INTRODUCTION, STRESSES AND STRAINS

Introduction

The study of the strength of materials is the study of the behavior of solid bodies under load. The way in which they react to applied forces, the deflections resulting and the stresses and strains set up within the bodies, are all considered in an attempt to provide sufficient knowledge to enable any component to be designed such that it will not fail within its service life. Typical components considered in this course include beams, shafts, cylinders and struts.

Analysis of Internal Forces

Consider a body of any shape acted upon by the forces shown in Figure 1-a, the forces $\vec{F}_1, \vec{F}_2, \vec{F}_3,$ and $\vec{F}_4$ are the external forces acted on the body. This body is considered in static equilibrium (i.e. remains at rest). To study the internal forces, a section *a-a* through the body will cut the body into two pieces, Figure 1-b shows one piece of section *a-a* of the body balanced by components of internal forces. If the X axis is normal to the section, Y and Z axes are chosen parallel to the section.

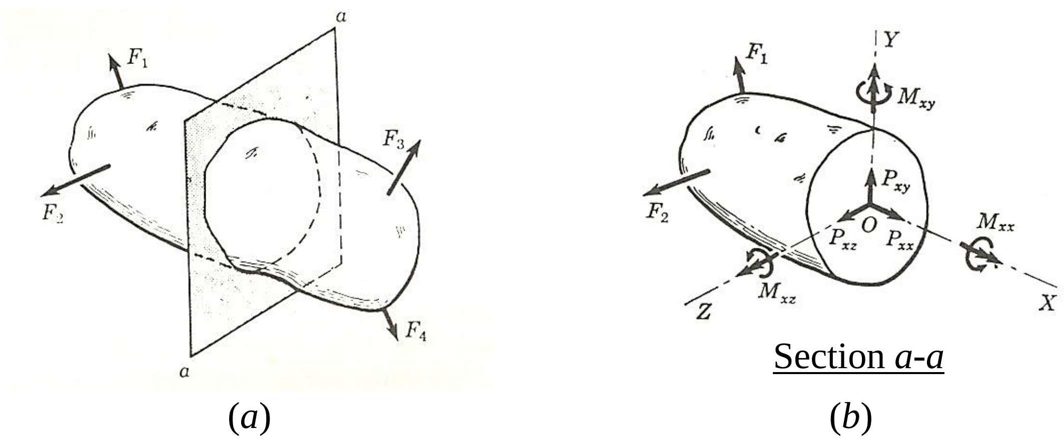


Figure 1: (a) Body of any shape subjected to external forces; (b) Balance of forces through section *a-a*.

Each component reflects a different effect of the applied loads on the member and is given a special name, as follows:

P_{xx}	*Axial forces*, if the forces try to pull the body, it is called tensile forces and called compressive if it tends to shorten the body.
P_{xy}, P_{xz}	*Shear forces*, usually designated by *V*, which acts parallel to the plane of section.
M_{xx}	*Torque T*, this component measure the resistance to twisting the member.
M_{xy}, M_{xz}	Bending moments, these components measure the resistance to bending the member about *Y* or *Z* axis.

Simple Stress

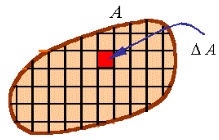
The unit strength of a material is usually defined as the stress in material. Stress is expressed as:

$$\sigma = \frac{Force}{Area} = \frac{F}{A}$$

where σ (Greek lower case letter sigma) is the intensity of forces per unit area, or stress (N/m^2), *F* is the applied load (*N*), and *A* is the cross sectional area (m^2).

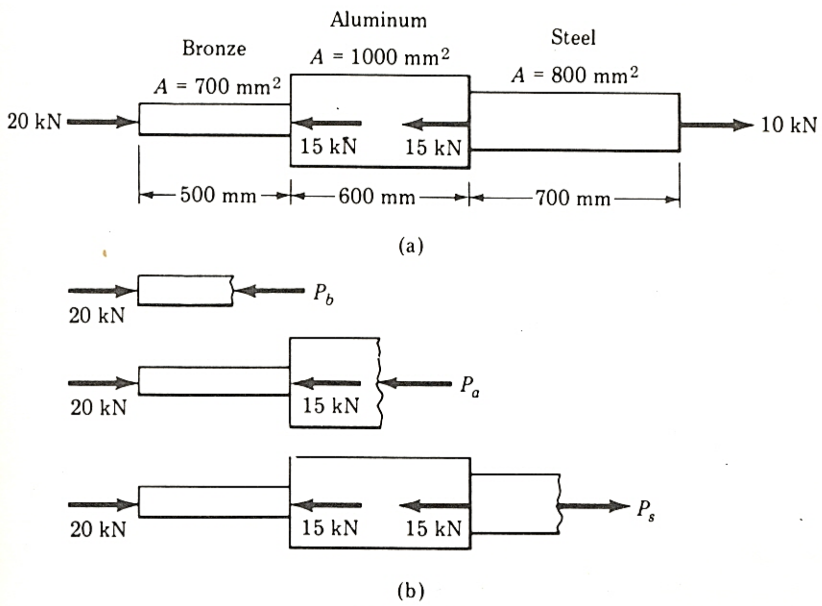
The above equation of stress represents an average stress, and means that the stress is known as a ***simple stress***. A more exact definition of stress is obtained by dividing the differential load, *dF*, by differential area, *dA*, over which it acts:

$$\sigma = \frac{dF}{dA}$$



Example 1:

An aluminum tube is rigidly fastened between a bronze rod and a steel rod as shown in the figure. Axial loads are applied at positions indicated. Determine the stress in each material.



Solution:

From the free body diagram

$$P_b = 20\ kN\ , \qquad P_a = 5\ kN\ ,\ P_s = 10\ kN$$

The stresses can be computed as:

$$\sigma = \frac{F}{A}$$

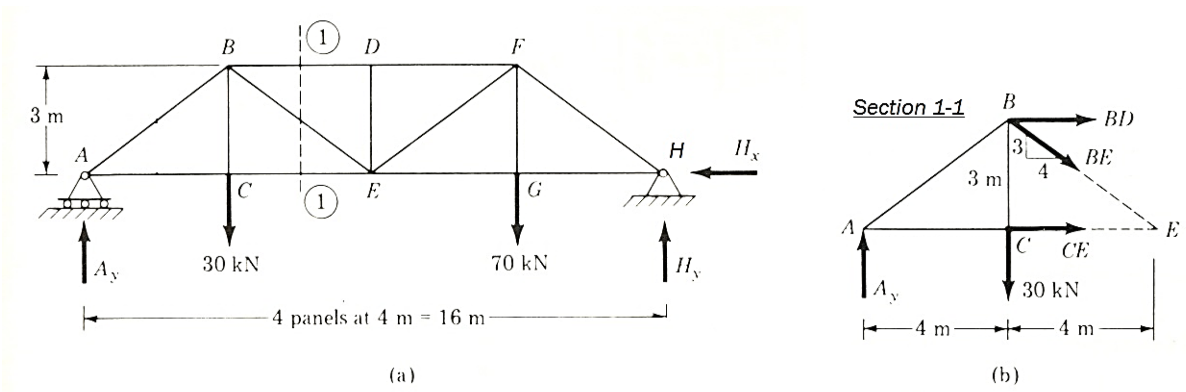
For Bronze: $\sigma_b = \frac{20\times10^3}{700\times10^{-6}} = 28 \times 10^6\ \frac{N}{m^2} = 28\ \text{MPa}$

For Aluminum: $\sigma_a = \frac{5\times10^3}{1000\times10^{-6}} = 5\ \text{MPa}$

For Steel: $\sigma_s = \frac{10\times10^3}{800\times10^{-6}} = 12.5\ \text{MPa}$

Example 2:

For the truss shown in figure, determine the stress in the member *BD*. the cross sectional area of each member is 900 mm^2



Solution

$$\sum M_H = 0: \qquad R_{AY} \times 16 - 30 \times 12 - 70 * 4 = 0$$

$$\therefore\ R_{AY} = 40\ kN$$

From figure (b), section 1-1,

$$\sum M_E = 0: \qquad F_{BD} \times 3 + 40 \times 8 - 30 \times 4 = 0$$

$$\therefore F_{BD} = -66.7\ kN$$

Therefore, the force in member BD is 66.7 kN ←,

$$\sigma_{BD} = \frac{66.7\times10^3}{900\times10^{-6}} = 74.1\ \text{MPa}$$

Home work:

Redo Example 2 by determining the stress in member *EG*.

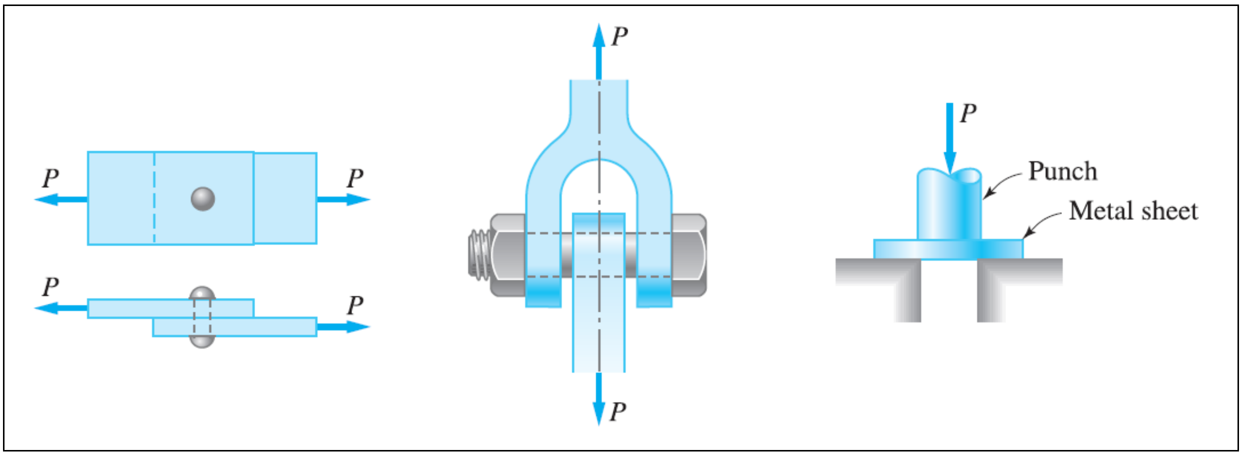
Shearing Stress

Shearing stress is the stress caused by force acting along or parallel to area resisting the forces and can be defined as:

$$\tau = \frac{V}{A}$$

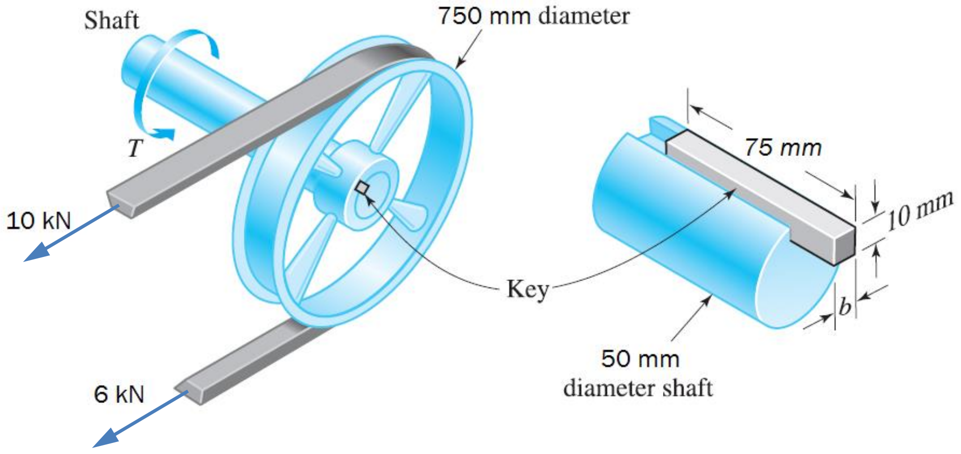
where τ (Greek lowercase letter tau) is the shearing stress (N/m^2), V is the shearing force (N), and A is the area (m^2).

Examples of shearing stresses:



Example 3:

A 750 mm pulley, loaded as shown in the figure, is keyed to a shaft 50mm diameter. Determine the width b of the 75mm-long key if the allowable shearing stress is 70 MPa.



Solution:

The torque on the pulley can be calculated as,

$$T = (10 \times 10^3 - 6 \times 10^3) \times \frac{0.75}{2} = 1.5 \times 10^3 \text{ N.m} = 1.5 \text{ kNm}$$

This torque will be transferred to shaft as,

$$T = 1.5 \times 10^3 = V \times \frac{50 \times 10^{-3}}{2}$$

$$V = \frac{2 \times 1.5 \times 10^3}{50 \times 10^{-3}} = 60 \text{ kN}$$

The shearing stress in the key is,

$$\tau = \frac{V}{A}$$

Therefore,

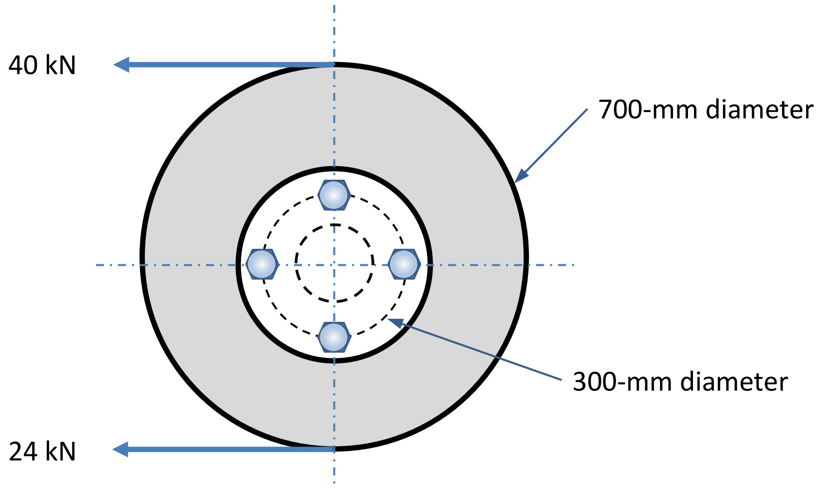
$$70 \times 10^6 = \frac{60 \times 10^3}{75 \times 10^{-3} \times b}$$

$$\therefore b = 0.0114 \text{ m} = 11.4 \text{ mm}$$

Homework

Home work:

For the pulley shown in the figure, determine the diameter of the four similar bolts needed to connect the shaft with the pulley via the coupling shown.



Bearing Stress

If two bodies are pressed against each other, compressive forces are developed on the area of contact. The pressure caused by these surface loads is called *bearing stress*. Examples of bearing stress are the soil pressure beneath a pier and the contact pressure between a rivet and the side of its hole. If the bearing stress is large enough, it can locally crush the material, which in turn can lead to more serious problems.

As an illustration of bearing stress, consider the lap joint formed by the two plates that are riveted together as shown in Figure 2(a). The bearing stress caused by the rivet is not constant; it actually varies from zero at the sides of the hole to a maximum behind the rivet as illustrated in Figure 2(b). The difficulty inherent in such a complicated stress distribution is avoided by the common practice of assuming that the bearing stress σ_b is uniformly distributed over a reduced area. The reduced area A_b is taken to be the *projected area* of the rivet:

$$A_b = t\, d$$

where t is the thickness of the plate and d represents the diameter of the rivet, as shown in the *free body diagram* (FBD) of the upper plate in Figure 2(c). From this FBD we see that the bearing force F_b equals the applied load P, so that the bearing stress becomes

$$\sigma_b = \frac{F_b}{A_b} = \frac{P}{t\, d}$$

where A_b is the projected area of the rivet hole.

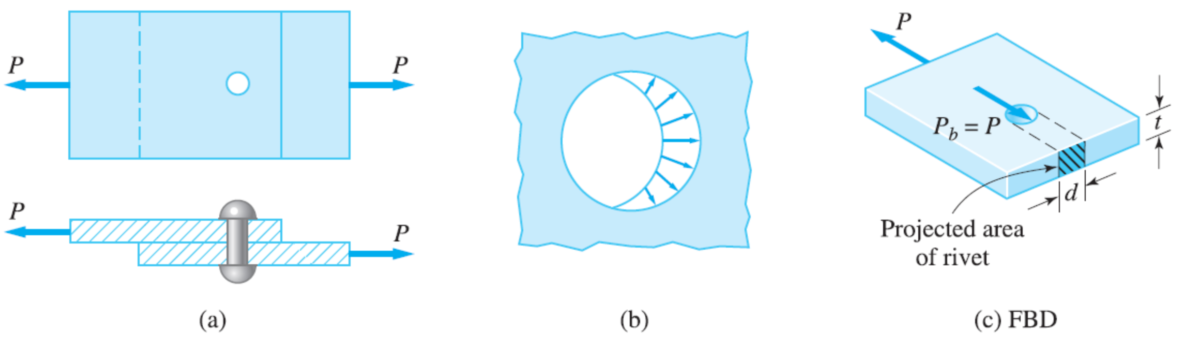
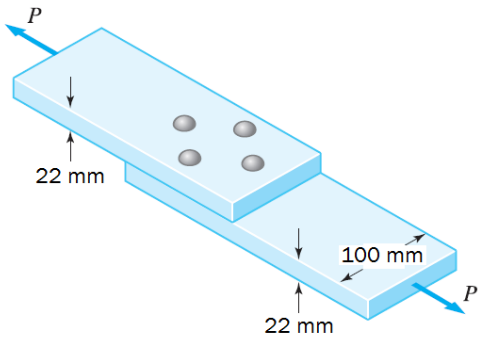


Figure 2: Bearing stress: (a) a rivet in a lap joint; (b) bearing stress is not constant; (c) bearing stress caused by the bearing force P_b is assumed to be uniform on projected area $t\ d$.

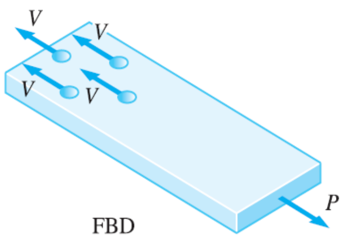
Example 4:

The lap joint shown in the figure is fastened by four rivets of 19 mm diameter. Find the maximum load *P* that can be applied if the working stresses are 96 MPa for shear in the rivet and 124 MPa for bearing in the plate.



Solution:

We will calculate *P* using each of the two design criteria. The largest safe load will be the smaller of the two values. The FBD is shown of the lower plate. This cut exposes the shear forces V that act on the cross sections of the rivets. We see that the equilibrium condition is $V = P/4$.



Design for Shear Stress in Rivets

The value of *P* that would cause the shear stress in the rivets to reach its working value is found as follows:

$$\tau = \frac{V}{A}$$

$$96 \times 10^6 = \frac{P/4}{\frac{\pi}{4}(19\times10^{-3})^2}$$

P=108.8 kN

Design for Bearing Stress in Plate

The shear force *V* =*P*/4 that acts on the cross section of one rivet is equal to the bearing force P_b due to the contact between the rivet and the plate. The value of *P* that would cause the bearing stress is,

$$\sigma_b = \frac{P_b}{A_b}$$

Therefore, $P_b = \sigma_b\ t\ d$

$$\frac{P}{4} = (124 \times 10^6)(22 \times 10^{-3})(19 \times 10^{-3})$$

P = 207.3 kN

Choose the Correct Answer

Comparing the above solutions, we conclude that the maximum safe load *P* that can be applied to the lap joint is

P=108.8 kN   Answer

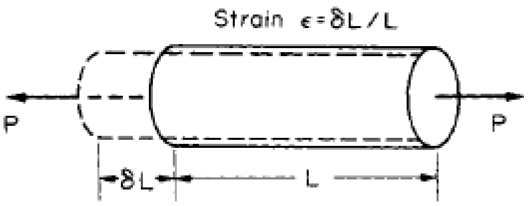
with the shear stress in the rivets being the governing design criterion.

Strain

If a bar is subjected to a direct load, the bar will change in length. If the bar has a length L and changes in length by ΔL, the strain produced is defined as

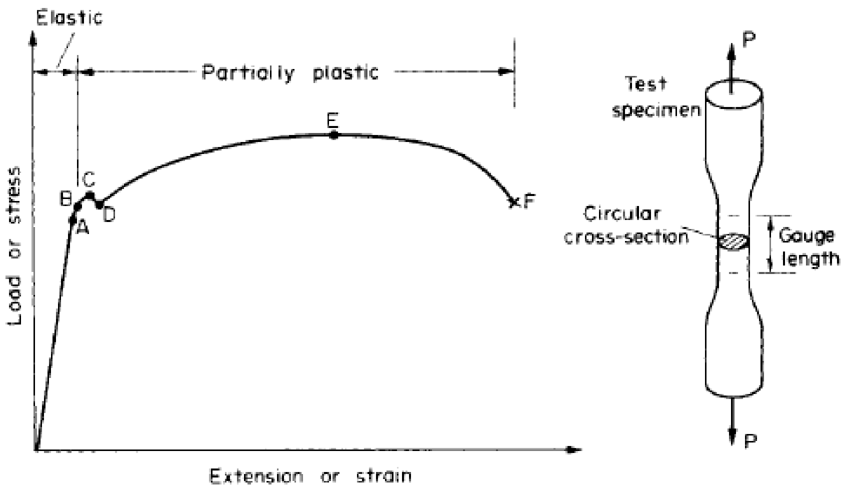
$$strain = \frac{change\ in\ length}{orginal\ length}$$

$$\varepsilon = \frac{\Delta L}{L} \quad or \quad \varepsilon = \frac{\delta}{L}$$



Stress – Strain Diagram

The strength of material is not the only criterion that must be considered in designing structures the stiffness of material is frequently of equal importance.

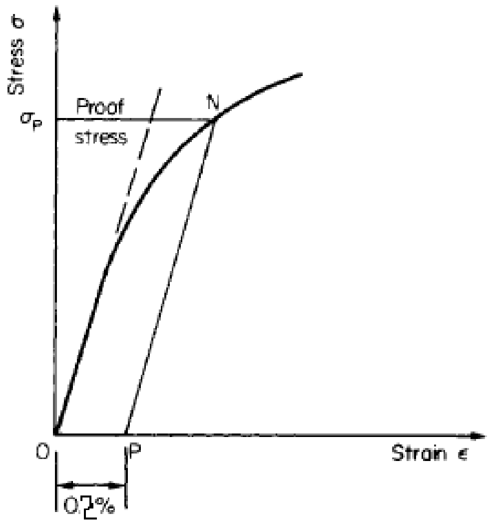


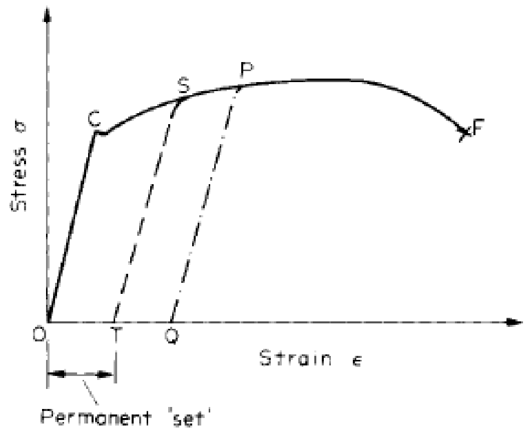
If a specimen of structural steel is gripped between the jaws of testing machine and the unit load or stress was plotted against unit elongation (or strain). The resulted diagram shown is called a stress-strain diagram.

Notes:

1. The material behaves elastically till *elastic limit B*.
2. Point *A* is called *proportionality limit* where stress is proportional to strain.
3. The material beyond elastic limit is *plastically deformed*.
4. *C* is called *upper yield point*, and *D* is the *lower yield point*.
5. *E* is where the *ultimate stress* occur, in this point necking occur.
6. *F* is the *fracture point*.

Discussion





Hooke's Law

A material is said to be elastic if it returns to its original, unloaded dimension, when load is removed. In most engineering materials this elastic behavior is linear, i.e. the stress is directly proportional with strain, Hooke's law states that:

$$\text{Stress } (\sigma) \propto \text{Strain } (\varepsilon)$$

Therefore,

$$\frac{stress}{strain} = constant$$

$$\sigma = E\varepsilon$$

where E is called the *modulus of elasticity* or *Young's modulus*.

Note: In most common engineering applications strain is rarely exceeded 0.001 or 0.1%.

$\sigma = E\varepsilon$, since $\sigma = \frac{F}{A}$ and $\varepsilon = \frac{\delta}{L}$

$$\frac{F}{A} = E\frac{\delta}{L}$$

$$\delta = \frac{FL}{AE}$$

Poisson's Ratio

When a specimen subjected to axial tensile loading a *reduction or lateral contraction induces* to the specimen's cross-sectional area. Similarly, a contraction owing to an axial compressive load is accompanied by a lateral extension. In the linearly elastic region, it is found experimentally that lateral strains, say in the y and z directions, are related by a constant of proportionality, ν, to the axial strain caused by *uniaxial stress only* $\varepsilon_x = \sigma_x/E$, in the x direction:

$$\varepsilon_y = \varepsilon_z = -\nu\frac{\sigma_x}{E} \qquad \textbf{(6)}$$

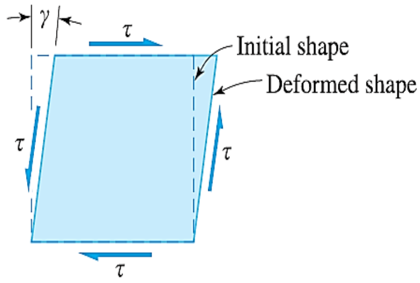
Alternatively, the definition of ν may be stated as

$$\nu = -\frac{lateral\ strain}{axial\ strain} \qquad \textbf{(7)}$$

Here ν is known as *Poisson's ratio*, after S. D. Poisson (1781-1840). The values of *Poisson's ratio* are 0.25 to 0.35 for most metals. Extreme cases range from a low of 0.1 (for some concretes) to a high of 0.5 (for rubber).

Hooke's Law in Shear

Shear stress causes the deformation shown in the figure. The lengths of the sides of the element do not change, but the element undergoes a distortion from a rectangle to a parallelogram. The shear strain, which measures the amount of distortion, is the angle γ (lowercase Greek gamma), always expressed in radians. It can be shown that the relationship between shear stress τ and shear strain γ is linear within the elastic range; that is,

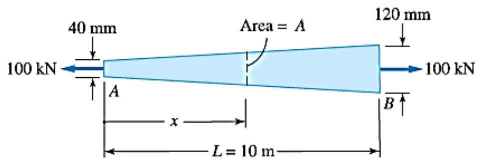


$$\tau = G\,\gamma$$

which is *Hooke's law for shear*. The material constant G is called the *shear modulus of elasticity* (or simply *shear modulus*), or the *modulus of rigidity*. The shear modulus has the same units as the modulus of elasticity (Pa).

Sample Problem **2.2**

The cross section of the 10-m-long flat steel bar AB has a constant thickness of 20 mm, but its width varies as shown in the figure. Calculate the elongation of the bar due to the 100-kN axial load. Use $E = 200$ GPa for steel.



Solution

Equilibrium requires that the internal axial force $P = 100$ kN is constant along the entire length of the bar. However, the cross-sectional area A of the bar varies with the x-coordinate, so that the elongation of the bar must be computed from Eq. (2.7).

We start by determining A as a function of x. The cross-sectional areas at A and B are $A_A = 20 \times 40 = 800$ mm^2 and $A_B = 20 \times 120 = 2400$ mm^2. Between A and B the cross-sectional area is a linear function of x:

$$A = A_A + (A_B - A_A)\frac{x}{L} = 800\ \text{mm}^2 + (1600\ \text{mm}^2)\frac{x}{L}$$

Converting the areas from mm^2 to m^2 and substituting $L = 10$ m, we get

$$A = (800 + 160x) \times 10^{-6}\ \text{m}^2 \qquad \text{(a)}$$

Substituting Eq. (a) together with $P = 100 \times 10^3$ N and $E = 200 \times 10^9$ Pa into Eq. (2.7), we obtain for the elongation of the rod

$$\delta = \int_0^L \frac{P}{EA}\,dx = \int_0^{10\ \text{m}} \frac{100 \times 10^3}{(200 \times 10^9)[(800 + 160x) \times 10^{-6}]}\,dx$$

$$= 0.5\int_0^{10\ \text{m}} \frac{dx}{800 + 160x} = \frac{0.5}{160}[\ln(800 + 160x)]_0^{10}$$

$$= \frac{0.5}{160}\ln\frac{2400}{800} = 3.43 \times 10^{-3}\ \text{m} = 3.43\ \text{mm}$$

Answer

THERMAL DEFORMATIONS AND STRESSES

Introduction

It is well known that changes in temperature cause dimensional changes in a body: An increase in temperature results in expansion, whereas a temperature decrease produces contraction. This deformation is isotropic (the same in every direction) and proportional to the temperature change. The strain caused by temperature change (°C) is denoted by α and is called the coefficient of thermal expansion. Thermal strain caused by a uniform increase in temperature ΔT is

$$\varepsilon_{th} = \alpha \Delta T$$

and

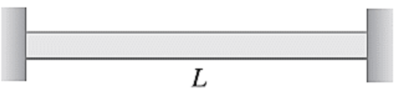
$$\delta_{th} = \alpha (\Delta T) L$$

where α is the coefficient of thermal expansion.

Example 1:

A steel rod of length L and uniform cross sectional area A is secured between two walls, as shown in the figure. Use L=1.5m, E=200 GPa, $\alpha = 11.7 \times 10^{-6}$ /°C and $\Delta T = 80$ °C. Calculate the stress for a temperature increase of ΔT for:

a) The walls are fixed.
b) The walls move apart a distance 0.5mm.



Solution:

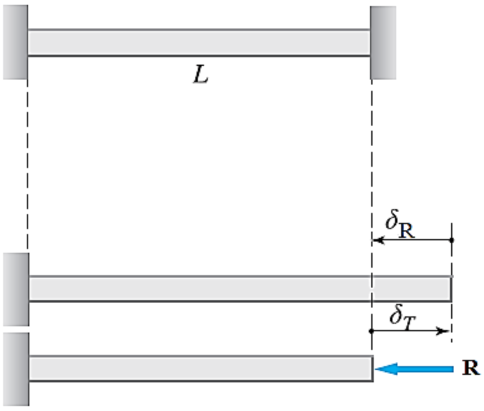
a) $\delta_{th} - \delta_R = 0$

$$\alpha(\Delta T)L - \frac{RL}{AE} = 0$$

$\therefore$ $R = A\,E\,\alpha\,(\Delta T)$

$$\sigma = \frac{R}{A} = E\alpha(\Delta T)$$

$$= 200 \times 10^9 \times 11.7 \times 10^{-6} \times 80 = 187.2 \text{ MPa}$$ (Answer)



b) $\delta_{th} - \delta_R = \delta_w$

$$\alpha(\Delta T)L - \frac{RL}{AE} = \delta_w$$

$R = AE\,(\alpha\,\Delta T - \frac{\delta_w}{L})$

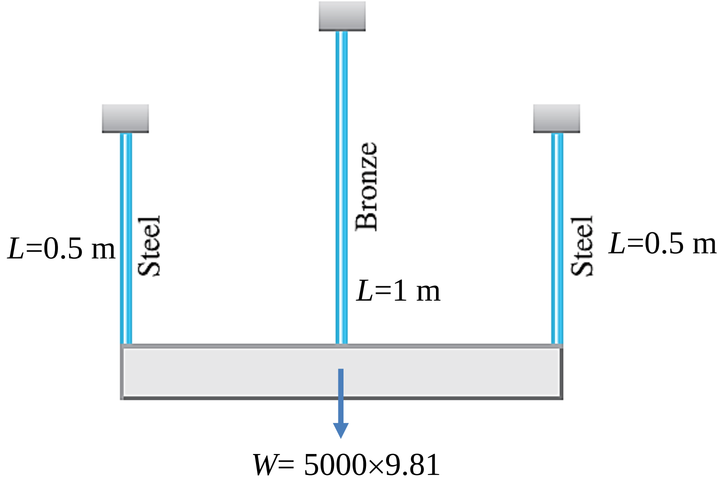
The compressive stress is then,

$$\sigma = \frac{R}{A} = E\left(\alpha\Delta T - \frac{\delta_w}{L}\right)$$

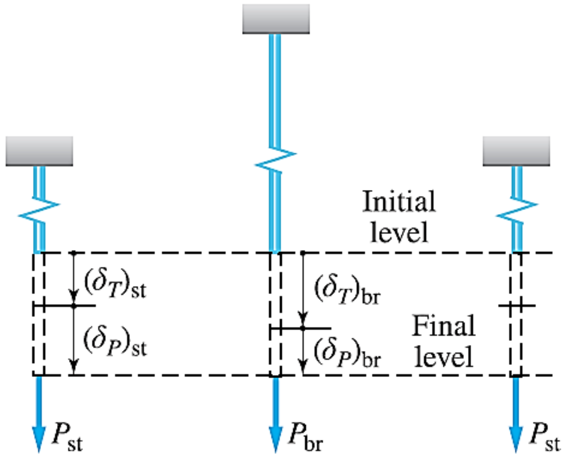
$$= 200 \times 10^9 \left(11.7 \times 10^{-6} \times 80 - \frac{0.5\times10^{-3}}{1.5}\right) = 120.52 \text{ MPa}$$ (Answer)

Example 2:

A rigid block having a mass 5 Mg is supported by three rods symmetrically placed, as shown in the figure. Determine the stress in each rod after a temperature rise of 40 °C. Use E_s=200 GPa, α_s=11.7 µm/m·°C, A_s=500 mm^2, E_b=83 GPa, α_b=18. 9 µm/m·°C, and A_b=900 mm^2.



Solution:



<u>Deformation</u>

$$\delta_{th_s} + \delta_{P_s} = \delta_{th_b} + \delta_{P_b}$$

$$\alpha_s(\Delta T)L_s + \frac{P_{st}L_s}{A_sE_s} = \alpha_b(\Delta T)L_b + \frac{P_{br}L_b}{A_bE_b}$$

$$11.7 \times 10^{-6} \times 40 \times 0.5 + \frac{P_{st}\times 0.5}{500\times10^{-6}\times200\times10^{9}} = 18.9 \times 10^{-6} \times 40 \times 1 + \frac{P_{br}\times 1}{900\times10^{-6}\times83\times10^{9}}$$

Simplifying the above equation,

$$P_{st} - 2.6P_{br} = 104 \times 10^3 \text{ N} \qquad (1)$$

<u>Statics</u> (Free Body Diagram, F.B.D)

$$2P_{st} + P_{br} = 5000 \times 9.81 = 49.05 \times 10^3 \quad \text{N} \qquad (2)$$

Solving equation (1) and (2),

$P_{st} = 37.0$ kN and $P_{br} = -25$ kN (compression)

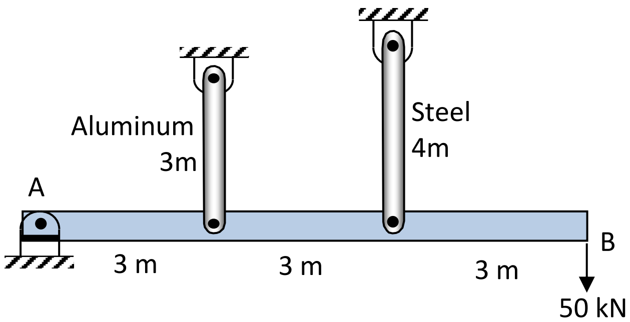
<u>Stresses</u>

$\sigma = \frac{F}{A}$, hence $\quad \sigma_s = \frac{P_{st}}{A_s} = \frac{37\times10^3}{500\times10^{-6}} = 74$ MPa (Answer)

$\sigma_b = \frac{P_{st}}{A_b} = \frac{25*10^3}{900*10^{-6}} = 27.8$ MPa (Answer)

Example 3:

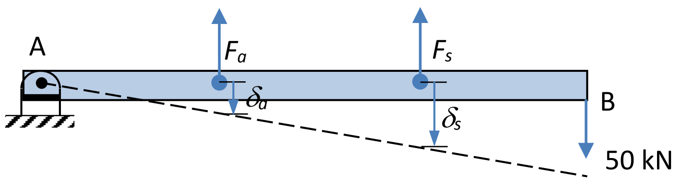
For assembly shown in the figure. Determine the stress in each of the two vertical rods if the temperature rises 40 °C after the load *P*=50 kN is applied. Neglect the deformation and mass of the horizontal bar *AB*. Use E_a=70 GPa, α_a=23.0 µm/m·°C, A_a=900 mm^2, E_s=200 GPa, α_s=11.7 µm/m·°C and A_s=600 mm^2.



Solution:

$$\sum M_A = 0: \quad 50\times10^3\times9 - F_s\times6 - F_a\times3 = 0$$

$$2F_s + F_a = 150\times10^3 \qquad (1)$$



<u>Deformation</u>

$$\frac{\delta_s}{6} = \frac{\delta_a}{3} \rightarrow \delta_s = 2\delta_a$$

$$\alpha_s(\Delta T)L_s + \frac{F_s L_s}{A_s E_s} = 2\left(\alpha_a(\Delta T)L_a + \frac{F_a L_a}{A_a E_a}\right)$$

$$11.7 \times 10^{-6} \times 40 \times 4 + \frac{F_s \times 4}{600\times10^{-6}\times200\times10^{9}} =$$

$$2\left(23 \times 10^{-6} \times 40 \times 3 + \frac{F_a\times3}{900\times10^{-6}\times70\times10^{9}}\right)$$

$$F_s - 2.857F_a = 109.44\ \text{kN} \qquad (2)$$

Solve (1) and (2) for F_s and F_a,

$F_s = 80.4$ kN and $F_a = -10.17$ kN

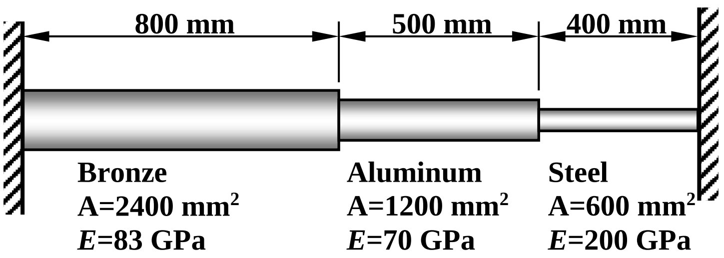
Stresses

$$\sigma_s = \frac{F_s}{A_s} = \frac{80.4\times10^{3}}{600\times10^{-6}} = 11.3\ MPa \text{ (Answer)}$$

$$\sigma_a = \frac{F_a}{A_a} = \frac{10.17\times10^{3}}{900\times10^{-6}} = 134 \text{ MPa (Answer)}$$

Example 4:

A rod is composed of three segments, as shown in the figure. Compute the stress induced in each material by a temperature drop 30 °C if (*a*) the walls are rigid and (*b*) the walls spring together by 0.3mm. Assume E_a=70 GPa, α_a=23.0 µm/m·°C, A_a=1200 mm^2, E_b=83 GPa, α_b=18.9µm/m·°C, A_b=2400 mm^2, E_s=200 GPa, α_s=11.7 µm/m·°C and A_s=600 mm^2.



Solution

a) $\sum(\delta_{th} + \delta_F) = 0$

$$18.9\times10^{-6}\times30\times0.8-\frac{F*0.8}{2400\times10^{-6}\times83\times10^{9}}+23\times10^{-6}\times30\times0.5-\frac{F\times0.5}{1200\times10^{-6}\times70\times10^{9}}+11.7\times10^{-6}\times30\times0.4-\frac{F\times0.4}{600\times10^{-6}\times200\times10^{9}}=0$$

F=70.592 kN

<u>Stresses</u>

$$\sigma_s=\frac{F_s}{A_s}=\frac{70.592\times10^3}{600\times10^{-6}}=117.65\text{ MPa}\qquad\text{(Answer)}$$
$$\sigma_a=\frac{F_a}{A_a}=\frac{70.592\times10^3}{1200\times10^{-6}}=58.82\text{ MPa}\qquad\text{(Answer)}$$
$$\sigma_b=\frac{F_b}{A_b}=\frac{70.592\times10^3}{2400\times10^{-6}}=29.41\text{ MPa}\qquad\text{(Answer)}$$

b) $\sum(\delta_{th}+\delta_F)=0.3\times10^{-3}$

$$18.9\times10^{-6}\times30\times0.8-\frac{F*0.8}{2400\times10^{-6}\times83\times10^{9}}+23\times10^{-6}\times30\times0.5-\frac{F\times0.5}{1200\times10^{-6}\times70\times10^{9}}+11.7\times10^{-6}\times30\times0.4-\frac{F\times0.4}{600\times10^{-6}\times200\times10^{9}}=0.3*10^{-3}$$

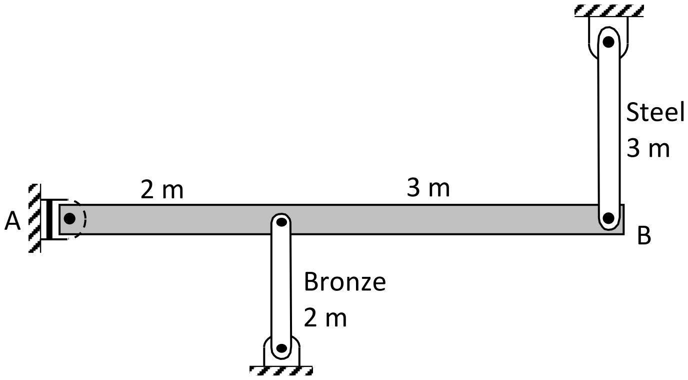
F=49.15KN

<u>Stresses</u>

$$\sigma_s=\frac{F_s}{A_s}=\frac{49.15\times10^3}{600\times10^{-6}}=81.91\text{ MPa}\qquad\text{(Answer)}$$
$$\sigma_a=\frac{F_a}{A_a}=\frac{49.15\times10^3}{1200\times10^{-6}}=40.95\text{ MPa}\qquad\text{(Answer)}$$
$$\sigma_b=\frac{F_b}{A_b}=\frac{49.15\times10^3}{2400\times10^{-6}}=20.47\text{ MPa}\qquad\text{(Answer)}$$

Example 5:

A rigid horizontal bar of negligible mass is connected to two rods as shown in the figure. If the system is initially stress-free; determine the temperature change that will cause a tensile stress of 60 MPa in the steel rod. Assume E_s=200 GPa, α_s=11.7 µm/m·°C and A_s=900 mm^2, E_b=83 GPa, α_b=18.9µm/m·°C, A_b=1200 mm^2.



Solution:

$$\sigma_s = \frac{F_s}{A_s} \rightarrow F_s = A_s\sigma_s$$

<u>Statics</u>

$$\sum M_A = 0: \qquad F_s * 5 = F_b * 2$$

$$\therefore F_b = \frac{5}{2}F_s \qquad (1)$$

Since $\sigma_s = 60$ MPa, then $F_s = A_s\sigma_s = 900 \times 10^{-6} \times 60 \times 10^6 = 54$ kN,

Use equation (1), $F_b = 135$ kN

<u>Deformation</u>

$$\frac{\delta_s}{5} = \frac{\delta_b}{2} \rightarrow \delta_b = \frac{5}{2}\delta_s$$

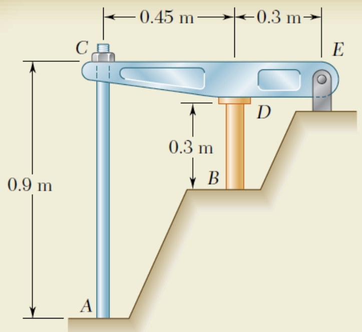
$$\alpha_b(\Delta T)L_b + \frac{F_b L_b}{A_b E_b} = \frac{5}{2}\left(\alpha_s(\Delta T)L_s + \frac{F_s L_s}{A_s E_s}\right)$$

$$18.9\times 10^{-6} \times \Delta T \times 2 + \frac{135\times10^3\times2}{1200\times10^{-6}\times83\times10^9} = \frac{5}{2}\left(11.7 \times 10^{-6} \times \Delta T \times 3 + \frac{54\times10^3\times3}{900\times10^{-6}\times200\times10^9}\right)$$

$$\Delta T = ?$$

The following example from:

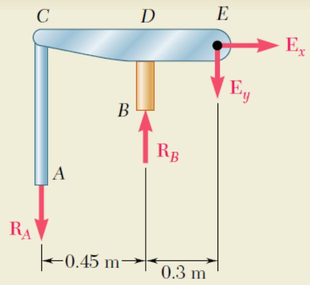
- Beer F.P., Johnston E.R., *Mechanics of Materials*, McGraw-Hill, New York, 2012.



SAMPLE PROBLEM 2.4

The rigid bar *CDE* is attached to a pin support at *E* and rests on the 30-mm-diameter brass cylinder *BD*. A 22-mm-diameter steel rod *AC* passes through a hole in the bar and is secured by a nut which is snugly fitted when the temperature of the entire assembly is 20°C. The temperature of the brass cylinder is then raised to 50°C while the steel rod remains at 20°C. Assuming that no stresses were present before the temperature change, determine the stress in the cylinder.

Rod *AC*: Steel	Cylinder *BD*: Brass
$E = 200$ GPa	$E = 105$ GPa
$\alpha = 11.7 \times 10^{-6}/°C$	$\alpha = 20.9 \times 10^{-6}/°C$



SOLUTION

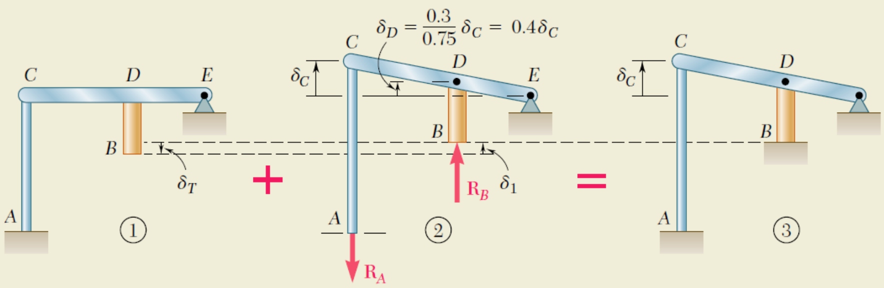
Statics. Considering the free body of the entire assembly, we write

$$+\circlearrowleft \Sigma M_E = 0: \quad R_A(0.75 \text{ m}) - R_B(0.3 \text{ m}) = 0 \qquad R_A = 0.4R_B \qquad (1)$$

Deformations. We use the method of superposition, considering $\mathbf{R}_B$ as redundant. With the support at *B* removed, the temperature rise of the cylinder causes point *B* to move down through δ_T. The reaction $\mathbf{R}_B$ must cause a deflection δ_1 equal to δ_T so that the final deflection of *B* will be zero (Fig. 3).

Deflection δ_T. Because of a temperature rise of 50° − 20° = 30°C, the length of the brass cylinder increases by δ_T.

$$\delta_T = L(\Delta T)\alpha = (0.3 \text{ m})(30°C)(20.9 \times 10^{-6}/°C) = 188.1 \times 10^{-6} \text{ m} \downarrow$$



Deflection δ_1. We note that $\delta_D = 0.4\delta_C$ and $\delta_1 = \delta_D + \delta_{B/D}$.

$$\delta_C = \frac{R_A L}{AE} = \frac{R_A(0.9 \text{ m})}{\frac{1}{4}\pi(0.022 \text{ m})^2(200 \text{ GPa})} = 11.84 \times 10^{-9}R_A \uparrow$$

$$\delta_D = 0.40\delta_C = 0.4(11.84 \times 10^{-9}R_A) = 4.74 \times 10^{-9}R_A \uparrow$$

$$\delta_{B/D} = \frac{R_B L}{AE} = \frac{R_B(0.3 \text{ m})}{\frac{1}{4}\pi(0.03 \text{ m})^2(105 \text{ GPa})} = 4.04 \times 10^{-9}R_B \uparrow$$

We recall from (1) that $R_A = 0.4R_B$ and write

$$\delta_1 = \delta_D + \delta_{B/D} = [4.74(0.4R_B) + 4.04R_B]10^{-9} = 5.94 \times 10^{-9}R_B \uparrow$$

But $\delta_T = \delta_1$: $\quad 188.1 \times 10^{-6} \text{ m} = 5.94 \times 10^{-9}R_B \qquad R_B = 31.7$ kN

Stress in Cylinder: $\sigma_B = \dfrac{R_B}{A} = \dfrac{31.7 \text{ kN}}{\frac{1}{4}\pi(0.03 \text{ m})^2} \qquad \sigma_B = 44.8$ MPa ◀

TORSION OF CIRCULAR SHAFT

Introduction

In many engineering applications, members are required to carry torsional loads. In this lecture, we consider the torsion of circular shafts. Because a circular cross section is an efficient shape for resisting torsional loads, circular shafts are commonly used to transmit power in rotating machinery. Derivation of the equations used in the analysis follows these steps:

- Make simplifying assumptions about the deformation based on experimental evidence.
- Determine the strains that are geometrically compatible with the assumed deformations.
- Use Hooke's law to express the equations of compatibility in terms of stresses.
- Derive the equations of equilibrium. (These equations provide the relationships between the stresses and the applied loads.)

Torsion of Circular Shafts

Consider the solid circular shaft, shown in the Figure 2.1, and subjected to a torque T at the end of the shaft. The fiber AB on the outside surface, which is originally straight, will be twisted into a helix AB' as the shaft is twist through the angle θ. During the deformation, the cross sections remain circular (NOT distorted in any manner) - they remain plane, and the radius r does not change.

Besides, the length L of the shaft remains constant. Based on these observations, the following assumptions are made:

- The material is homogeneous, i.e. of uniform elastic properties throughout.
- The material is elastic, following Hooke's law with shear stress proportional to shear strain.
- The stress does not exceed the elastic limit or limit of proportionality.
- Circular cross sections remain plane (do not warp) and perpendicular to the axis of the shaft.
- Cross sections do not deform (there is no strain in the plane of the cross section).
- The distances between cross sections do not change (the axial normal strain is zero).

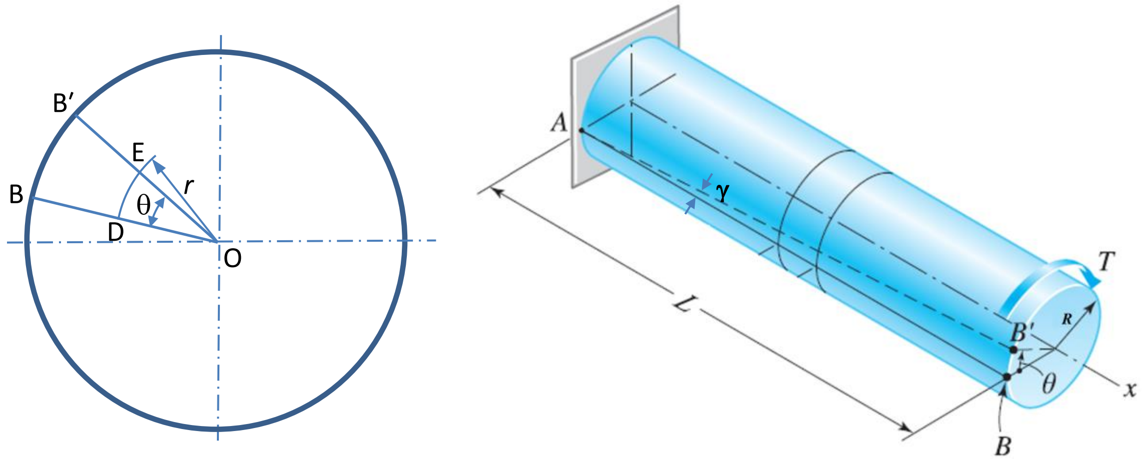


Figure 2.1: Deformation of a circular shaft caused by the torque T.

$$\delta_s = DE = r\theta \qquad (1)$$

where the subscript s denotes *shear*, r is the distance from the origin to any interested fiber, and θ is the angle of twist.

From Figure 2.1,

$$\gamma L = r\theta$$

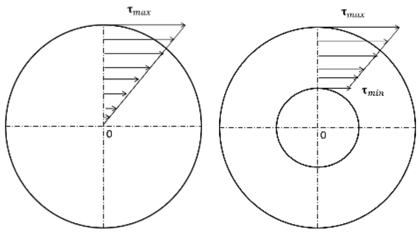
The unit deformation of this fiber is,

$$\gamma = \frac{\delta_s}{L} = \frac{r\theta}{L} \qquad (2)$$

Shear stress can be determined using Hooke's law as:

$$\tau = G\gamma = G\left(\frac{r\theta}{L}\right) \quad (3)$$

Note: since $\tau = \left(\frac{G\theta}{L}\right)r = const.r,$ therefore, the conclusion is that the shearing stress at any internal fiber varies linearly with the radial distance from the axis of the shaft.



For the shaft to be in equilibrium, the resultant of the shear stress acting on a cross section must be equal to the internal torque T acting on that cross section. Figure 2.2 shows a cross section of the shaft containing a differential element of area dA located at the radial distance r from the axis of the shaft. The shear force acting on this area is $dF = \tau\, dA$, directed perpendicular to the radius. Hence, the torque of dF about the center O is:

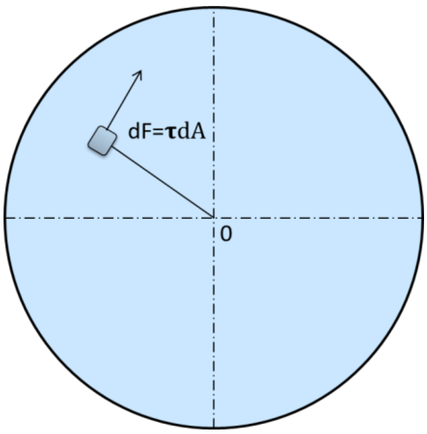


Figure 2.2: The resultant of the shear stress acting on the cross section.

$$T = \int r dF \quad = \int r\, \tau\, dA \quad (4)$$

Substituting equation (3) into equation (4),

$$T_r = \int r\left(\frac{G\theta}{L}\right) r dA \; = \frac{G\theta}{L}\int r^2 dA$$

Since $\int r^2 dA = J$, the polar 2nd moment of area (or polar moment of inertia) of the cross section

$$T = \frac{G\theta}{L}J$$

Rearranging the above equation,

$$\theta = \frac{TL}{JG} \quad (5)$$

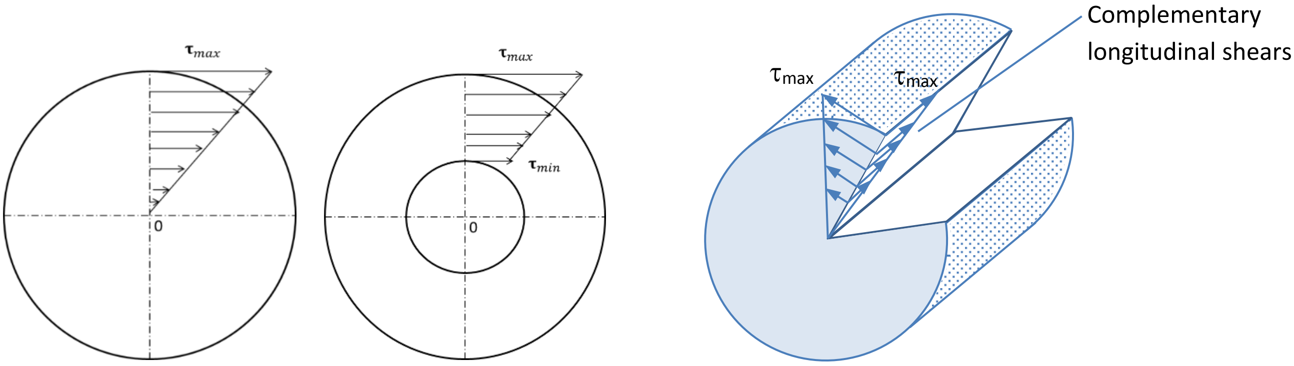
where T is the applied torque (N.m), L is length of the shaft (m), G is the shear modulus (N/m^2), J is the polar moment of inertia (m^4), and θ is the angle of twist in radians.

From equations (5) and (3),

$$\tau = \left(\frac{G\theta}{L}\right) r = \frac{T}{J} r$$

or

$$\tau = \frac{Tr}{J} \quad (6)$$



Polar Moment of Inertia

- Solid Shaft

  Consider the solid shaft shown, therefore,

$$J = \int r^2 dA = \int_0^R r^2 (2\pi r dr) = 2\pi \int_0^R r^3 dr$$

which yields,

$$J = 2\pi [\frac{r^4}{4}]_0^R = \frac{\pi}{2} R^4$$

or

$$J = \frac{\pi d^4}{32}$$

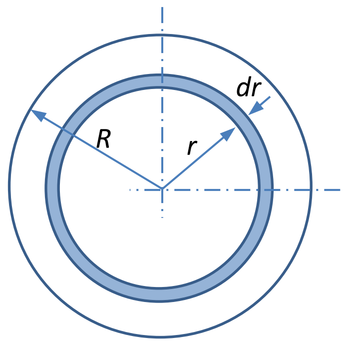


Figure 2.3: Shaft cross-section

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Lecturer Rasha Mohammed

- Hollow Shaft

  The above procedure can be used for calculating the polar moment of inertia of the hollow shaft of inner radius R_i and outer radius R_o,

$$J = 2\pi \int_{R_i}^{R_o} r^3 dr = \frac{\pi}{2}(R_o^4 - R_i^4)$$

or

$$J = \frac{\pi}{32}(D_o^4 - D_i^4)$$

- Thin-Walled Hollow Shaft

  For thin-walled hollow shafts the values of D_o and D_i may be nearly equal, and in such cases there can be considerable errors in using the above equation involving the difference of two large quantities of similar value. It is therefore convenient to obtain an alternative form of expression for the polar moment of area. Therefore,

$$J = \int_0^R 2\pi r^3 dr = \sum(2\pi\, r\, dr) r^2 = \sum A\, r^2$$

where $A = (2\pi\, r\, dr)$ is the area of each small element of Figure 2.3, i.e. ***J*** is the sum of the $\boldsymbol{Ar^2}$ terms for all elements.

If a thin hollow cylinder is therefore considered as just one of these small elements with its wall thickness $t = dr$, then

$$J = A\, r^2 = (2\pi\, r\, t) r^2 = 2\pi r^3 t \quad (approximately)$$

Notes: The maximum shear stress is found (at the surface of the shaft) by replacing r by the radius R, for solid shaft, or by R_o , for the hollow shaft, as

$$\tau_{max} = \frac{2T}{\pi R^3} = \frac{16T}{\pi D^3} \quad \rightarrow solid\ shaft$$

$$\tau_{max} = \frac{2TR}{\pi(R^4 - r^4)} = \frac{16TD_o}{\pi(D_o^4 - D_i^4)} \quad \rightarrow hollow\ shaft$$

Composite Shafts - Series Connection

If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is therefore termed ***series-connected***, as shown in Figure 2.4. In such cases the composite shaft strength is treated by considering each component shaft separately, applying the torsion theory to each in turn; the composite shaft will therefore be as weak as its weakest component. If relative dimensions of the various parts are required then a solution is usually effected by equating the torques in each shaft, e.g. for two shafts in series

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2}$$

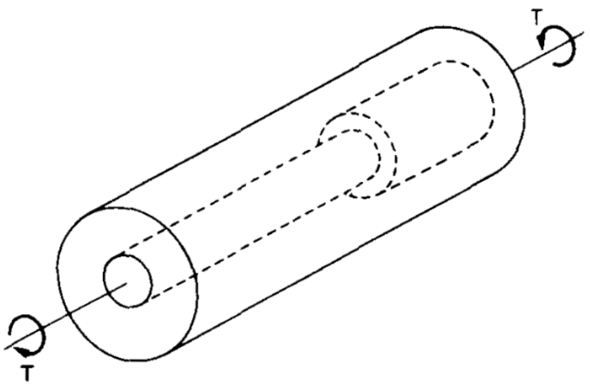

Figure 2.4: "Series connected" shaft - common torque

Composite Shafts - Parallel Connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel (Figure 2.5).

For parallel connection,

Total Torque $T = T_1 + T_2$ (7)

In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1\, J_1} = \frac{T_2 L_2}{G_2\, J_2} \qquad (8)$$

or

$$\frac{T_1}{T_2} = \frac{G_1\, J_1}{G_2\, J_2}\left(\frac{L_2}{L_1}\right)$$

Thus two equations are obtained in terms of the torques in each part of the composite shaft and these torques can therefore be determined.

In case of equal lengths, equation (8) becomes

$$\frac{T_1}{T_2} = \frac{G_1\, J_1}{G_2\, J_2}$$

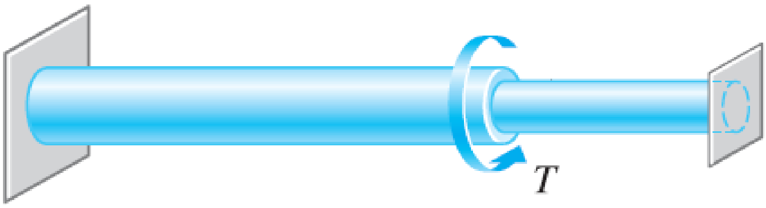

Figure 2.5: "Parallel connected" shaft - shared torque.

Power Transmitted by Shafts

If a shaft carries a torque *T* Newton meters and rotates at ω rad/s it will do work at the rate of

$$T\omega \text{ Nm/s (or joule/s).}$$

Now the rate at which a system works is defined as its power, the basic unit of power being the Watt (1 Watt = 1 Nm/s).

Thus, the power transmitted by the shaft:

$$= T\omega \text{ Watts.}$$

Since the Watt is a very small unit of power in engineering terms use is normally made of S.I. multiples, i.e. kilowatts (kW) or megawatts (MW).

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Lecturer Rasha Mohammed

Example1:

A solid shaft in a rolling mill transmits 20 kW at 120 r.p.m. Determine the diameter of the shaft if the shearing stress is not to exceed 40MPa and the angle of twist is limited to 6° in a length of 3m. Use G=83GPa.

Solution

Power = $T\,\omega$

$$20\times10^3 = T\times120\times\frac{2\pi}{60}$$

$$\therefore T = \frac{20\times10^3}{4\pi} = 1590\ N.m$$

Since two design conditions have to be satisfied, i.e. strength (stress) consideration, and rigidity (angle of twist) consideration. The calculations will be as:

- Based on strength consideration $\left(\tau_{max} = \frac{16T}{\pi D^3}\right)$

$$40\times10^6 = \frac{16\times1590}{\pi D^3}$$

$$\therefore D = 0.0587 = 58.7mm$$

- Based on rigidity consideration $\left(\theta = \frac{TL}{JG}\right)$

$$\theta = \frac{TL}{\frac{\pi d^4}{32}G}$$

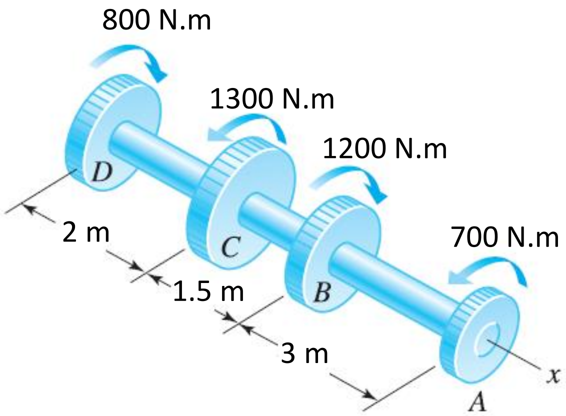
$$\therefore\ 6°\times\frac{\pi}{180} = \frac{32\times1590\times3}{\pi d^4\times83\times10^9}$$

$$\therefore D=0.0465\ m= 46.5\ mm$$

Therefore, the minimum diameter that satisfy both the strength and rigidity considerations is D=58.7mm. (Answer)

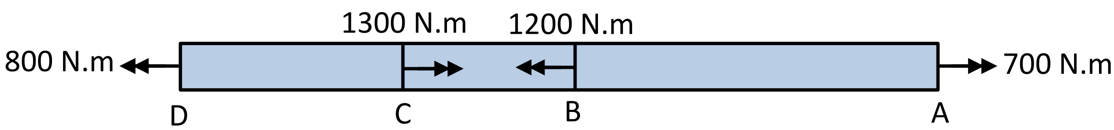
Example 2:

A steel shaft with constant diameter of 50 mm is loaded as shown in the figure by torques applied to gears fastened to it. Using G= 83 GPa, compute in degrees the relative angle of rotation between gears A and D.



Solution:

It is convenient to represent the torques as vectors (using the right-hand rule) on the free body diagram, as shown in the figure.



Using the equations of statics (i.e. $\sum T = 0$), the internal torques are: T_{AB}=700N.m, T_{BC}=-500N.m and T_{CD}=800N.m.

$$J_{AB} = J_{BC} = J_{CD} = J = \frac{\pi(0.05)^4}{32}$$

$$\theta_{A/D} = \sum \frac{TL}{JG} = \frac{T_{AB}L_{AB}}{J_{AB}G} + \frac{T_{BC}L_{BC}}{J_{BC}G} + \frac{T_{CD}L_{CD}}{J_{CD}G}$$

$$= \frac{1}{\frac{\pi(0.05)^4}{32}\times 83\times 10^9}(700\times 3 - 500\times 1.5 + 800\times 2) = 0.0579\, rad.$$

$\therefore \theta_{A/D} = 3.32^{\circ}$ (Answer)

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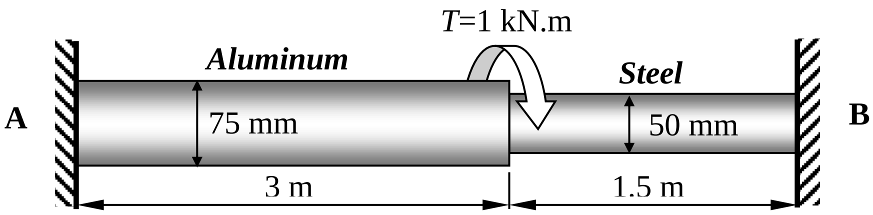
Lecturer Rasha Mohammed

Example3:

A compound shaft made of two segments: solid steel and solid aluminum circular shafts. The compound shaft is built-in at A and B as shown in the figure. Compute the maximum shearing stress in each shaft. Given G_{al}=28GPa, G_{st} = 83 GPa.



Solution:

This type of problem is a statically indeterminate problem, where the equation of statics (or equilibrium) is not enough to solve the problem. Therefore, one equation will be obtained from statics, and the other from the deformation.

- Statics

$$T_s + T_a = T = 1000 \qquad (1)$$

- Deformation $(\theta_s = \theta_a)$

Since $\theta_s = \theta_a$, then $\frac{T_s L_s}{J_s G_s} = \frac{T_a L_a}{J_a G_a}$, which yield,

$$\frac{T_s \times 1.5}{\frac{\pi(0.05)^4}{32} \times 83 \times 10^9} = \frac{T_a \times 3}{\frac{\pi(0.075)^4}{32} \times 28 \times 10^9}$$

from which,

$$T_s = 1.17 T_a \qquad (2)$$

Solving equation (1) and (2):

$$T_a = 461\ N.m \quad and \quad T_s = 539 N.m$$

- Stresses $\left(\tau = \frac{Tr}{J}\right)$

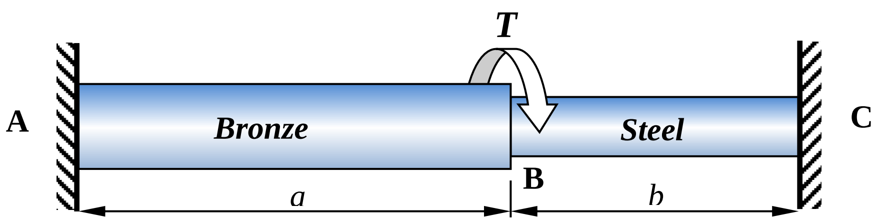
The maximum stress occur at the surface, i.e. $\tau_{max} = \frac{16T}{\pi D^3}$

$$\tau_a = \frac{16*461}{\pi(0.075)^3} = 5.57 \text{ MPa}$$ (Answer)

$$\tau_s = \frac{16*539}{\pi(0.05)^3} = 22.0 \text{ MPa}$$ (Answer)

Example 4:

The compound shaft, shown in the figure, is attached to rigid supports. For bronze (*AB*) *d*=75mm, *G*=35GPa, $\tau \leq 60$MPa. For steel (*BC*), *d*=50mm, *G*=83GPa, $\tau \leq 80 MPa$. Determine the ratio of lengths *b*/*a* so that each material will be stressed to its permissible limit, also find the torque *T* required.



Solution:

- For bronze

$$\tau_b = \frac{T_b r}{J_b} \quad \rightarrow 60\times10^6 = \frac{T_b \times 0.075/2}{\frac{\pi}{32}\times(0.075)^4}$$

From which

$$T_b = 4970 \ N.m$$

For steel

$$\tau_s = \frac{T_s r}{J_s} \quad \rightarrow 80\times10^6 = \frac{T_s \times 0.05/2}{\frac{\pi}{32}\times(0.05)^4}$$

From which

$$T_s = 1963.5 \ N.m$$

Applied torque $T = T_b + T_s = 6933.6 \ N.m$ (Answer)

From the deformation $\theta_s = \theta_b$,

$$\frac{T_sL_s}{J_sG_s} = \frac{T_bL_b}{J_bG_b} \rightarrow \frac{1963.5\times b}{\frac{\pi}{32}\times(0.05)^4\times83\times10^9} = \frac{4970\times a}{\frac{\pi}{32}\times(0.075)^4\times35\times10^9}$$

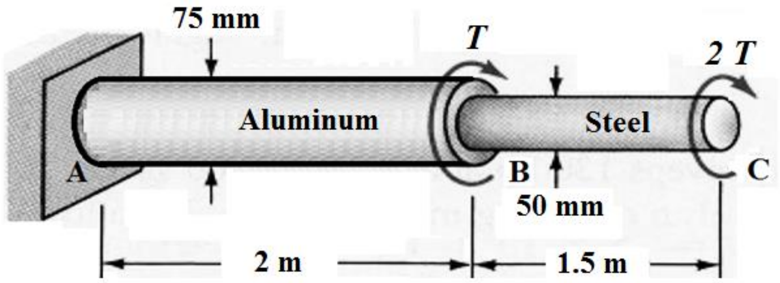
From which

(b/a)=1.1856 (Answer)

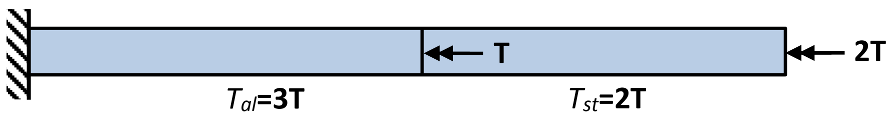
Example 5:

A compound shaft consisting of an aluminum segment and a steel is acted upon by two torque as shown in the figure. Determine the maximum permissible value of T subjected to the following conditions:

$\tau_s \leq 100\text{MPa}, \tau_a \leq 70\text{MPa}$, and the angle of rotation of the free end limited to 12°. Use $G_s = 83GPa \;\; and \; G_a = 28 \; GPa$.



Solution:



$$J_{st} = \frac{\pi}{32}\times(0.05)^4 = 6.136\times10^{-7}m^4$$

$$J_{al} = \frac{\pi}{32}\times(0.075)^4 = 3.106\times10^{-6}m^4$$

- For steel $\left(\tau = \frac{Tr}{J}\right)$

$$100\times10^6 = \frac{2T\times0.025}{6.136\times10^{-7}}$$

From which, $T = 1.23 \; kN.m$

- For aluminum

$$70\times10^{6}=\frac{3T\times0.075/2}{3.106\times10^{-6}}$$

From which, $T=1.93\ kNm$

- Deformation

$$\theta=\sum_{i=1}^{2}\frac{TL}{JG}=\frac{T_aL_a}{J_aG_a}+\frac{T_sL_s}{J_sG_s}$$

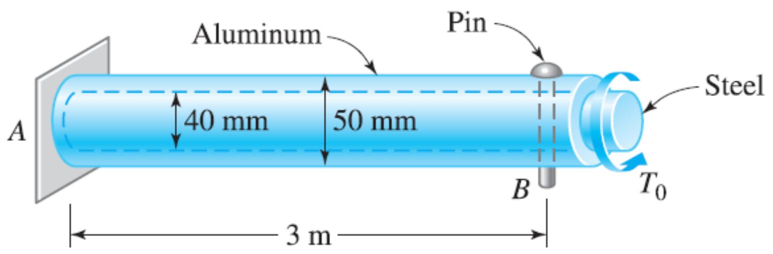
$$12\times\frac{\pi}{180}=\frac{3T\times2}{3.106\times10^{-6}\times28\times10^{9}}+\frac{2T\times1.5}{6.136\times10^{-7}\times83\times10^{9}}$$

From which, T=1.64 kN.m

Therefore, the maximum safe value of torque (T) is T=1.23 kN.m (Answer)

Example 6:

The steel rod fits loosely inside the aluminum sleeve. Both components are attached to a rigid wall at A and joined together by a pin at B. Because of a slight misalignment of the pre-drilled holes, the torque T_o = 750 N.m was applied to the steel rod before the pin could be inserted into the holes. Determine the torque in each component after T_o was removed. Use G = 80 GPa for steel and G = 28 GPa for aluminum.

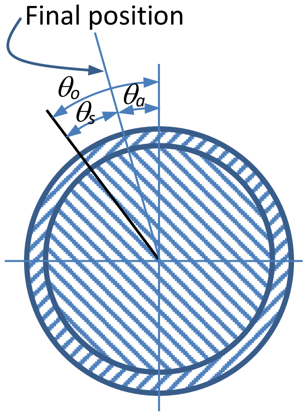


Solution:

The initial torque T_o will cause an initial angle of twist to the steel rod,

$$\theta_o=\frac{T_oL}{J_sG_s}=\frac{750\times3}{\frac{\pi}{32}(0.04)^4\times80\times10^{9}}=0.1119058\,rad.$$

When the pin was inserted into the holes with the removal of T_o, the system will stabilize in static equilibrium. This will cause some of the deformation of steel rod to be recovered, as shown in the figure. This relation may be expressed as,



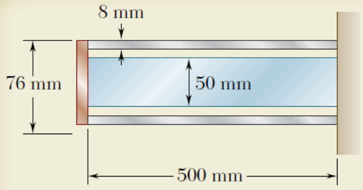
$$\theta_o = \theta_s + \theta_a$$

$$0.1119058 = \frac{T\times 3}{\frac{\pi}{32}(0.04)^4\times 80\times 10^9} + \frac{T\times 3}{\frac{\pi}{32}((0.05)^4-(0.04)^4)\times 28\times 10^9}$$

From which, $T = 251.5\ N.m$ (Answer)

The following example from:

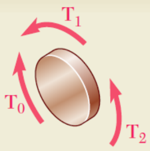
- Beer F.P., Johnston E.R., *Mechanics of Materials*, McGraw-Hill, New York, 2012.



SAMPLE PROBLEM 3.5

A steel shaft and an aluminum tube are connected to a fixed support and to a rigid disk as shown in the cross section. Knowing that the initial stresses are zero, determine the maximum torque $\mathbf{T}_0$ that can be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use G = 77 GPa for steel and G = 27 GPa for aluminum.

SOLUTION



Statics. ***Free Body of Disk.*** Denoting by $\mathbf{T}_1$ the torque exerted by the tube on the disk and by $\mathbf{T}_2$ the torque exerted by the shaft, we find

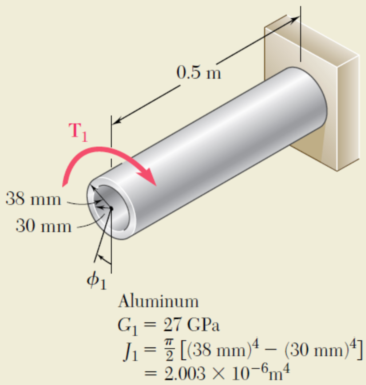
$$T_0 = T_1 + T_2 \qquad (1)$$

Deformations. Since both the tube and the shaft are connected to the rigid disk, we have

$$\phi_1 = \phi_2: \quad \frac{T_1 L_1}{J_1 G_1} = \frac{T_2 L_2}{J_2 G_2}$$

$$\frac{T_1(0.5\text{ m})}{(2.003 \times 10^{-6}\text{ m}^4)(27\text{ GPa})} = \frac{T_2(0.5\text{ m})}{(0.614 \times 10^{-6}\text{ m}^4)(77\text{ GPa})}$$

$$T_2 = 0.874 T_1 \qquad (2)$$



Shearing Stresses. We *assume* that the requirement $\tau_{alum} \le 70$ MPa is critical. For the aluminum tube, we have

$$T_1 = \frac{\tau_{alum} J_1}{c_1} = \frac{(70\text{ MPa})(2.003 \times 10^{-6}\text{ m}^4)}{0.038\text{ m}} = 3690\text{ N}\cdot\text{m}$$

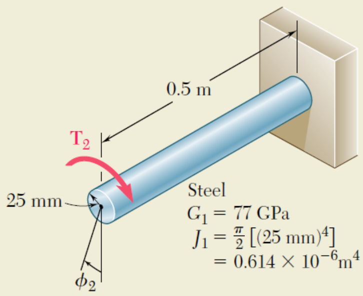
Using Eq. (2), we compute the corresponding value T_2 and then find the maximum shearing stress in the steel shaft.

$$T_2 = 0.874 T_1 = 0.874(3690) = 3225\text{ N}\cdot\text{m}$$

$$\tau_{steel} = \frac{T_2 c_2}{J_2} = \frac{(3225\text{ N}\cdot\text{m})(0.025\text{ m})}{0.614 \times 10^{-6}\text{ m}^4} = 131.3\text{ MPa}$$

We note that the allowable steel stress of 120 MPa is exceeded; our assumption was *wrong*. Thus the maximum torque $\mathbf{T}_0$ will be obtained by making τ_{steel} = 120 MPa. We first determine the torque $\mathbf{T}_2$.

$$T_2 = \frac{\tau_{steel} J_2}{c_2} = \frac{(120\text{ MPa})(0.614 \times 10^{-6}\text{ m}^4)}{0.025\text{ m}} = 2950\text{ N}\cdot\text{m}$$



From Eq. (2), we have

$$2950\text{ N}\cdot\text{m} = 0.874 T_1 \qquad T_1 = 3375\text{ N}\cdot\text{m}$$

Using Eq. (1), we obtain the maximum permissible torque

$$T_0 = T_1 + T_2 = 3375\text{ N}\cdot\text{m} + 2950\text{ N}\cdot\text{m}$$

$$T_0 = 6.325\text{ kN}\cdot\text{m} \quad ◀$$

TORSION OF THIN-WALLED TUBES

Consider the thin-walled tube subjected to the torque T shown in Figure 1(a). We assume the tube to be of constant cross section, but the wall thickness t is allowed to vary within the cross section. The surface that lies midway between the inner and outer boundaries of the tube is called the middle surface. If t is small compared to the overall dimensions of the cross section, the shear stress τ induced by torsion can be shown to be almost constant through the wall thickness of the tube and directed tangent to the middle surface, as shown in Figure 1(b). It is convenient to introduce the concept of shear flow q, defined as the shear force per unit edge length of the middle surface. Thus, the shear flow is

$$q = \tau\, t \qquad (1)$$

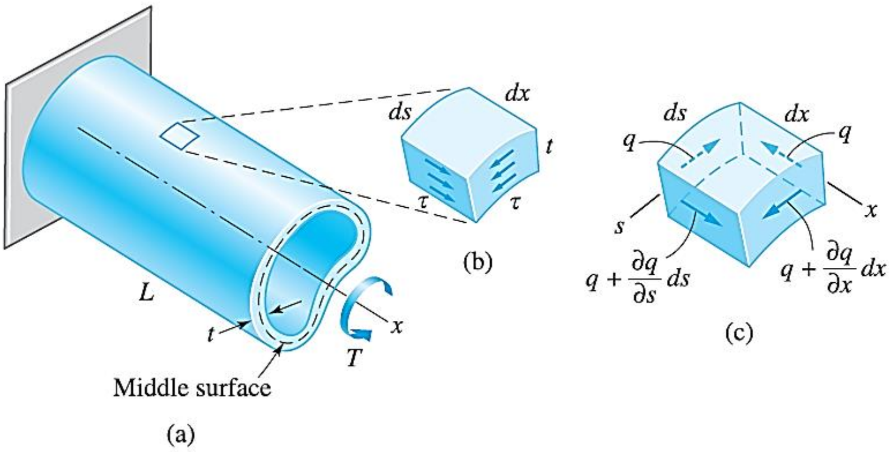


Figure 1: (a) Thin-walled tube in torsion; (b) shear stress in the wall of the tube; (c) shear flows on wall element.

The shear flow is constant throughout the tube, as explained in what follows. Considering the equilibrium of the element shown in Figure 1(c). In labeling the shear flows, we assume that q varies in the longitudinal (x) as well as the circumferential (s) directions. Thus, the terms $(\partial q/\partial x)\,dx$ and $(\partial q/\partial s)\,ds$ represent the changes in the shear flow over the distances dx and ds, respectively. The force acting on each side of the element is equal to the shear flow multiplied by the edge length, resulting in the equilibrium equations

$$\sum F_x = 0: \qquad \left(q + \frac{\partial q}{\partial s} ds\right) dx - q\, dx = 0$$

$$\sum F_s = 0: \qquad \left(q + \frac{\partial q}{\partial x} dx\right) ds - q\, ds = 0$$

which yield $\frac{\partial q}{\partial s} = \frac{\partial q}{\partial x} = 0$, thus proving that the shear flow is constant throughout the tube.

To relate the shear flow to the applied torque T, consider the cross section of the tube in Figure 2. The shear force acting over the infinitesimal edge length ds of the middle surface is $dP = q\, ds$. The moment of this force about an arbitrary point O in the cross section is $r\, dP = (q\, ds)\, r$, where r is the perpendicular distance of O from the line of action of dP. Equilibrium requires that the sum of these moments must be equal to the applied torque T; that is,

$$T = \oint_S q\, r\, ds \qquad (2)$$

where the integral is taken over the closed curve formed by the intersection of the middle surface and the cross section, called the *median line*.

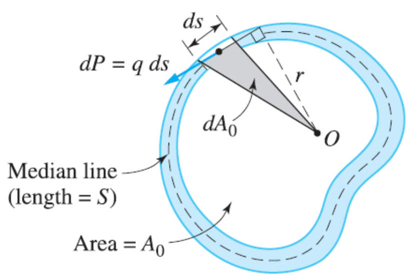


Figure 2: Calculating the torque T on the cross section of the tube.

Since q is constant, equation (2) can be written as $T = q \oint_S r\, ds$. From Figure 2 it can be seen that $dA_0 = \frac{1}{2} r\, ds$, where dA_0 is the area of the shaded triangle. Therefore, $\oint_S r\, ds = 2A_0$, where A_0 is the area of the cross section that is enclosed by the *median line*. Consequently, equation (2) becomes

$$T = 2A_0\, q$$

from the shear flow is

$$q = \frac{T}{2A_0} \qquad (3)$$

The angle of twist of the tube cab found by equating the work done by the shear stress in the tube to the work of the applied torque T. From Figure 3, the work done on the element is,

$$dU = \frac{1}{2}\,(force \times distance) = \frac{1}{2}\,(q\, ds \times \gamma\, dx)$$

where $q\, ds$ is the elemental shear force which moves a distance $\gamma\, dx$, Figure 3. Using Hooke's law, i.e. $\gamma = \frac{\tau}{G} = q/(Gt)$, the above equation may be written as,

$$dU = \frac{q^2}{2G\, t}\, ds\, dx \qquad (4)$$

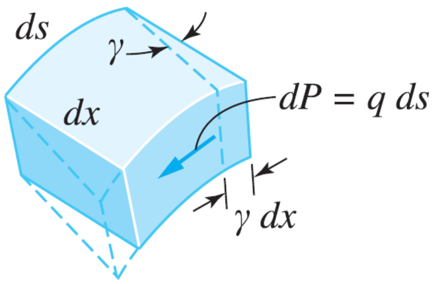


Figure 3: Deformation of element caused by shear flow.

Since q and G are constants and t is independent of x, the work U is obtained from equation (4) over the middle surface of the tube,

$$U = \frac{q^2}{2G} \int_0^L \left(\oint_S \frac{ds}{t} \right) dx = \frac{q^2 L}{2G} \left(\oint_S \frac{ds}{t} \right) \qquad (5)$$

Conservation of energy requires U to be equal to the work of the applied torque; that is, $U = T\,\theta/2$. Then, using equation (3), equation (5) will be,

$$\left(\frac{T}{2A_0}\right)^2 \frac{L}{2G}\left(\oint_S \frac{ds}{t}\right) = \frac{1}{2}T\,\theta$$

from which the angle of twist of the tube is

$$\theta = \frac{T\,L}{4\,G\,A_0^2}\left(\oint_S \frac{ds}{t}\right) \qquad (6)$$

If t is constant, we have $\oint_S (ds/t) = S/t$, where S is the length of the *median line*. Therefore, equation (6) becomes

$$\theta = \frac{T\,L\,S}{4\,G\,A_0^2\,t} = \frac{\tau\,L\,S}{2\,A_0\,G} \qquad (7)$$

For closed sections which have constant thickness over specified lengths but varying from one part of the perimeter to another:

$$\theta = \frac{T\,L}{4\,G\,A_0^2}\left(\frac{S_1}{t_1} + \frac{S_2}{t_2} + \frac{S_3}{t_3} + \cdots \text{etc.}\right) \qquad (8)$$

Thin-Walled Cellular Sections

The above theory may be applied to the solution of problems involving cellular sections of the type shown in Figure 4.

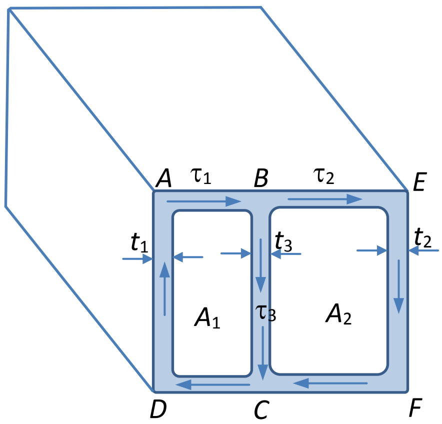


Figure 4: Thin-walled cellular section.

Assume the length *CDAB* is of constant thickness t_1 and subjected therefore to a constant shear stress τ_1. Similarly, *BEFC* is of thickness t_2 and stress τ_2 with *BC* of thickness t_3 and stress τ_3.

Considering the equilibrium of complementary shear stresses on a longitudinal section at *B*, it follows that

$$q_1 = q_2 + q_3$$

or

$$\tau_1 t_1 = \tau_2 t_2 + \tau_3 t_3 \qquad (9)$$

The total torque for the section is then found as the sum of the torques on the two cells by application of equation (3) to the two cells and adding the result,

$$T = 2q_1 A_1 + 2q_2 A_2 = 2(\tau_1 t_1 A_1 + \tau_2 t_2 A_2) \qquad (10)$$

The angle of twist will be common to both cells, i.e.,

$$\theta = \frac{L}{2G}\left(\frac{\tau_1 S_1 + \tau_3 S_3}{A_1}\right) = \frac{L}{2G}\left(\frac{\tau_2 S_2 - \tau_3 S_3}{A_2}\right) \qquad (11)$$

where S_1, S_2 and S_3 are the median line perimeters *CDAB*, *BEFC* and *BC* respectively.

Note: The negative sign appears in the final term because the shear flow along *BC* for this cell opposes that in the remainder of the perimeter.

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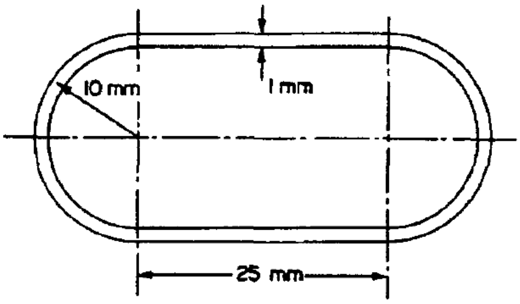
Example 1:

A thin-walled member 1.2 m long has the cross-section shown in the figure. Determine the maximum torque which can be carried by the section if the angle of twist is limited to 10°. What will be the maximum shear stress when this maximum torque is applied? For the material of the member $G = 80$ GN/m^2.



Solution:

Now, perimeter of median line $= s = (2 \times 25 + 2\pi \times 10)$ mm

$$= 112.8 \text{ mm}$$

area enclosed by median $= A = (20 \times 25 + \pi \times 10^2)$ mm^2

$$= 814.2 \text{ mm}^2$$

From eqn (7),
$$\theta = \frac{TLs}{4A^2\,Gt}$$

$$\therefore \quad \frac{10 \times 2\pi}{360} = \frac{T \times 1.2 \times 112.8 \times 10^{-3}}{4(814.2 \times 10^{-6})^2 \times 80 \times 10^9 \times 1 \times 10^{-3}}$$

i.e. maximum torque possible,

$$T = \frac{20\pi \times 4 \times 814.2^2 \times 80 \times 10^{-6}}{360 \times 1.2 \times 112.8 \times 10^{-3}}$$

$$= \mathbf{273\ Nm}$$

From eqn. (3),
$$\tau_{max} = \frac{T}{2At}$$

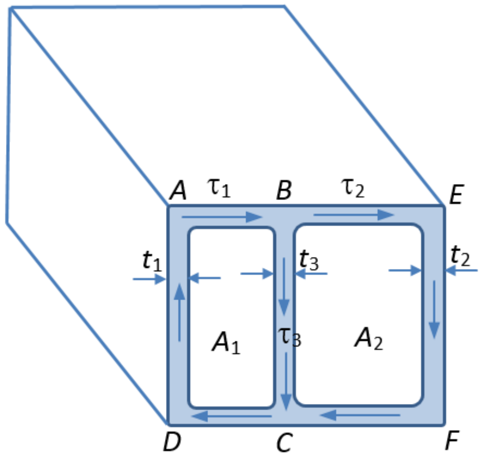
$$= \frac{273}{2 \times 814.2 \times 10^{-6} \times 1 \times 10^{-3}}$$

$$= 168 \times 10^6 = \mathbf{168\ MN/m^2}$$

The maximum stress produced is **168 MN/m^2**.

Example 2:

The median dimensions of the two cells shown in the cellular section of the figure below are A_1 =20 mm × 40 mm and A_2 = 50 mm × 40 mm with wall thicknesses t_1 = 2 mm, t_2 = 1.5 mm and t_3 = 3 mm. If the section is subjected to a torque of 320 Nm, determine the angle of twist per unit length and the maximum shear stress set up. The section is constructed from a light alloy with a modulus of rigidity G = 30 GN/m^2.



Solution:

From eqn. (10),

$$320 = 2(\tau_1 \times 2 \times 20 \times 40 + \tau_2 \times 1.5 \times 50 \times 40) \times 10^{-9} \quad (1)$$

From eqn. (11),

$$2 \times 30 \times 10^9 \times \theta = \frac{1}{20 \times 40 \times 10^{-6}}(\tau_1(40 + 2 \times 20)10^{-3} + \tau_3 \times 40 \times 10^{-3}) \quad (2)$$

and,

$$2 \times 30 \times 10^9 \times \theta = \frac{1}{50 \times 40 \times 10^{-6}}(\tau_2(40 + 2 \times 50)10^{-3} - \tau_3 \times 40 \times 10^{-3}) \quad (3)$$

Equating (2) and (3),

$$40\tau_1 = 28\tau_2 - 28\tau_3 \quad (4)$$

From eqn. (9),

$$2\tau_1 = 1.5\tau_2 + 3\tau_3 \quad (5)$$

The negative sign indicates that the direction of shear flow in the wall of thickness t_3 is reversed from that shown in the figure.

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Asst. Lecturer Rasha Mohammed

Solving equations (1), (4) and (5) for τ_1, τ_2 and τ_3,

$\tau_1 = 27.6$ MPa, $\tau_2 = 38.6$ MPa and $\tau_3 = -0.9$ MPa

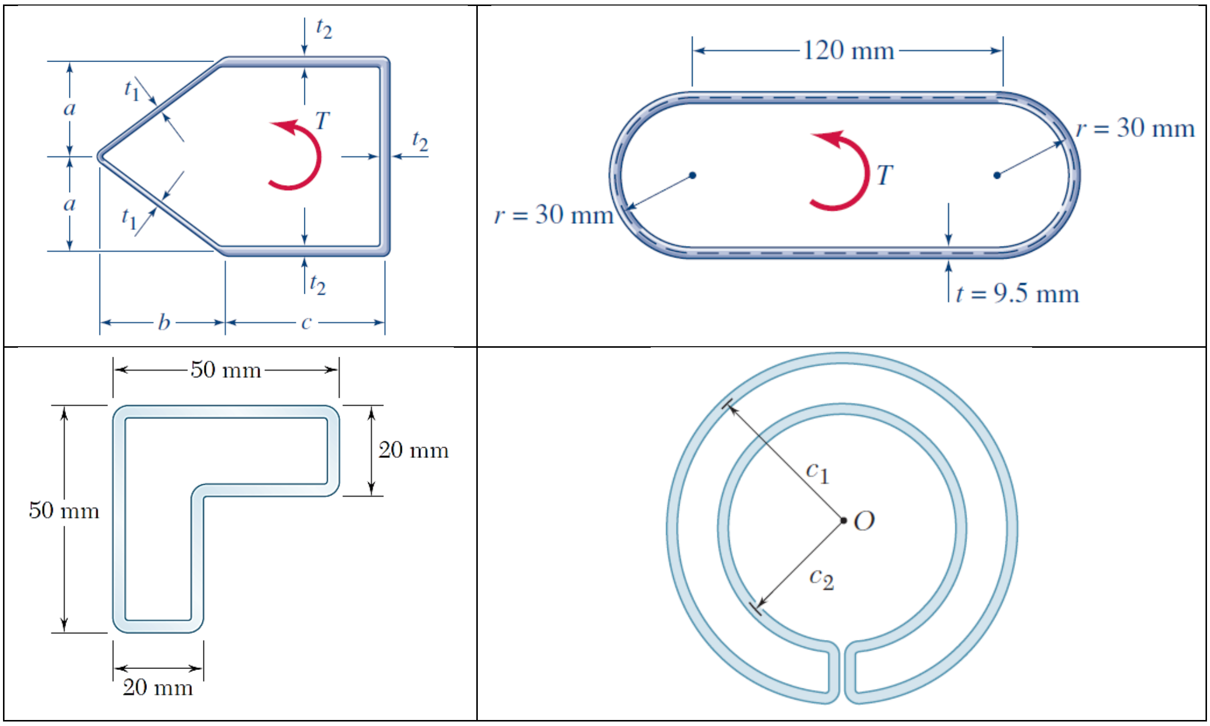
The maximum shear stress present in the section is thus **38.6** MN/m^2 in the 1.5 mm wall thickness.

From eqn. (3),

$$2 \times 30 \times 10^9 \times \theta = \frac{1\times10^3}{20\times40\times10^{-6}}(27.6 \times (40 + 2 \times 20) - 0.9 \times 40)$$

$$\therefore \theta = 0.04525\ rad. = 2.592° \text{ (Answer)}$$

Some Available Cross-Section



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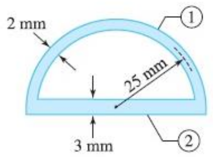
The following example from:

- Pytel A., Kiusalaas J., *Mechanics of Materials*, 2nd Edition, Cengage Learning, Stamford, 2010.

Sample Problem 3.7



An aluminum tube, 1.2 m long, has the semicircular cross section shown in the figure. If stress concentrations at the corners are neglected, determine (1) the torque that causes a maximum shear stress of 40 MPa, and (2) the corresponding angle of twist of the tube. Use $G = 28$ GPa for aluminum.

Solution

Part 1

Because the shear flow is constant in a prismatic tube, the maximum shear stress occurs in the thinnest part of the wall, which is the semicircular portion with $t = 2$ mm. Therefore, the shear flow that causes a maximum shear stress of 40 MPa is

$$q = \tau t = (40 \times 10^6)(0.002) = 80 \times 10^3 \text{ N/m}$$

The cross-sectional area enclosed by the median line is

$$A_0 = \frac{\pi r^2}{2} = \frac{\pi(0.025)^2}{2} = 0.9817 \times 10^{-3} \text{ m}^2$$

which results in the torque—see Eq. (3.8a):

$$T = 2A_0 q = 2(0.9817 \times 10^{-3})(80 \times 10^3) = 157.07 \text{ N} \cdot \text{m} \qquad \textit{Answer}$$

Part 2

The cross section consists of two parts, labeled ① and ② in the figure, each having a constant thickness. Hence, we can write

$$\oint_S \frac{ds}{t} = \frac{1}{t_1}\int_{S_1} ds + \frac{1}{t_2}\int_{S_2} ds = \frac{S_1}{t_1} + \frac{S_2}{t_2}$$

where S_1 and S_2 are the lengths of the median lines of parts ① and ②, respectively. Therefore,

$$\oint_S \frac{ds}{t} = \frac{\pi r}{t_1} + \frac{2r}{t_2} = \frac{\pi(25)}{2} + \frac{2(25)}{3} = 55.94$$

and Eq. (3.9a) yields for the angle of twist

$$\theta = \frac{TL}{4GA_0^2}\oint_S \frac{ds}{t} = \frac{157.07(1.2)}{4(28 \times 10^9)(0.9817 \times 10^{-3})^2}(55.94)$$

$$= 0.0977 \text{ rad} = 5.60^\circ \qquad \textit{Answer}$$

Asst. Prof. Dr. Mohsin Noori Hamzah
Asst. Lecturer Rasha Mohammed

THIN WALLED PRESSURE VESSELS

Introduction

A pressure vessel is a pressurized container, often cylindrical or spherical. The pressure acting on the inner surface is resisted by tensile stresses in the walls of the vessel. If the wall thickness t is sufficiently small compared to the inner diameter of the vessel, d_i, these stresses are almost uniform throughout the wall thickness. It can be shown that if $(t/d_i) < (1/20)$, the stresses between the inner and outer surfaces of the wall vary by less than 5%. Thin wall pressure vessels are widely used in industry for storage and transportation of liquids and gases when configured as tanks. See Figure 1.

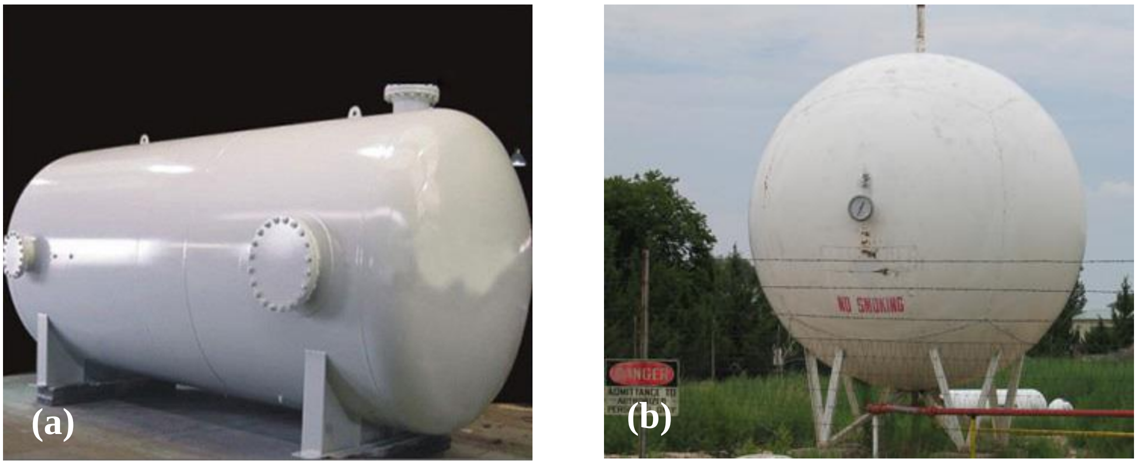


Figure 1: Pressure vessels: (a) cylindrical tank, (b) spherical tanks.

Asst. Prof. Dr. Mohsin Noori Hamzah
Asst. Lecturer Rasha Mohammed

Thin Cylinder under Internal Pressure

When a thin-walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder materials, these stresses are

1. Circumferential or hoop stress
2. Radial stress
3. Longitudinal stress

Note: a cylinder is consider to be thin when the ratio $\frac{t}{d_i} < \frac{1}{20}$, where t is the thickness and d_i is the inner diameter of the cylinder.

Assumptions

- Hoop and longitudinal stress are considered constant along thickness.
- Radial stress is small for thin cylinder assumption and may be neglected.

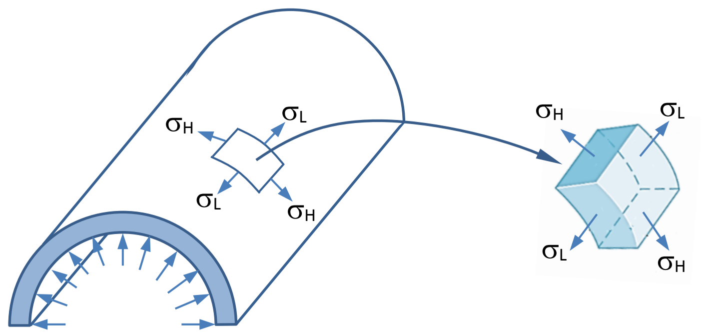


Figure 1: Thin cylindrical pressure vessel subjected to internal pressure.

Hoop or circumferential stress

Total force on half cylinder$= p \times \ projected\ area = \ p \times (dL)$

Total resisting force $=2\sigma_H \times tL$

$2\sigma_H \times tL = pdL$

$$\text{Hoop stress} \ \ \sigma_H = \frac{pd}{2t}$$

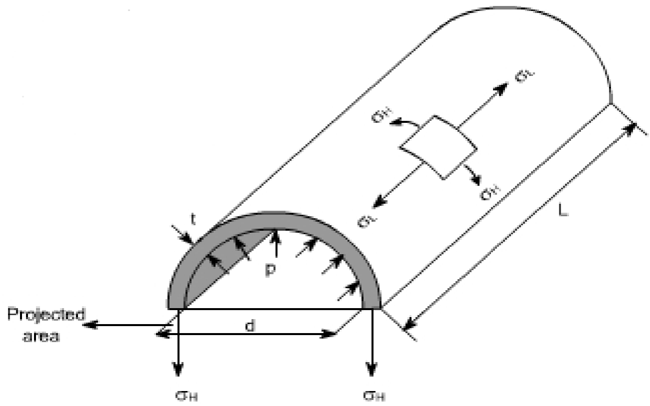


Figure 2: Balance of forces along circumferential to find hoop stress.

Longitudinal stress

Total force on the end of cylinder owing to internal pressure = pressure × area, i.e. $p \times \frac{\pi d^2}{4}$

Area of metal resisting this force=πdt

$\therefore \sigma_L = \frac{force}{area}=p\frac{\pi d^2/4}{\pi dt}=\frac{pd}{4t}$

$$\text{Longitudinal stress } \sigma_L = \frac{pd}{4t}$$

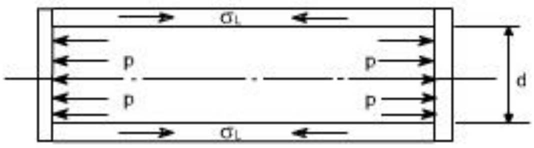


Figure 3: Balance of forces along longitudinal to find longitudinal stress.

Change in dimensions

1- Change in length

The change in length of the cylinder may be determined from the longitudinal strain,

$$\text{Longitudinal strain}=\frac{1}{E}(\sigma_L - \nu\sigma_H)$$

Note: In the above equation the radial stress is neglected because the cylinder is considered thin.

Change in length= longitudinal strain × original length

$$=\frac{1}{E}(\sigma_L - v\sigma_H)L$$

$$=\frac{pd}{4tE}(1-2v)L$$

2- Change in diameter

$$\text{Change in diameter} = \frac{pd^2}{4tE}(2-v)$$

3- Change in internal volume

$$\text{Change in internal volume} = \frac{pd}{4tE}(5-4v)V$$

Note: v is the Poisson's ratio, as stated before, which is a material property, defined as $v = -(\varepsilon_{lateral}/\varepsilon_{longitudinal})$

Thin Spherical Shell under Internal Pressure

Because of the symmetry of the sphere, the stresses set up owing to internal pressure will be two mutually perpendicular hoop or circumferential stresses of equal value and a radial stress.

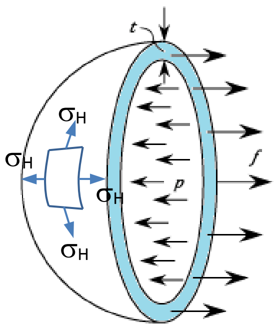


Figure 4: Thin spherical shell subjected to internal pressure.

Note: for $\frac{t}{d} < \frac{1}{20}$, the spherical vessel is considered thin, and the radial stress, σ_R, can be neglected

Force on half-sphere owing to internal pressure = *pressure* × *projected area*

$=p\frac{\pi d^2}{4}$

Resisting force $=\sigma_H \times \pi dt$

$$p\frac{\pi d^2}{4} = \sigma_H \times \pi dt$$

Circumferential or hoop stress= $\frac{pd}{4t}$

Change in internal volume

Change in internal volume=$\frac{3pd}{4tE}(1-v)V$

Cylindrical Vessel with Hemispherical Ends

a) For the cylindrical portion

Hoop stress $\quad \sigma_{Hc} = \frac{pd}{2t_c}$

and

Longitudinal stress $\sigma_{Lc} = \frac{pd}{4t_c}$

Hoop or circumferential strain =$\frac{1}{E}(\sigma_{Hc} - v\sigma_{Lc})$

$=\frac{pd}{4t_cE}(2-v)$

b) For the hemispherical ends

Hoop stress $\quad \sigma_{Hs} = \frac{pd}{4t_s}$

Hoop strain =$\frac{1}{E}(\sigma_{Hs} - v\sigma_{Hs})$

$=\frac{pd}{4t_sE}(1-v)$

Thus equating the two strains in order that there shall be no distortion of the junction,

$$\frac{pd}{4t_cE}(2-v) = \frac{pd}{4t_sE}(1-v)$$

$$\frac{t_s}{t_c} = \frac{1-v}{2-v}$$

Note: for $v = 0.3$ (steel), $\frac{t_s}{t_c} = \frac{0.7}{1.7} = 0.4117$

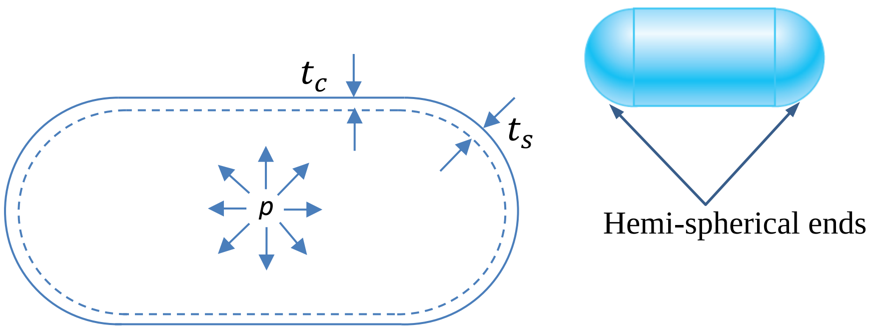


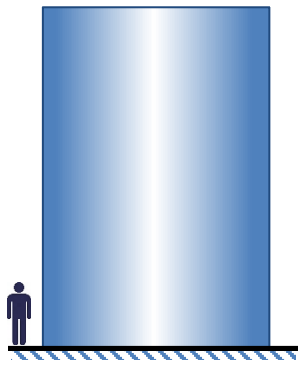
Figure 5: Cylindrical vessel with hemispherical ends

Example 1:

A water tank of 8 m diameter and 12 m high. If the tank is to be completely filled, determine the thickness of the tank plating if the stress is limited to 40 MPa.

Solution:

The maximum stress in cylindrical pressure vessel occur at hoop



$$\sigma_H = \frac{pd}{2t}$$

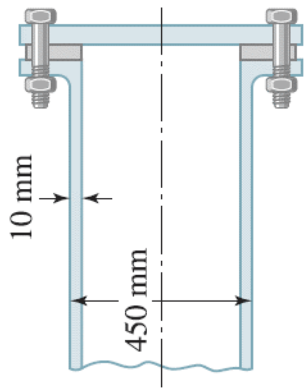
Pressure $(p)=\rho g h = 1000 \times 9.81 \times 12 = 117.72\ kPa$

$$40 \times 10^6 = \frac{117.72 \times 10^3 \times 8}{2t}$$

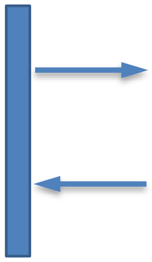
$\therefore\ t$ =11.8mm (Answer)

Example 2:

The pipe carrying steam at 3.5 MPa has an outer diameter of 450 mm and a wall thickness of 10 mm. A gasket is inserted between the flange at one end of the pipe, and a flat plate is used to cap the end. (a) How many 40-mm-diameter bolts must be used to hold the cap on if the allowable stress in the bolts is 80 MPa, of which 55 MPa is the initial stress? (b) What circumferential stress is developed in the pipe?



Solution:



$$F_{bolts} = n \times (\sigma_{bolt} A_{bolt})$$

$$F_{steam} = \frac{p\pi D^2}{4}$$

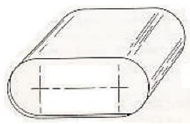
$$\frac{p\pi D^2}{4} = n\sigma_{bolt}A_{bolt}$$

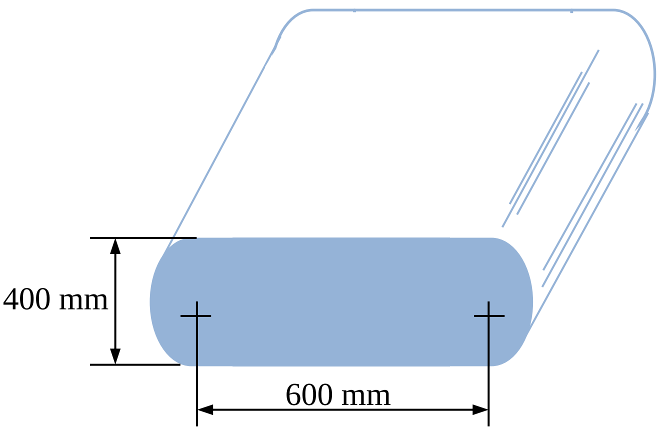
$$n = \frac{pD^2}{\sigma_{bolt}d^2} = \frac{(3.5\times10^6\times(0.45-0.02)^2}{(80-55)\times10^6\times(0.04)^2} = 16.2$$ Use 17 bolts (Answer)

$$\sigma_H = \frac{pD}{2t} = \frac{3.5\times10^6\times(0.45-0.02)}{2\times0.01} = 75.3 \text{ MPa}$$ (Answer)

Example 3:

The tank, shown in the figure, is fabricated from steel plate. Determine the minimum thickness of plate which may be used if the stress is limited to 40 MPa and the internal pressure is 1.5 MPa.





Solution:

Checking for hoop:

Hoop stress= $\frac{p*\left(600\times10^{-3}+2\times(400*10^{-3}/2)\right)L}{2Lt}$

$$40\times10^{6}=\frac{1.5\times10^{6}\times(0.6+0.4)L}{2Lt}$$

t= 0.01875m= 18.75 mm

Checking for longitudinal stress: $\sigma_L=\frac{p\left(\frac{\pi}{4}(0.4)^2+0.6\times0.4\right)}{(\pi\times0.4+2\times0.6)t}$

$$40\times10^{6}=\frac{1.5\times10^{6}\left(\frac{\pi}{4}(0.4)^2+0.6\times0.4\right)}{(\pi\times0.4+2\times0.6)t}$$

t=5.58 mm

Therefore, the minimum safe thickness is *t* =18. 75 mm (Answer)

Asst. Prof. Dr. Mohsin Noori Hamzah
Asst. Lecturer Rasha Mohammed