 0$, $y = 0$ in Eq. (9.49) and noting that all the brackets contain negative quantities and, therefore, are equal to zero, we conclude that $C_2 = 0$. Letting now $x = 3.6$, $y = 0$, and $C_2 = 0$ in Eq. (9.49), we write

$$0 = -0.0625\langle 3.0\rangle^4 + 0.0625\langle 1.8\rangle^4 + 0.4333(3.6)^3 - 0.2\langle 3.0\rangle^3 - 0.72\langle 1.0\rangle^2 + C_1(3.6) + 0$$

Since all the quantities between brackets are positive, the brackets can be replaced by ordinary parentheses. Solving for C_1, we find $C_1 = -2.692$.

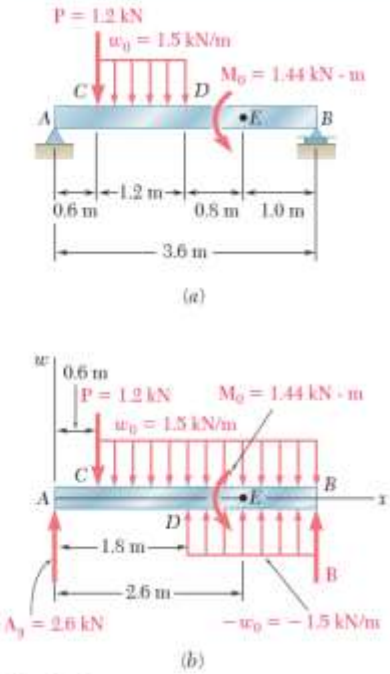


Fig. 9.29

Fig. 9.30

(*b*) Substituting for C_1 and C_2 into Eq. (9.49) and making $x = x_D = 1.8$ m, we find that the deflection at point D is defined by the relation

$$EIy_D = -0.0625\langle 1.2\rangle^4 + 0.0625\langle 0\rangle^4 + 0.4333(1.8)^3 - 0.2\langle 1.2\rangle^3 - 0.72\langle -0.8\rangle^2 - 2.692(1.8)$$

The last bracket contains a negative quantity and, therefore, is equal to zero. All the other brackets contain positive quantities and can be replaced by ordinary parentheses. We have

$$EIy_D = -0.0625(1.2)^4 + 0.0625(0)^4 + 0.4333(1.8)^3 - 0.2(1.2)^3 - 0 - 2.692(1.8) = -2.794$$

Recalling the given numerical values of E and I, we write

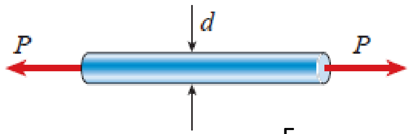
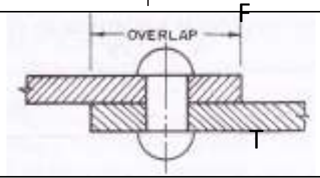
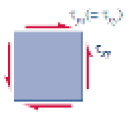
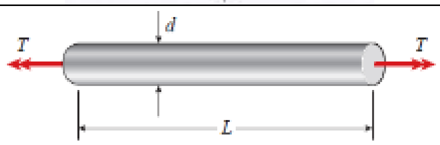
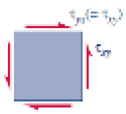
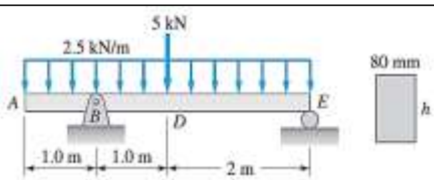
$$(200 \text{ GPa})(6.87 \times 10^{-6} \text{ m}^4)y_D = -2.794 \text{ kN} \cdot \text{m}^3$$

$$y_D = -13.64 \times 10^{-3} \text{ m} = -2.03 \text{ mm}$$

STRESSES DUE TO COMBINED LOADS

Introduction

In preceding lectures, we studied stress of various structural members carrying fundamental loads: bars with axial loading, different components subjected to direct shear stress, torsion of circular shafts, thin-walled pressure vessels subjected to biaxial loading, and bending of beams. In this lecture, we will be considering combined axial and lateral loading, and stress at a point. And the following table reviews these types of loading:

Loading Types	Stress	Example	Stress State
Axial Loading	$\sigma_a = \dfrac{P}{A}$		
Shearing Loading	$\tau = \dfrac{V}{A}$		
Torsional Loading	$\tau = \dfrac{Tr}{J}$		
Flexural Loading	$\sigma = \dfrac{My}{I}$		

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Lecturer Rasha Mohammed

Combined Axial and Flexural Loads

For the beam shown in the figure the beam subjected to axial load P combined with lateral load Q. The axial load cause a stress:

$$\sigma_a = \frac{P}{A} \qquad (1)$$

And the lateral load Q cause a bending stress of:

$$\sigma_b = \frac{My}{I} \qquad (2)$$

It is shown that the top surface subjected to *compression* while bottom surface subjected to *tension*. Therefore, the combine stress will be:

$$\sigma = \frac{P}{A} \mp \frac{My}{I} \qquad (3)$$

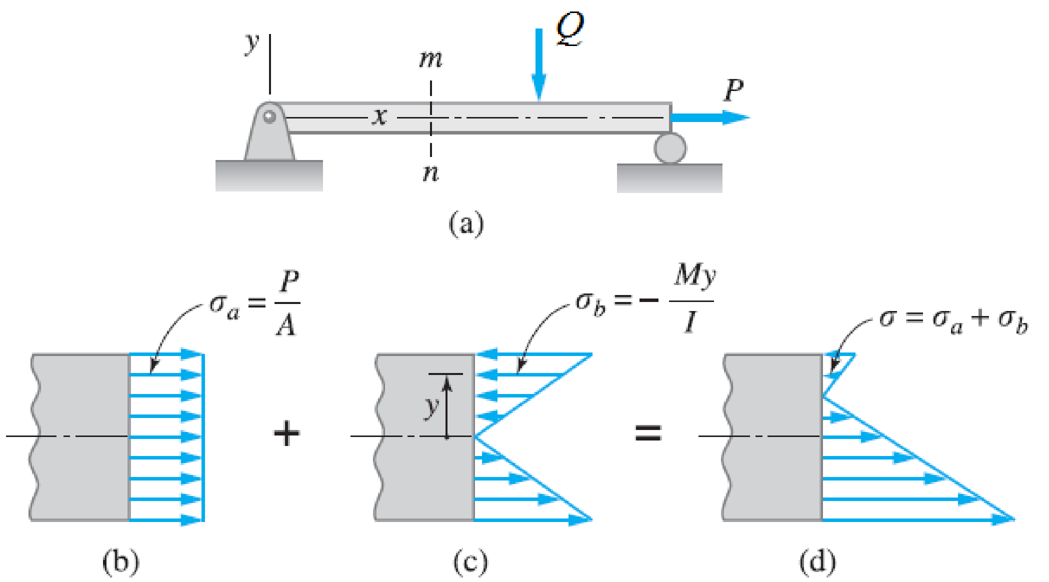
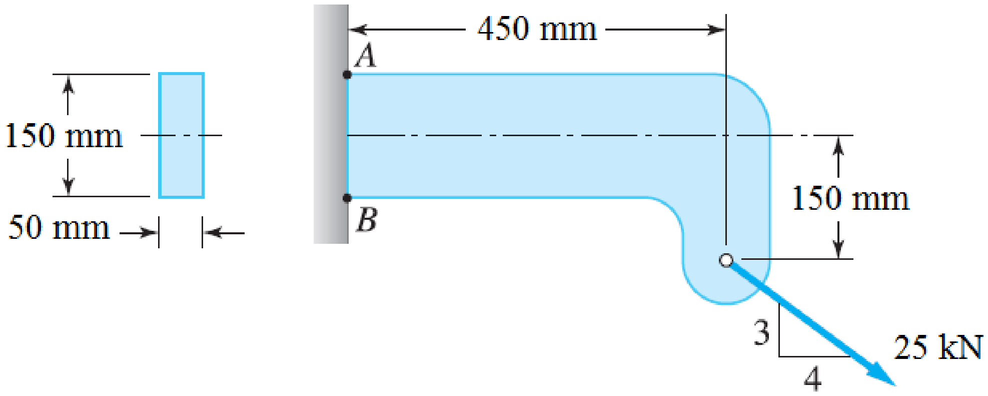


Figure 1: (a) Rectangular bar carrying axial and lateral loads; (b)–(d) stress distribution obtained by superimposing stresses due to axial load and bending.

Example 1:

Determine the resultant normal stress at A and B near the wall



Solution:

Horizontal component of the force = $25 \times 10^3 \times \frac{4}{5} = 20\ kN$

Vertical component of the force = $25 \times 10^3 \times \frac{3}{5} = 15\ kN$

Bending moment at the wall

$$M = 15 \times 10^3 \times 0.45 - (20 \times 10^3) \times 0.15 = 3750\ Nm$$

From which the beam is concave downward, thereby causing tension at *A* and compression at *B*.

Stress at A

$$\sigma = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{M\frac{h}{2}}{\left(\frac{bh^3}{12}\right)} = \frac{P}{A} + \frac{6\,M}{bh^2}$$

$$= \frac{20\times10^3}{0.05\times0.15} + \frac{6\times3750}{(0.05)\times(0.15)^2} = 22.67\ \text{MPa}$$

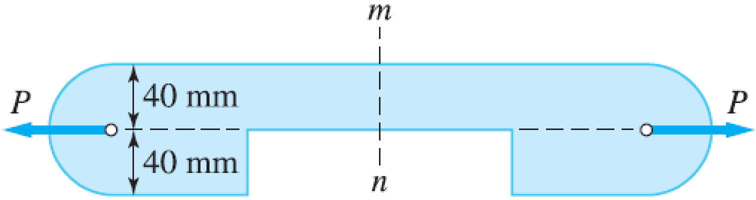
Stress at B

$$\sigma = \frac{P}{A} - \frac{6M}{bh^2} = \frac{20 \times 10^3}{0.05 \times 0.15} - \frac{6 \times 3750}{0.05 \times (0.15)^2}$$

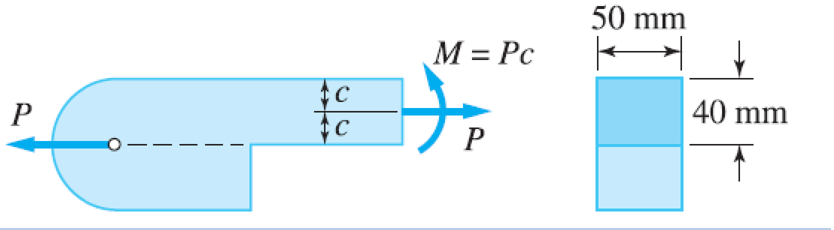
$$= -17.33\ \text{MPa}$$

Example 2:

For the link shown in the figure, the thickness of the link is 50 mm. Given that the force P = 40 kN, determine the maximum and minimum values of the normal stress acting on section m-n.



Solution:



Bending moment

$$M = P\,c = 0.02\ \times 40 \times 10^3\ = 800\ \text{Nm}$$

From which the beam is concave downward, thereby causing tension at A and compression at B.

Maximum normal stress

$$\sigma = \frac{P}{A} + \frac{Mc}{I} = \frac{40 \times 10^3}{(50 \times 10^{-3}\ \times\ 40 \times 10^{-3})} + \frac{800 \times 20 \times 10^{-3}}{\left(\frac{(50 \times 10^{-3})(40 \times 10^{-3})^3}{12}\right)}$$

$$= 80.0\ \text{MPa}$$

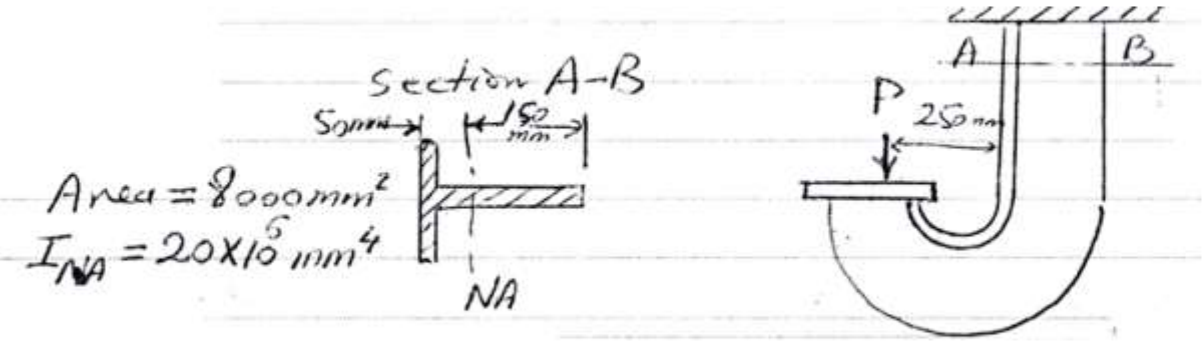
Minimum normal stress

$$\sigma = \frac{P}{A} - \frac{Mc}{I} = \frac{40 \times 10^3}{(50 \times 10^{-3}\ \times\ 40 \times 10^{-3})} - \frac{800 \times 20 \times 10^{-3}}{\left(\frac{(50 \times 10^{-3})(40 \times 10^{-3})^3}{12}\right)}$$

$$= -40.0\ \text{MPa}$$

Example 3:

Determine the largest load P that can be supported by the platform of the cast-iron bracket shown in the figure if $\sigma_t \leq 30$ MPa and $\sigma_c \leq 70$ MPa.



Solution:

Bending moment at section A-B

$$M = P \times (0.25 + 0.05) = 0.3P$$

Stress at A

$$\sigma = \frac{P}{A} + \frac{My}{I} = \frac{P}{8000 \times 10^{-6}} + \frac{0.3P \times 0.05}{20 \times 10^{-6}}$$

$\therefore P = 34.285$ kN

Stress at B

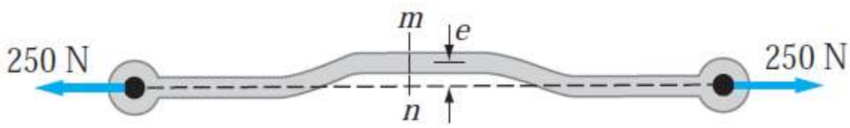
$$\sigma = \frac{P}{A} - \frac{My}{I} = \frac{P}{8000 \times 10^{-6}} - \frac{0.3P \times 0.15}{20 \times 10^{-6}}$$

$\therefore P = 32.94$ kN

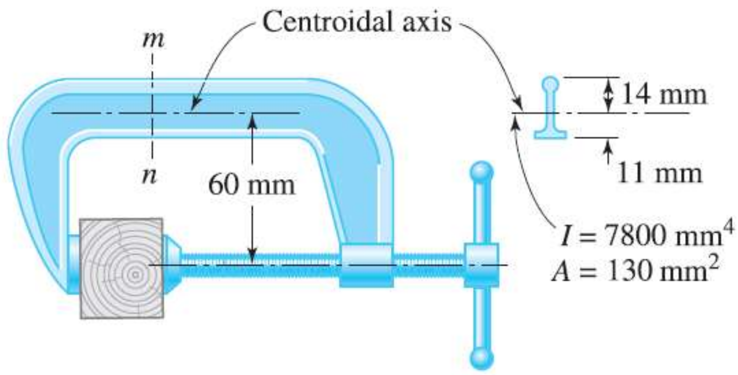
Therefore, the largest safe load is 32.94 kN

Home work

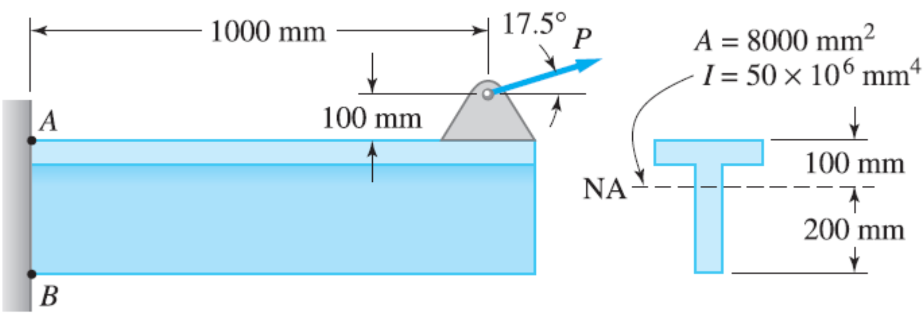
1- The cross section of the machine part is a square, 5 mm on a side. If the maximum stress at section *m-n* is limited to 150 MPa, determine the largest allowable value of the eccentricity *e*.



2- Find the largest clamping force that can be applied by the cast iron C-clamp if the allowable normal stresses on section *m-n* are 15 MPa in tension and 30 MPa in compression.



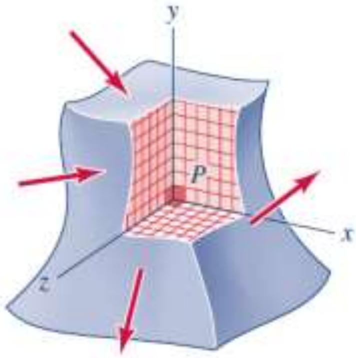
3- The force P = 100 kN is applied to the bracket as shown in the figure. Compute the normal stresses developed at points *A* and *B*.

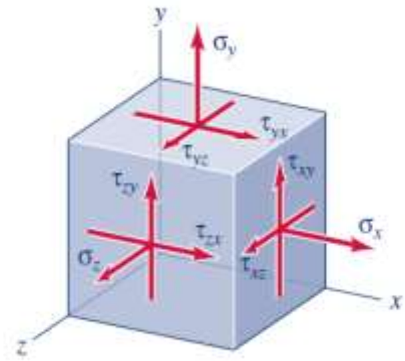


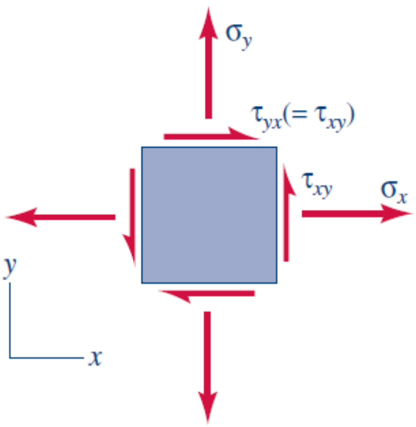
End of Home work

Stress at a Point

The average stress over an area is obtained by dividing the force by the area over which it acts. If the average stress is constant over the area, the stress is said to be uniform. If the stress is not uniform, the stress at any point is found by permitting the area enclosing the point to approach zero as a limit. In other words, stress at a point really defines the uniform stress distributed over a differential area.







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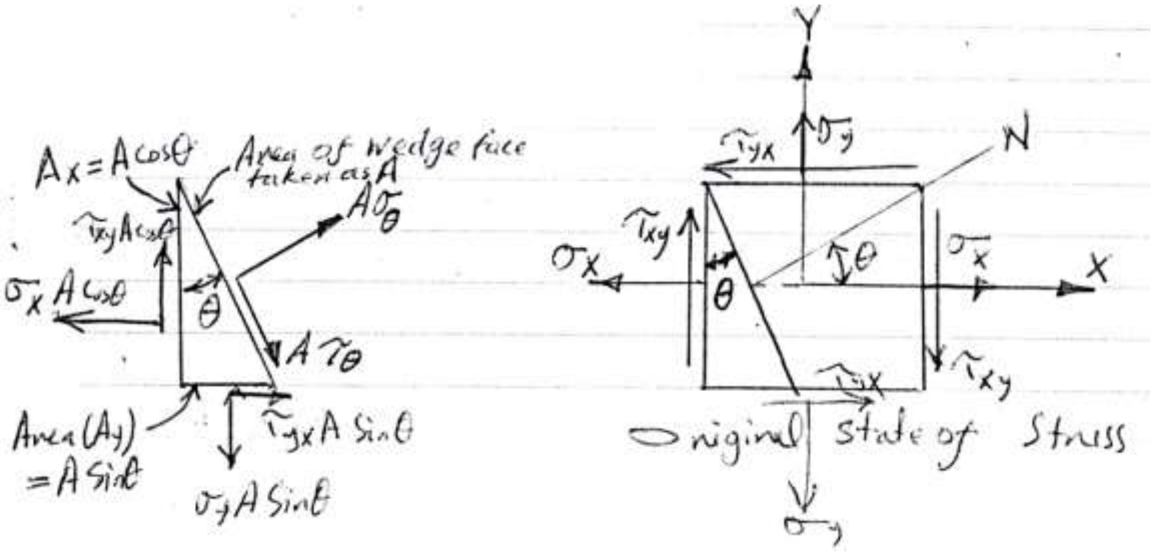
Stress at Inclined Plane



Applying the conditions of equilibrium in normal *N* and tangential *T* direction

$$\sum F_N = 0 \ : \quad A\sigma_\theta = (\sigma_x A\cos\theta)\cos\theta + \left(\sigma_y\, A\sin\theta\right)\sin\theta - \left(\tau_{xy}\, A\cos\theta\right)\sin\theta - \left(\tau_{yx}\, A\sin\theta\right)\cos\theta \qquad (1)$$

And

$$\sum F_T = 0 \ : \quad A\tau_\theta = (\sigma_x\, A\cos\theta)\sin\theta - \left(\sigma_y\, A\sin\theta\right)\cos\theta + \left(\tau_{xy}\, A\cos\theta\right)\cos\theta - \left(\tau_{xy}\, A\sin\theta\right)\sin\theta \qquad (2)$$

Since the common term *A* can be canceled, and since $\tau_{yx} = \tau_{xy}$, and if we use the relations

$$\cos^2\theta = \frac{1+cos2\theta}{2}, \quad \sin^2\theta = \frac{1-cos2\theta}{2}, \qquad \text{and} \quad \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

Equation 1 and 2 will be reduced as:

$$\sigma_\theta = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (3)$$

And

$$\tau_\theta = \frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (4)$$

The plane of maximum normal stress, σ_θ, can be found by differentiating Equation 3 with respect to θ and equating the derivative to zero, whence

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (5)$$

Similarly, the plane of maximum shearing stress are defined by

$$\tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} \qquad (6)$$

The plane of zero shearing stress may be determined by setting τ_θ to zero, this gives

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (7)$$

Note: Equation 7 is identical with Equation 5. Hence the maximum and minimum stress occur on plane of zero shearing stress.

Substitute the value of 2θ from Equation 5 and 6 into Equation 3 and 4 we get

$$\sigma_{\substack{max \\ min}} = \frac{\sigma_x + \sigma_y}{2} \mp \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \mp\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Note: σ_{max} and σ_{min} are called principal stresses and located in a plane of zero shear

Mohr's Circle

Mohr interprets Equation 3 and Equation 4 graphically, he used circle for this interpretation, accordingly, the construction is called Mohr's circle.

Mohr showed that Equations 3 and 4 define a circle by first rewriting them as follows

$$\sigma_\theta - \left(\frac{\sigma_x + \sigma_y}{2}\right) = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (8)$$

And

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (9)$$

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By squaring both these equations, i.e. Equations 8 and 9 adding the results, and simplifying, we obtain

$$\left(\sigma_\theta - \left(\frac{\sigma_x+\sigma_y}{2}\right)\right)^2 + \tau_\theta^2 = \left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2 \qquad (10)$$

Since σ_x, σ_y and τ_{xy} are known constants defining the specified state of stress, whereas σ_θ and τ_θ are variable. Equation 10 may be rewritten as

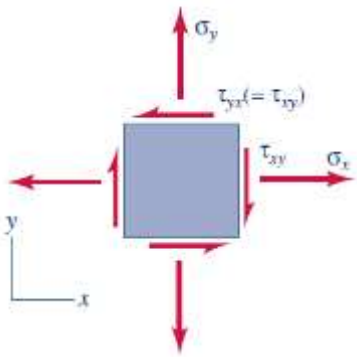
$$(\sigma_\theta - C)^2 + \tau_\theta^2 = R^2 \qquad (11)$$

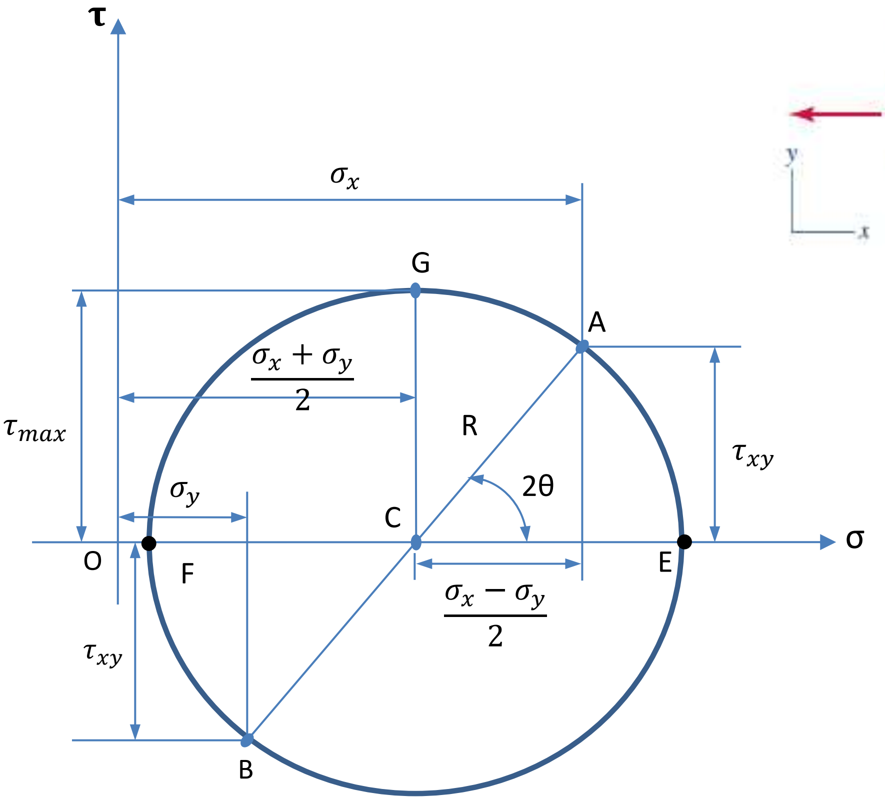
where $C = \frac{\sigma_x+\sigma_y}{2}$, and $R^2 = \left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2$ or $R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Equation 11 is similar to the equation:

$$(x-c)^2 + y^2 = r^2$$

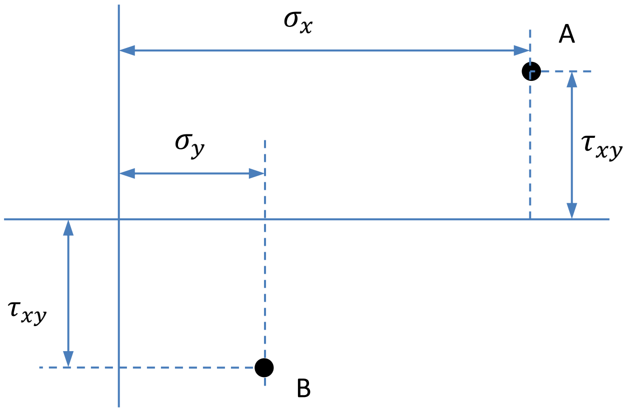
Which is an equation of circle, therefore Equation11 represents equation of circle shifted C distance from $\sigma_\theta -$ axis.



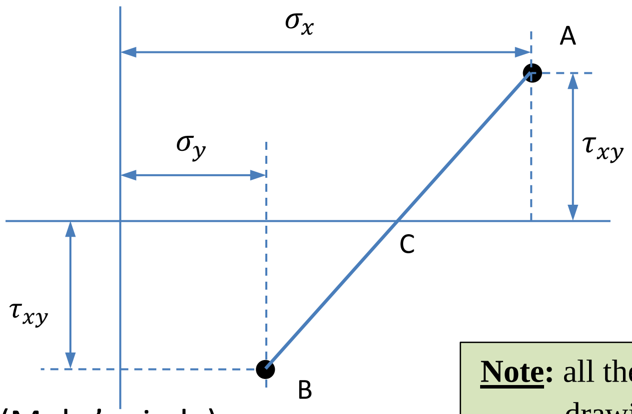


Rules for applying Mohr's circle:

1- Plot points that having coordinates (σ_x, τ_{xy}) and (σ_y, τ_{xy}) as shown in the figure

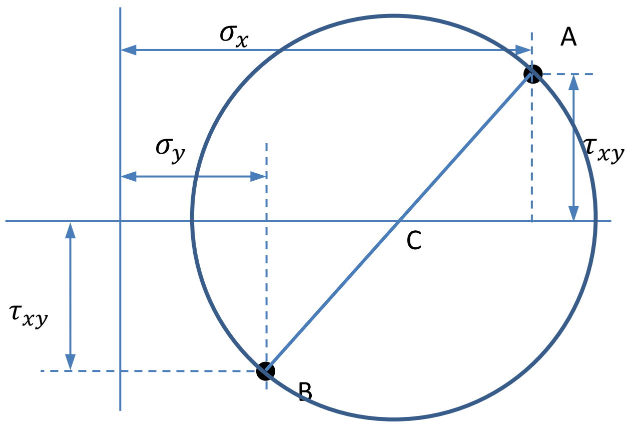


2- Join the points just plotted by straight line the line is the diameter of a circle whose center is on σ axis.



Note: all the dimension and drawing are to scale.

3- Draw the circle (Mohr's circle)



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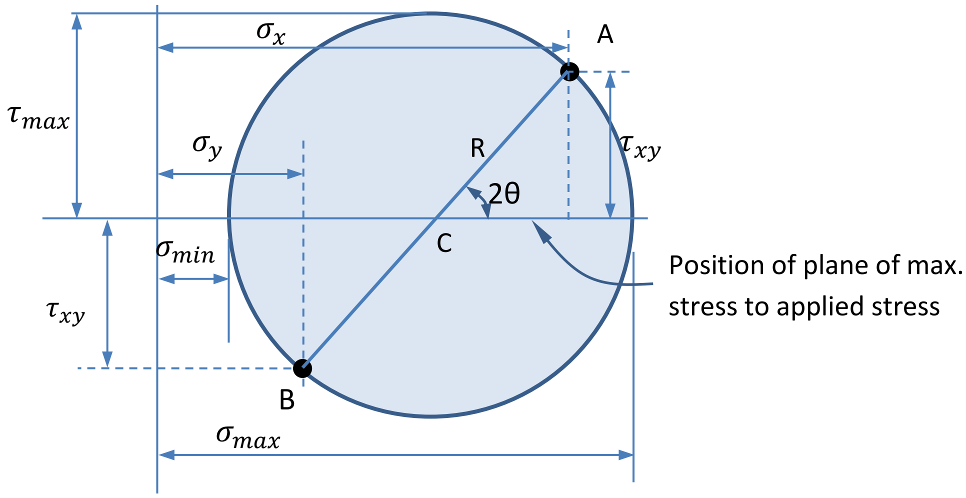
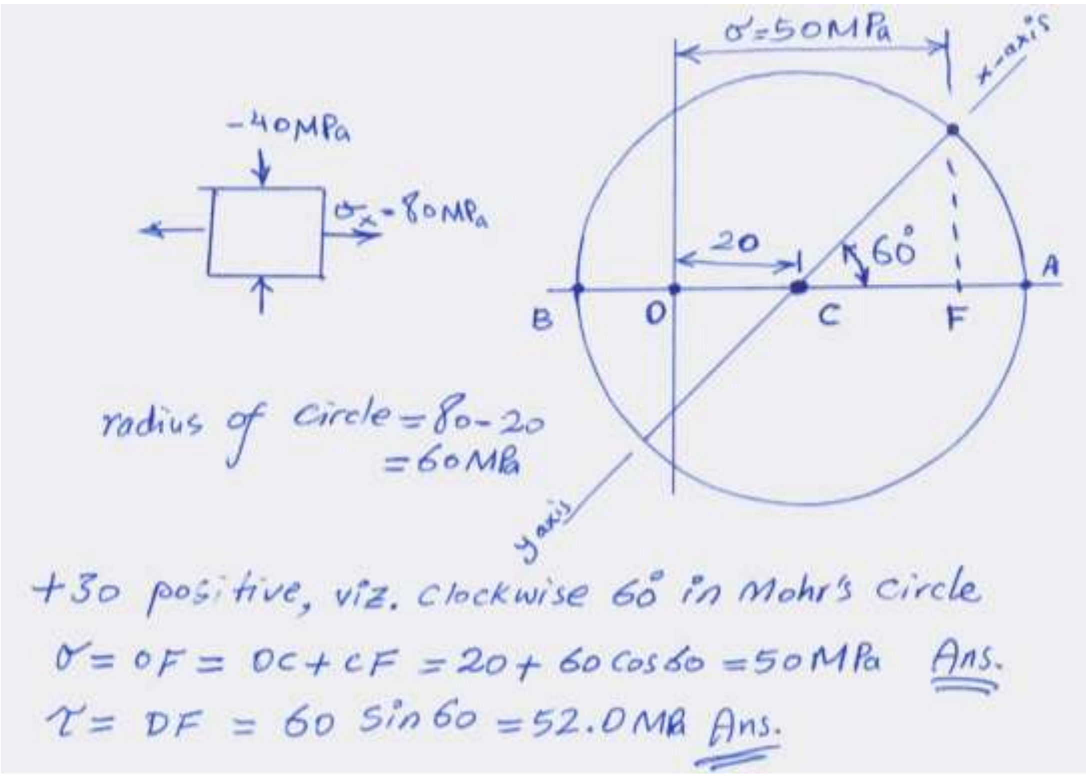
4- The angle between the radii to selected points on Mohr's circle is twice the angle between the normal to the actual planes represented by these points.



Example:

At a certain point a stressed body the principle stresses are $\sigma_{max} = 80$ MPa and $\sigma_{min} = -40$ MPa. Determine σ and **τ** on the planes whose normal are at +30° and 120° with the x-axis.

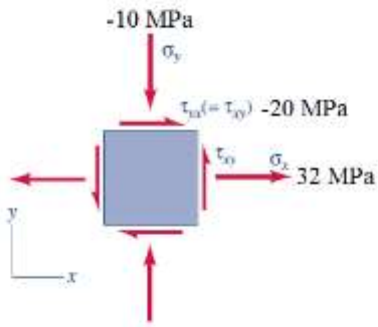
Solution



Example:

For the state of stress shown, determine the normal and shearing stresses on:

1- The principal plane
2- The plane of maximum shearing stress
3- The planes whose normal are at 36.8° and 126.8° with the x-axis.



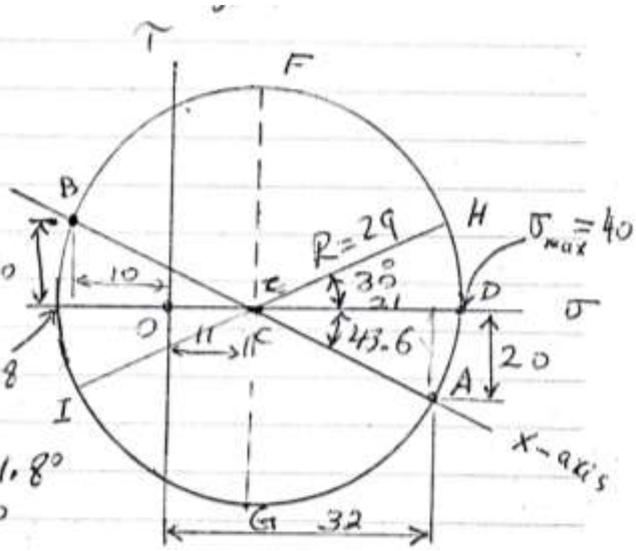
Solution

$R = 40 - 11 = 29$

a) Principal stresses

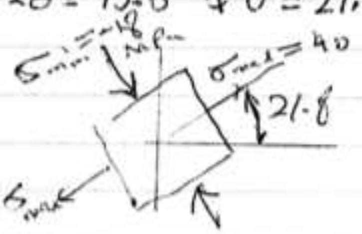
$\sigma_{max} = OD = 11 + 29 = 40\ MPa$

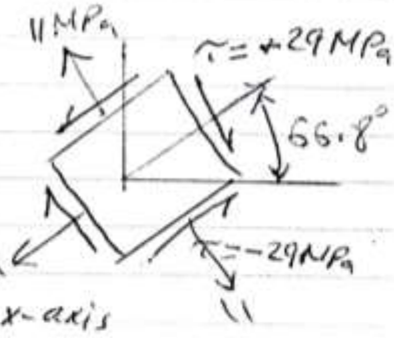
$\sigma_{min} = 11 - 29 = -18\ MPa$



$\tan 2\theta = \frac{20}{21} = 0.952$

$\therefore 2\theta = 43.6^\circ \ \& \ \theta = 21.8^\circ$





b) Max. Shear Stress

$\tau_{max/min} = \pm R = \pm 29\ MPa$

θ of max. shear $= \frac{90 + 43.6}{2} = 66.8^\circ$ with the x-axis

c) Normal is at 36.8°

$\measuredangle ACH = 2 * 36.8 = 73.6$

Continue →

$\therefore \measuredangle HCD = 73.6 - 43.6 = 30^\circ$

Coordinates of point ~~of~~ H are

$\sigma = 11 + 29 \cos 30^\circ = 36.1$ MPa

$\tau = 29 * \sin 30^\circ = 14.5$ MPa

Normal at 126.8°

$\sigma' = 11 - 29 * \cos 30^\circ = -14.1$ MPa

$\tau' = -29 * \sin 30^\circ = -14.5$ MPa

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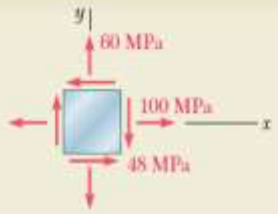
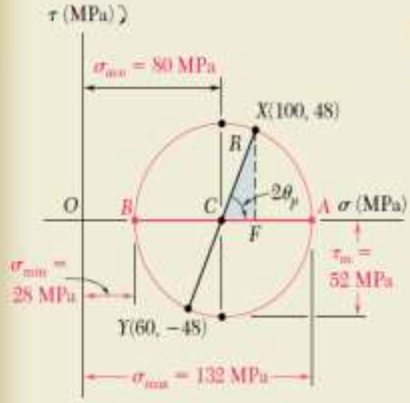
SAMPLE PROBLEM 7.2



For the state of plane stress shown, determine (*a*) the principal planes and the principal stresses, (*b*) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30°.

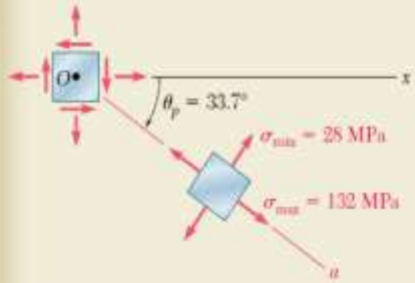
SOLUTION



Construction of Mohr's Circle. We note that on a face perpendicular to the x axis, the normal stress is tensile and the shearing stress tends to rotate the element clockwise; thus, we plot X at a point 100 units to the right of the vertical axis and 48 units above the horizontal axis. In a similar fashion, we examine the stress components on the upper face and plot point $Y(60, -48)$. Joining points X and Y by a straight line, we define the center C of Mohr's circle. The abscissa of C, which represents σ_{ave}, and the radius R of the circle can be measured directly or calculated as follows:

$$\sigma_{ave} = OC = \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(100 + 60) = 80 \text{ MPa}$$

$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52 \text{ MPa}$$



a. Principal Planes and Principal Stresses. We rotate the diameter XY clockwise through $2\theta_p$ until it coincides with the diameter AB. We have

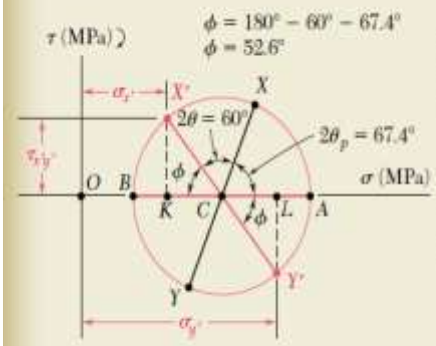
$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4 \quad 2\theta_p = 67.4° \downarrow \quad \theta_p = 33.7° \downarrow \blacktriangleleft$$

The principal stresses are represented by the abscissas of points A and B:

$$\sigma_{max} = OA = OC + CA = 80 + 52 \qquad \sigma_{max} = +132 \text{ MPa} \blacktriangleleft$$

$$\sigma_{min} = OB = OC - BC = 80 - 52 \qquad \sigma_{min} = +\ 28 \text{ MPa} \blacktriangleleft$$

Since the rotation that brings XY into AB is clockwise, the rotation that brings Ox into the axis Oa corresponding to σ_{max} is also clockwise; we obtain the orientation shown for the principal planes.



b. Stress Components on Element Rotated 30°↺. Points X' and Y' on Mohr's circle that correspond to the stress components on the rotated element are obtained by rotating XY counterclockwise through $2\theta = 60°$. We find

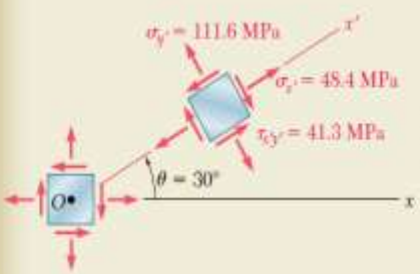
$$\phi = 180° - 60° - 67.4° \qquad \phi = 52.6° \blacktriangleleft$$

$$\sigma_{x'} = OK = OC - KC = 80 - 52 \cos 52.6° \qquad \sigma_{x'} = +\ 48.4 \text{ MPa} \blacktriangleleft$$

$$\sigma_{y'} = OL = OC + CL = 80 + 52 \cos 52.6° \qquad \sigma_{y'} = +111.6 \text{ MPa} \blacktriangleleft$$

$$\tau_{x'y'} = KX' = 52 \sin 52.6° \qquad \tau_{x'y'} = \ 41.3 \text{ MPa} \blacktriangleleft$$

Since X' is located above the horizontal axis, the shearing stress on the face perpendicular to Ox' tends to rotate the element clockwise.



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