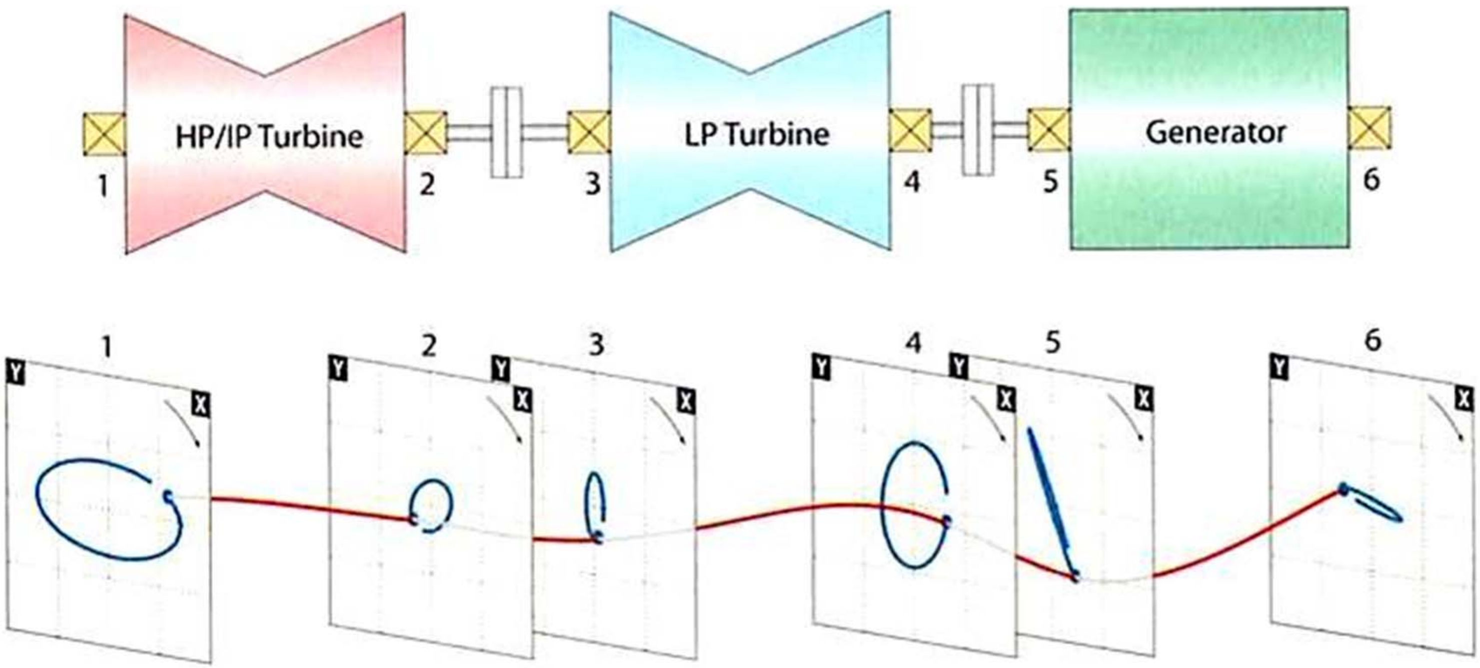

Rotor Dynamics

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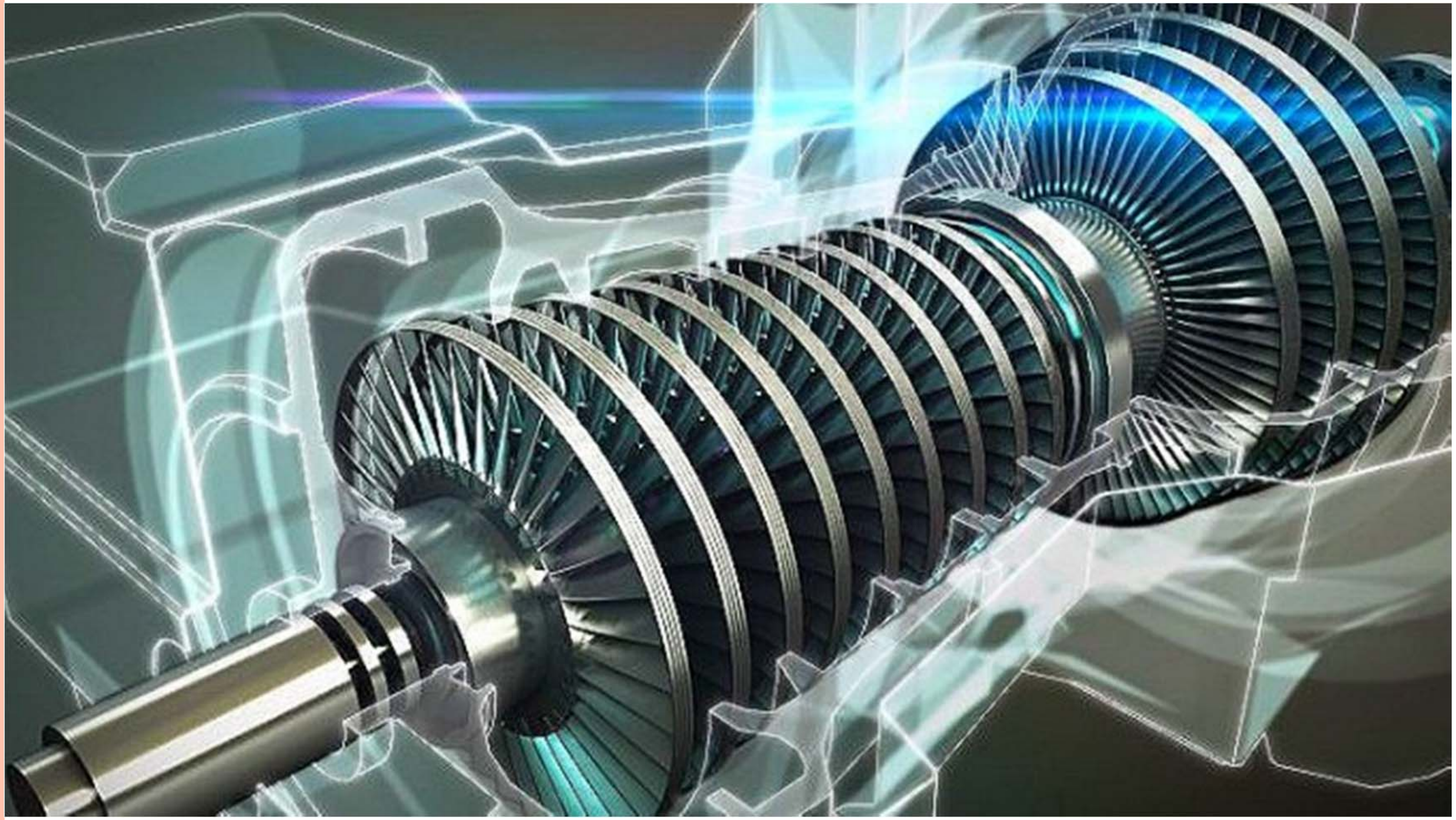
Fourth Year: First Semester

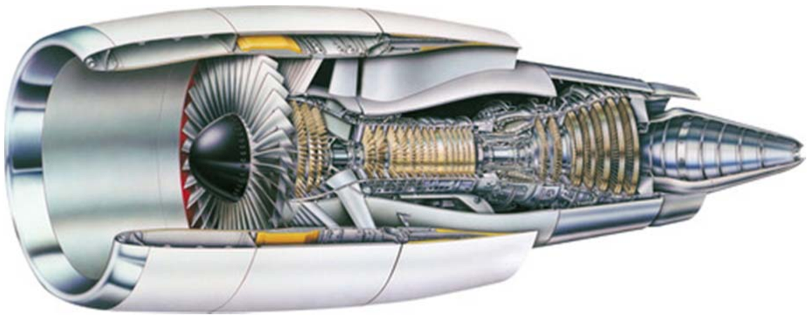
1ST LECTURE

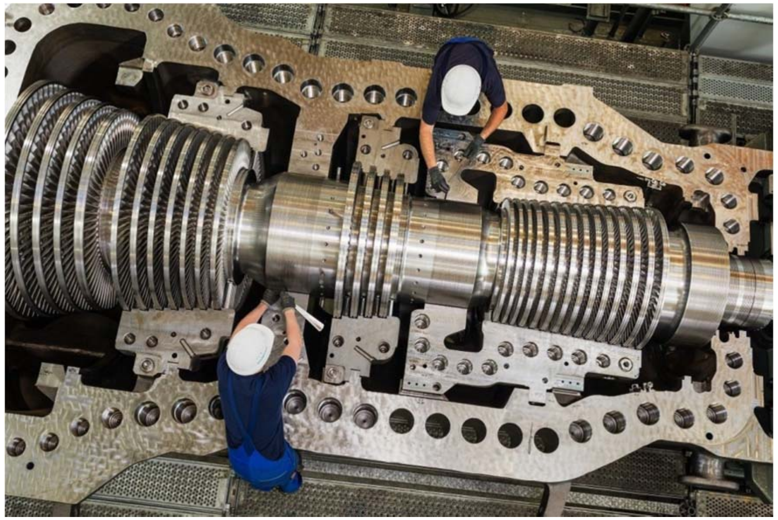


Basic Concepts

Rotor dynamics

is a specialized branch of applied mechanics concerned with the behavior and diagnosis of rotating structures. It is commonly used to analyze the behavior of structures ranging from jet engines and steam turbines to auto engines and computer disk storage.



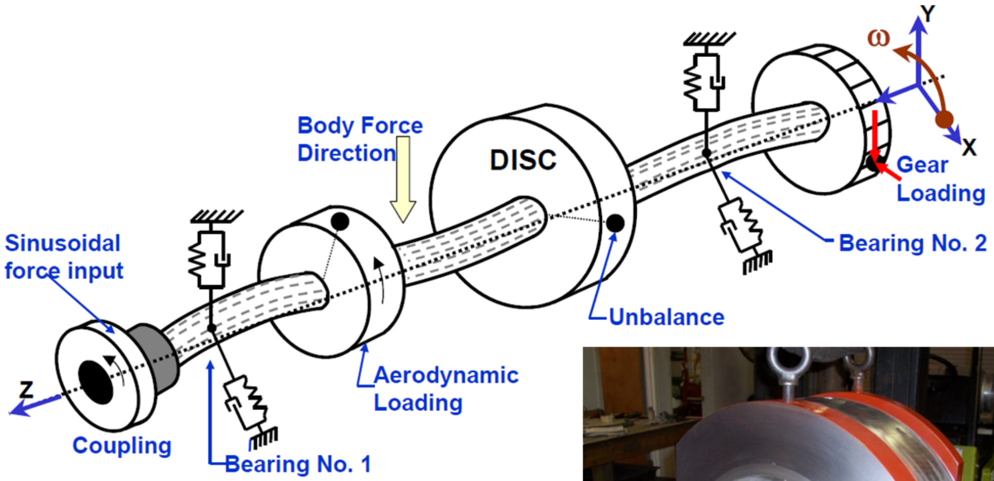


- **Rotor**

  is a moving component of an mechanical rotating system in the turbine, electric motor, electric generator, or alternator. Its rotation is due to the produced torque around the rotor's axis.

- **Bearings**

In rotor systems where bearings are far more flexible than the shaft, it is the bearings which will have the greatest influence on the motion of the rotor. Such rotors may be idealized as the rigid rotor. It is assumed that the shaft has no flexibility, and bearings are assumed to behave as linear springs and dampers,



Couplings

Turbine to Gear box : Most of the manufacturers are using flexible coupling , the reason might be offset alignment, High speed (5000–10000rpm) and the flexible couplings with the distance piece can bare up to 2–3 mm Misalignment.

Gear Box to Generator : Rigid coupling and Flexible coupling..

In Higher Capacity turbines

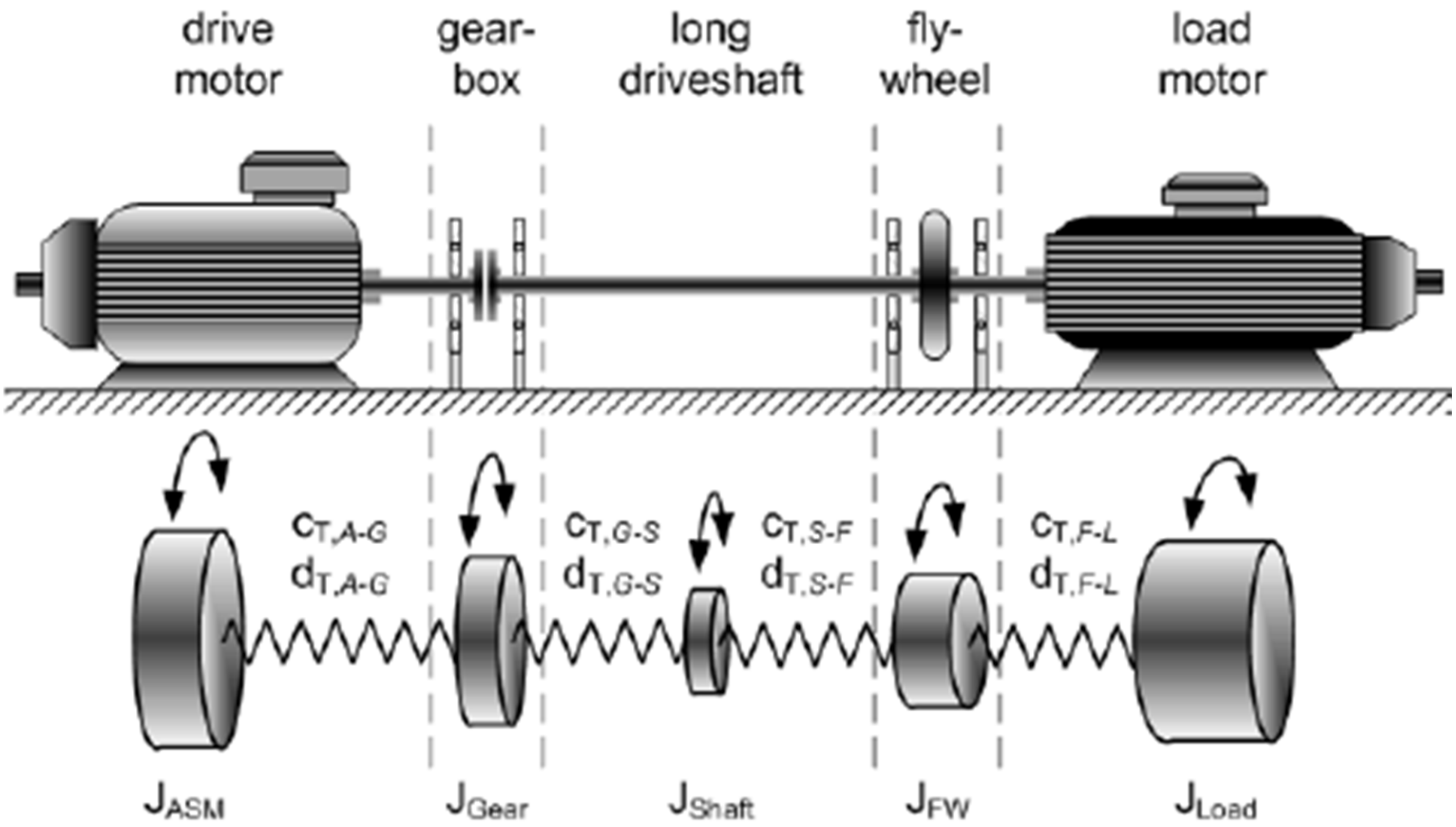
, Turbines are Directly

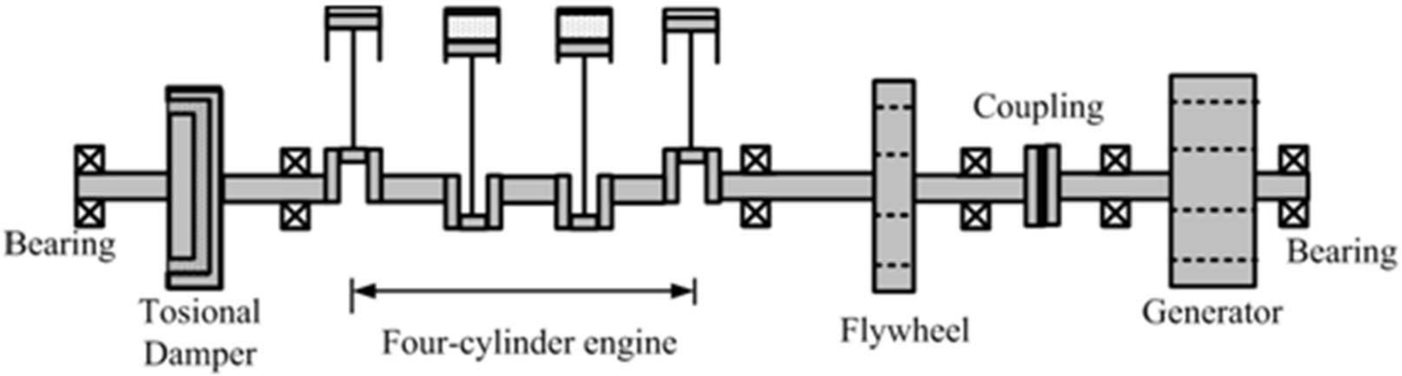
Coupled to Generator at

3000Rpm , so Rigid couplings are used.

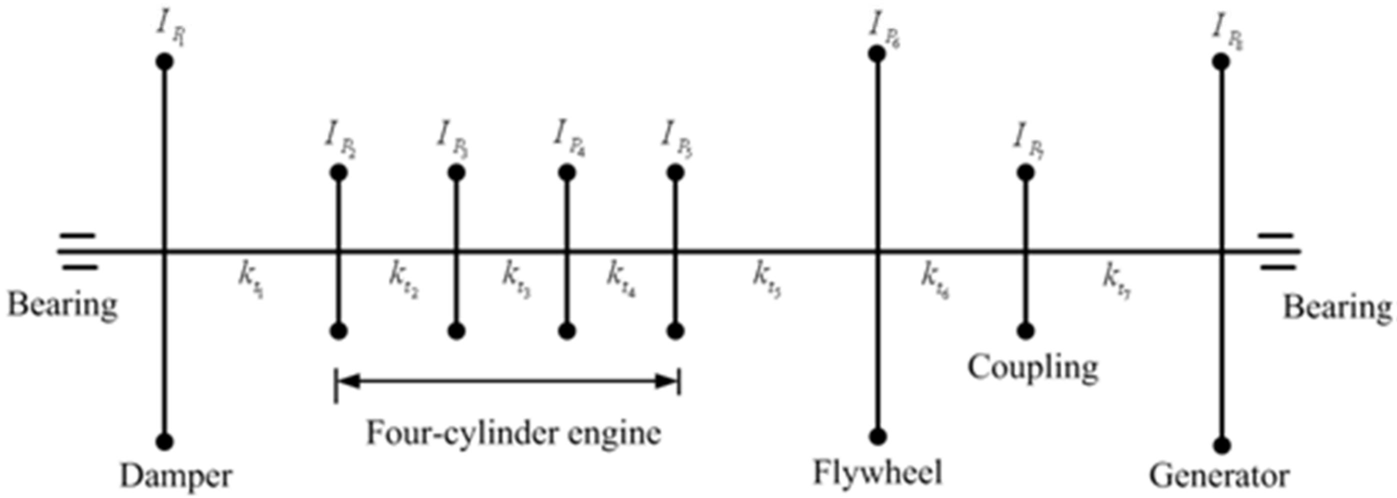
Angular and parallel alignment must be more accurate in Rigid coupling shafts than Flexible couplings.

- **Mathematical Model**

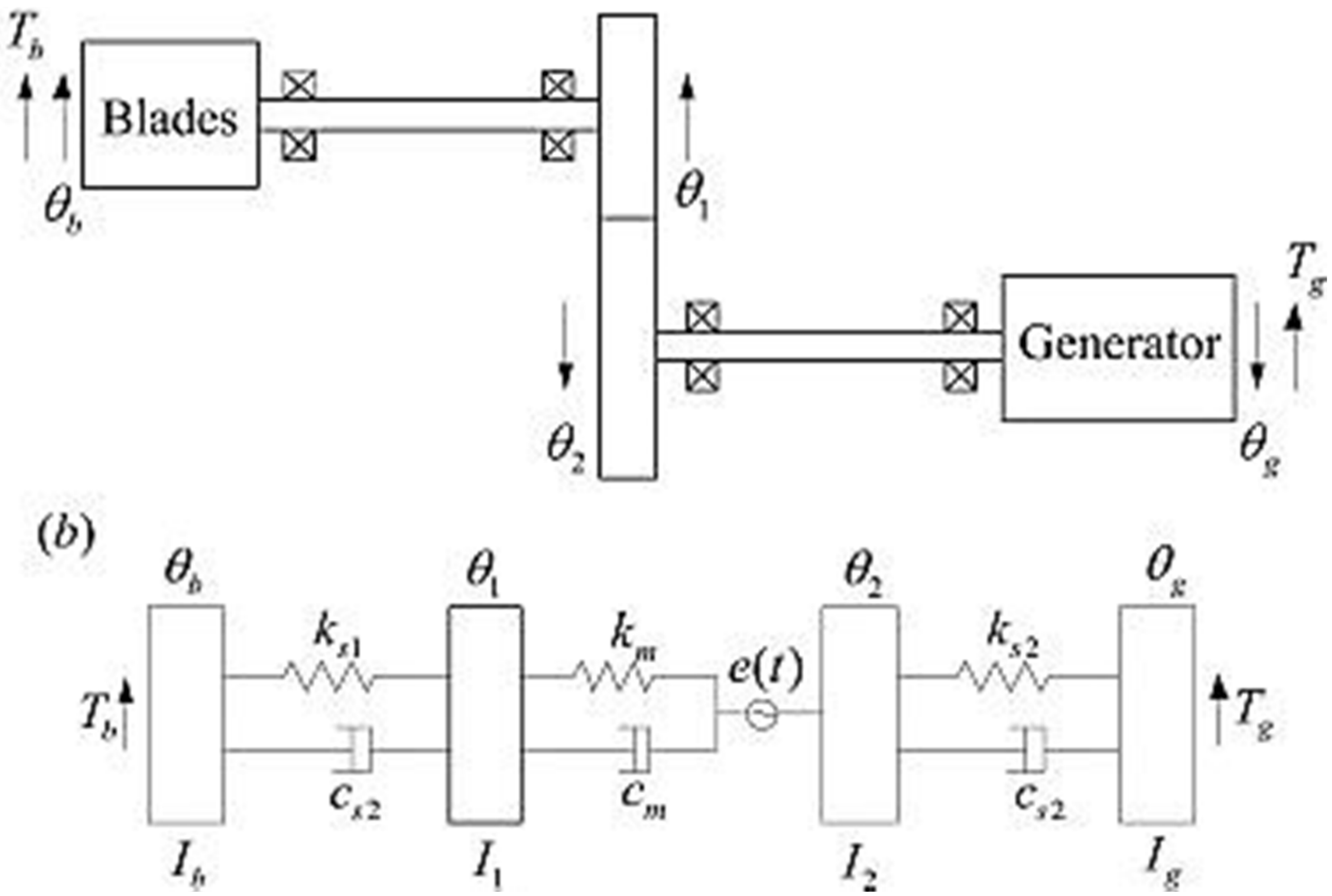



Coupling
Bearing
Bearing
Tosional
Damper
Four-cylinder engine
Flywheel
Generator

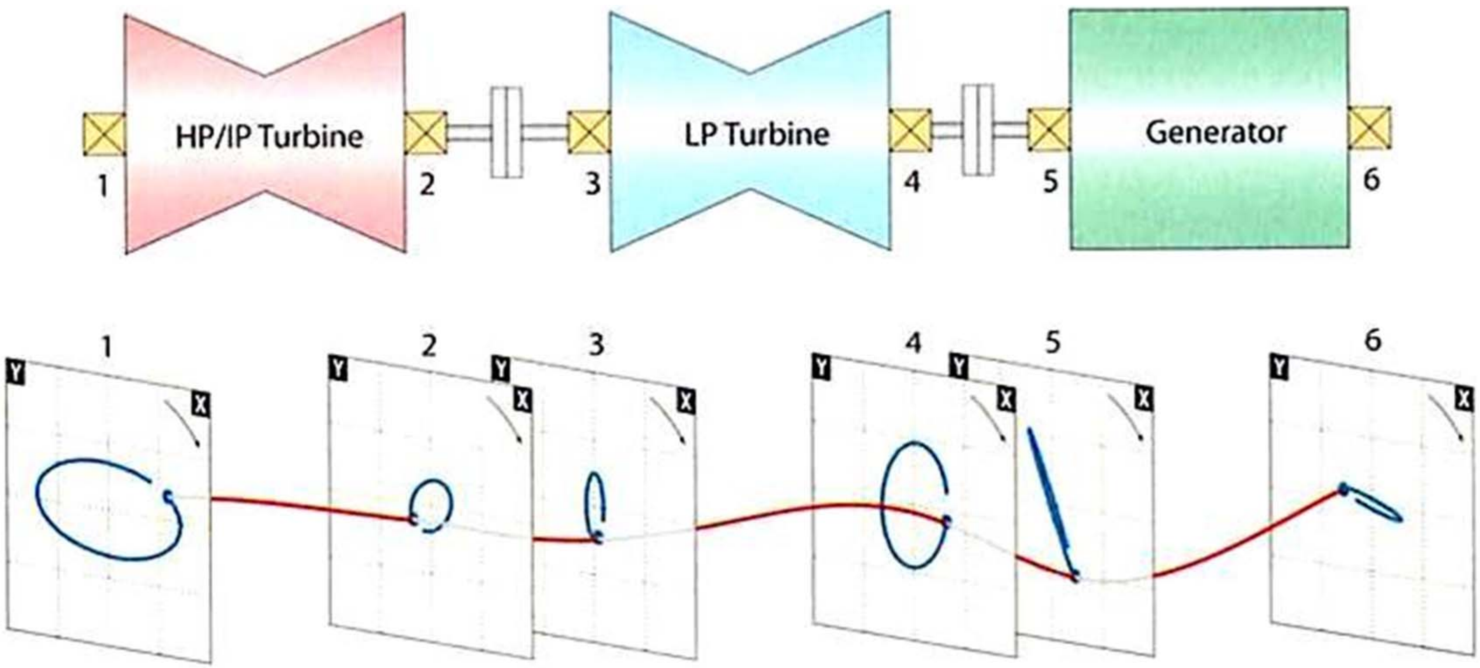
(a)


Bearing
Bearing
Coupling
Four-cylinder engine
Damper
Flywheel
Generator

(b)



T_b
Blades
θ_b
θ_1
θ_2
Generator
T_g
θ_g
(b)
θ_b
θ_1
θ_2
θ_g
k_{s1}
k_m
$e(t)$
k_{s2}
T_b
T_g
c_{s2}
c_m
c_{s2}
I_b
I_1
I_2
I_g



ROTOR DYNAMICS

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2ND LECTURE

Calculation of Equivalent Second Moment of Area for Stepped Shaft

Equivalent inertia technique

An approximate method is provided where by the fundamental natural frequency of a non-uniform cross-sectional shaft may be determined with reasonable accuracy.

The essential part of Rayleigh's method is the assumption of a dynamic deflection curve and the application of the energy method to determine the fundamental frequency.

If therefore, the resulting deflection curve for a stepped shaft could be replaced with a uniform shaft, having an equivalent second moment of area, such that the resulting deflection is not too far removed from the true shape, then the calculated natural frequency using the approximate deflected shape will be sufficiently accurate for general purposes.

Equivalent 2nd moment of area for over hanged rotor

Consider the stepped cantilever as illustrated in Fig.(1). The cantilever may be sub-divided into divisions equal to the number of changes in the shaft cross-section, applying to the right hand side of each section the equivalent static loading and couple produced by end load (W) The equations for deflection and slope for a uniform cantilever loaded at one end are:

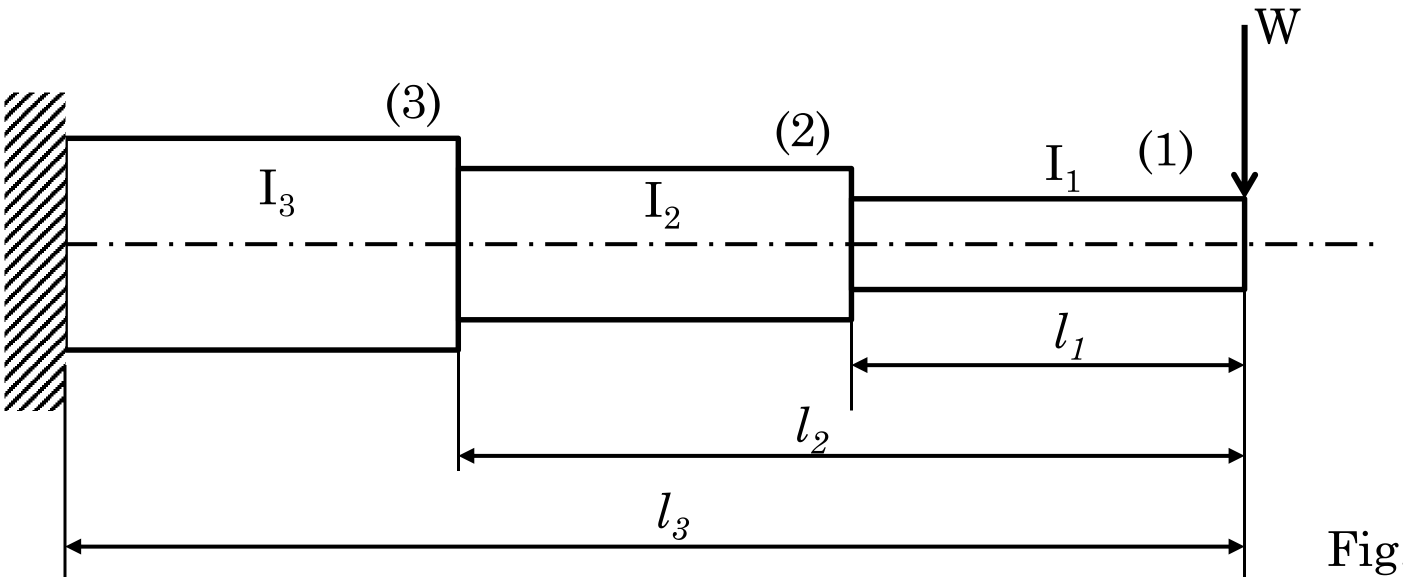


Fig.(1a)

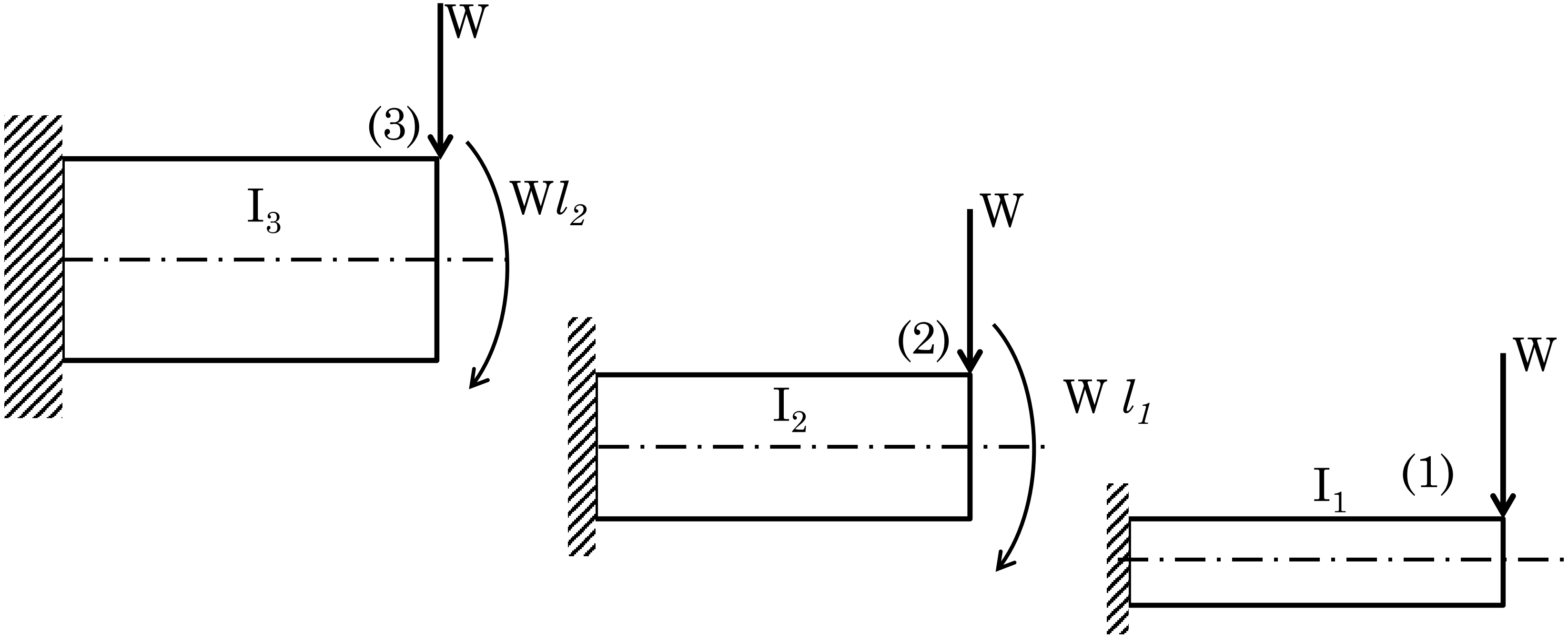


Fig.(1b)

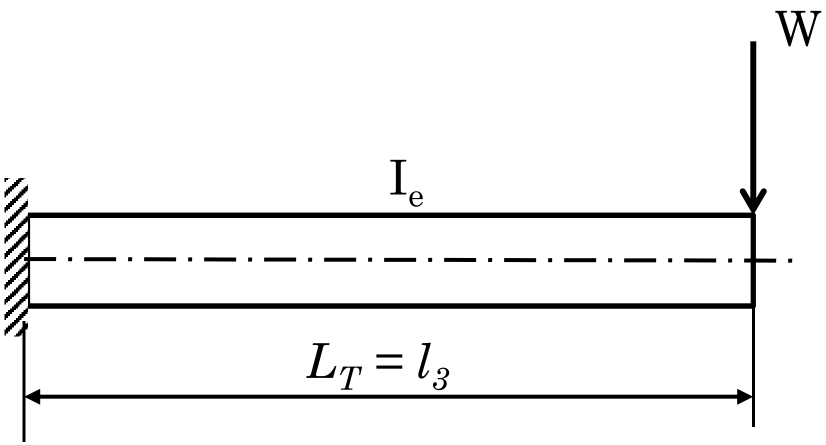


Fig.(2)

Deflection due to an end load= $\frac{WL^3}{3EI}$

Deflection due to an end couple = $\frac{ML^2}{2EI}$

Slope due to an end load= $\frac{WL^2}{2EI}$

Slope due to an end load= $\frac{ML}{EI}$

eq.1

Where:

W = applied end load (N)

L = beam length (m)

E = young modulus of the beam material

I = second moment of area of a uniform cross-sectioned beam

M = applied end moment

By using eq.1 , it is possible to determine the deflections for a stepped shaft at stations (1) , (2) and (3) in fig.1 as follows:

At station (3):

$$Y_3 = Deflection = \frac{W(L_3-L_2)^3}{3EI_3} + \frac{(WL_2)(L_3-L_2)^2}{2EI_3} \qquad \text{...............eq.2}$$

$$\theta_3 = Slope = \frac{W(L_3-L_2)^2}{2EI_3} + \frac{(WL_2)(L_3-L_2)}{EI_3} \qquad \text{...............eq.3}$$

At station (2):

$$Y_2 = Deflection = \frac{W(L_2-L_1)^3}{3EI_2} + \frac{(WL_1)(L_2-L_1)^2}{2EI_2} + Y_3 + \theta_3(L_2 - L_1) \quad \text{...............eq.4}$$

$$\theta_2 = Slope = \frac{W(L_2-L_1)^2}{2EI_2} + \frac{(WL_1)(L_2-L_1)}{EI_2} + \theta_3 \qquad \text{...............eq.5}$$

At station (1):

$$Y_1 == \frac{WL^3}{3EI} + Y_2 + \theta_2 L_1 \quad \text{...............eq.6}$$

If equations (2 up to 5) are substituted into eq.6 , then expanding the produced equation and cancelling the identical terms, the equation of deflection can be written as:

$$Y_1 = \frac{W}{3E}\left[\frac{L_1{}^3}{I_1} + \frac{(L_2{}^3 - L_1{}^3)}{I_2} + \frac{(L_3{}^3 - L_2{}^3)}{I_3}\right] \text{.............. eq.7}$$

Consider fig.2 a cantilever with a uniform cross-section, the deflection at the free end will be expressed as:

$$Y_e = \frac{W L_T{}^3}{3 E I_e} \text{ .... eq.8} \quad (L_T = L_3)$$

After equating eqs. 7 & 8, the equivalent second moment of inertia for the stepped shaft will be obtained from the following equation:

$$I_e = \frac{L_T{}^3}{\sum_1^n \frac{L_n{}^3 - L_{n-1}{}^3}{I_n}} \text{ .... eq.9} \ \text{(n = the number of rotor stations)}$$

Equivalent 2nd moment of area for the rotor between two bearing

If the shaft section between the two bearings in fig.(3) is now considered, the initial assumption made for this section of shaft was at the position where dy/dx = 0 the shaft could be considered as being encastre, in practice the problem would be in finding this position. However, provided the loading is spread approximately over the entire shaft system, it would not be unreasonable to use the center of gravity as the position where the cantilever may be considered as encastre.

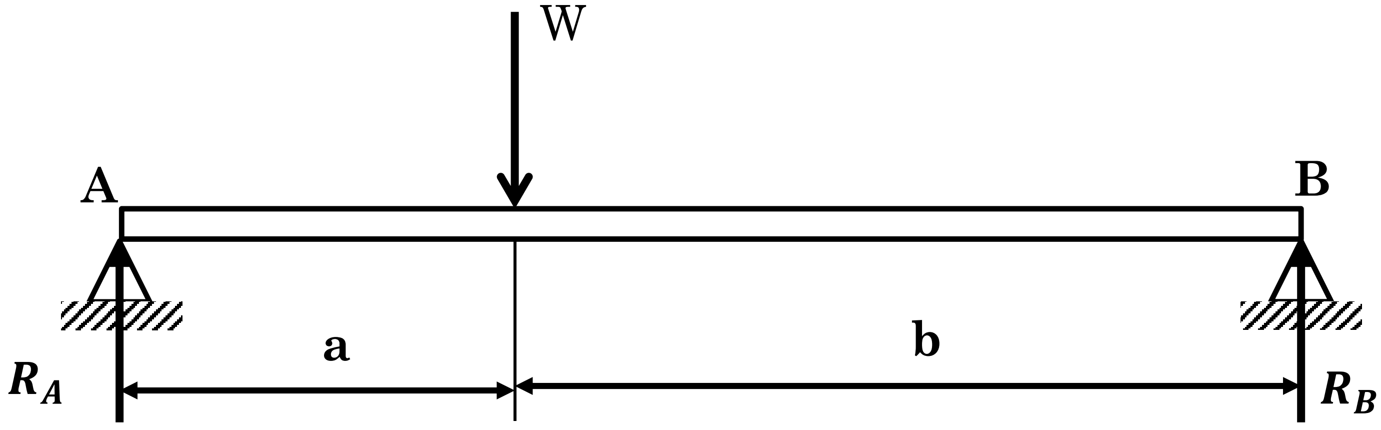


Fig.3

Hence the deflected cantilever can be equated to the deflection of a simply supported beam with the loading off-set to one side.

Further, by averaging the inertias of the two cantilevers the equivalent inertia for the complete system may be deduced.

Consider a simply supported beam subjected under load W as illustrated in Fig. (3). The reactions R_A and R_B can be determined by taking the algebraic summation of the moments about the point B and A respectively. Thus:

$$R_A = \frac{Wb}{a+b} \quad and \qquad R_B = \frac{Wa}{a+b} \qquad \text{…………… eq.10}$$

The center of gravity for the simply supported beam shown in Fig. (3) is at the point of the applied load W.

Hence, relaying on the basic assumption designated previously, the deflected beam can be considered as two cantilevers being encastre at the section where the load W is applied.

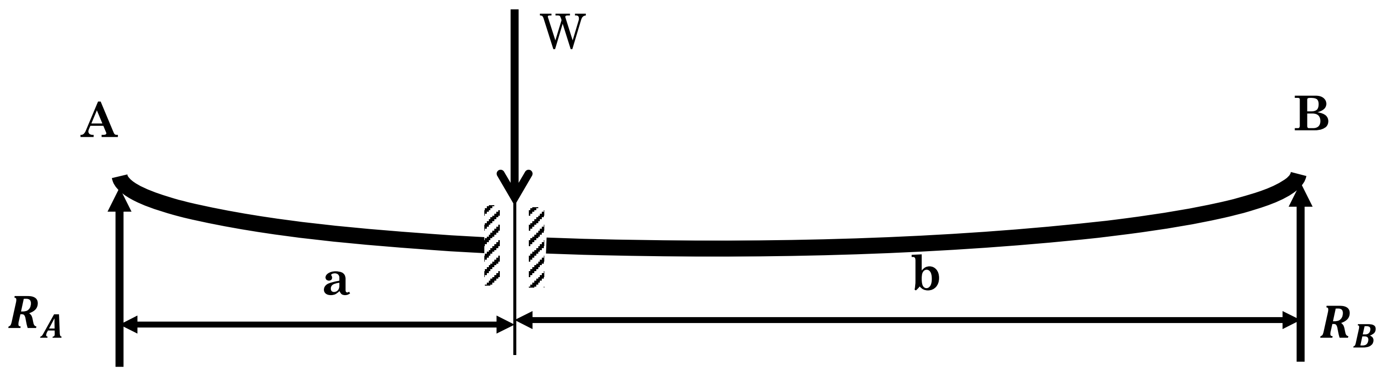


Fig.4

Consider Fig.(4). The deflection of the left cantilever due to an end load R_A is:

$$\delta_A = \frac{R_A a^3}{3EI_{eL}} \text{ .... eq.11}$$

Similarly, the deflection of the Right Cantilever due the end load R_B is:

$$\delta_B = \frac{R_B b^3}{3EI_{eR}} \text{ .... eq.12}$$

where, I_{eL} is the equivalent second moment of area for the left. stepped cantilever.

I_{eR} is the. equivalent second moment of area for the right stepped cantilever.

By substituting eqs.10 into eqs. 11 & 12 , δ_a and δ_b become:

$$\delta_A = \frac{Wba^3}{3EI_{eL}(a+b)} \quad \& \quad \delta_B = \frac{Wab^3}{3EI_{eR}(a+b)} \quad \ldots\ldots\ldots\ldots\ldots \text{eq.13}$$

If fig.(4) is again plotted with the lines AC, EG, BD, CB and CGD as illustrated in fig.(5)

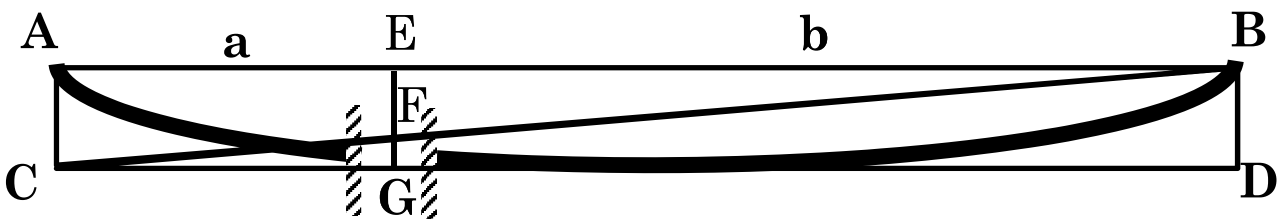


Fig.5

The line EG will represent the deflection of the simply supported beam at the point E where the load W is applied. Hence from the similarity of the of the two triangles ABC and EBF, EF can be determined. That is:

$$\frac{EF}{AC} = \frac{EB}{AB} \text{ gives , } EF = \frac{(EB)(AC)}{(AB)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{eq.14}$$

Similarly, from the similarity of the triangles GCF and DCB, FG can be found. Thus:

$$FG = \frac{(CG)(BD)}{(CD)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{eq.15}$$

But $AB = CD = AE + EB = a + b$

$AC = \delta_a$ and $BD = \delta_b$

$CG = AE = a$ and $GD = EB = b$ eq.16

By sub. eq.16 into eqs. 14 & 15 , EF and FG become:

$$EF = \frac{b\delta_a}{a+b} \quad \text{and} \quad FG = \frac{a\delta_b}{a+b} \quad \ldots\ldots\ldots\ldots\ldots \text{eq.17}$$

But $EG = EF + FG$ eq.18

Then;

$$EG = \frac{b\delta_a}{a+b} + \frac{a\delta_b}{a+b} \quad \ldots\ldots\ldots\ldots\ldots \text{eq.19}$$

And from eq.13 and eq.19 , EG becomes:

$$EG = \frac{Wb^2a^3}{3EI_{eL}(a+b)^2} + \frac{Wa^2b^3}{3EI_{eR}(a+b)^2} \quad \ldots\ldots\ldots\ldots\ldots \text{eq.20}$$

From eq.9 , I_{eL} and I_{eR} can be expressed as follows:

$$I_{eL} = \frac{a^3}{\left(\sum_1^n \frac{L_n{}^3 - L_{n-1}{}^3}{I_n}\right)_a} \quad and \quad I_{eR} = \frac{b^3}{\left(\sum_1^n \frac{L_n{}^3 - L_{n-1}{}^3}{I_n}\right)_b} \text{ .... eq.21}$$

Then from eq.21 and eq.20 , eq.21 becomes:

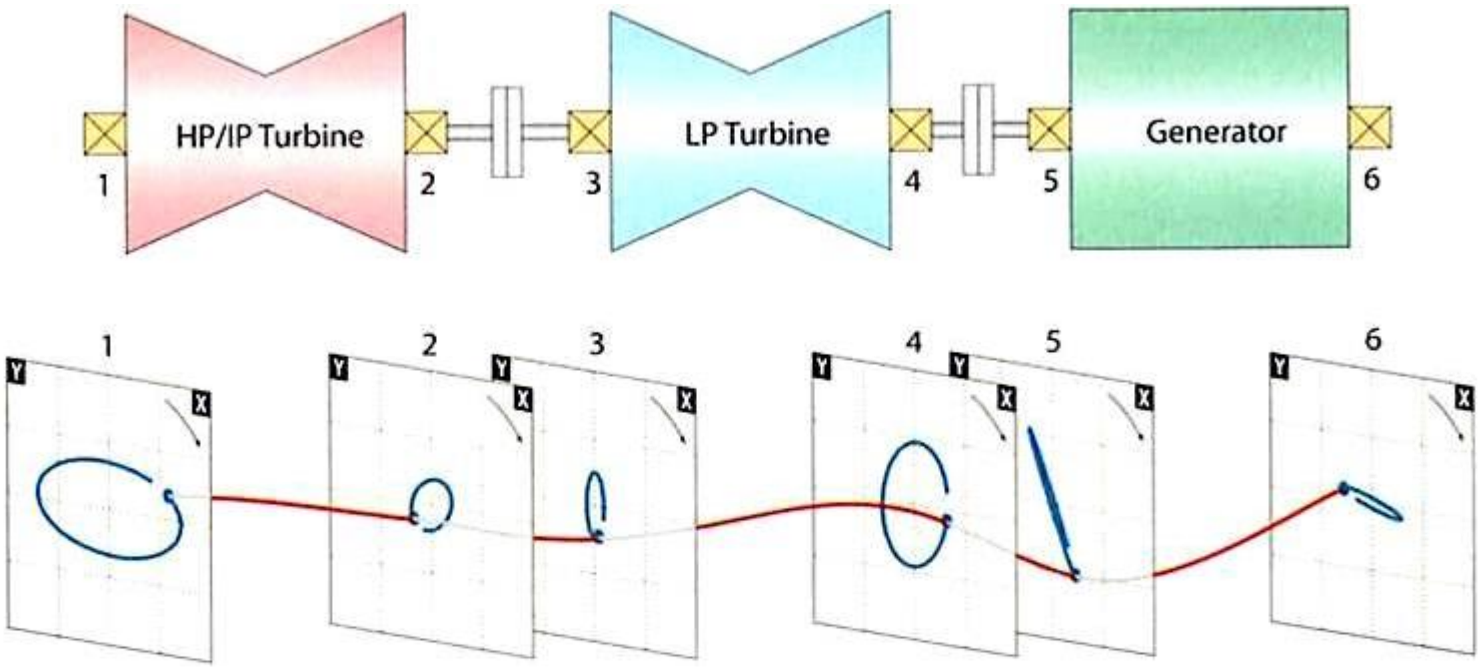
$$EG = \frac{W}{3E(a+b)^2}\left[b^2\left(\sum_1^n \frac{L_n{}^3 - L_{n-1}{}^3}{I_n}\right)_a + a^2\left(\sum_1^n \frac{L_n{}^3 - L_{n-1}{}^3}{I_n}\right)_b\right] \text{ .... eq.22}$$

The deflection of a uniform simply supported beam subjected under load W can be expressed as follows:

$$\delta_W = \frac{W}{3E}\frac{a^2 b^2}{(a+b)I_E} \text{ .... eq.23}$$

From equating eqs. 22 and 23 and cancelling the identical terms, I_E can be expressed as follows:

$$I_E = \frac{(a+b)a^2 b^2}{b^2\left(\sum_1^n \frac{L_n{}^3 - L_{n-1}{}^3}{I_n}\right)_a + a^2\left(\sum_1^n \frac{L_n{}^3 - L_{n-1}{}^3}{I_n}\right)_b} \text{ .... eq.24}$$



Rotor Dynamics

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Fourth Year: First Semester

3RD & 4TH LECTURES

Rayleigh method for estimation of the fundamental natural frequency

Rayleigh's method can be applied to find the fundamental natural frequency of continuous systems. This method is much simpler than exact analysis for systems with varying distributions of mass and stiffness. Although the method is applicable to all continuous systems,

we shall apply it only to beams in this section, consider the beam shown in Fig. (1).

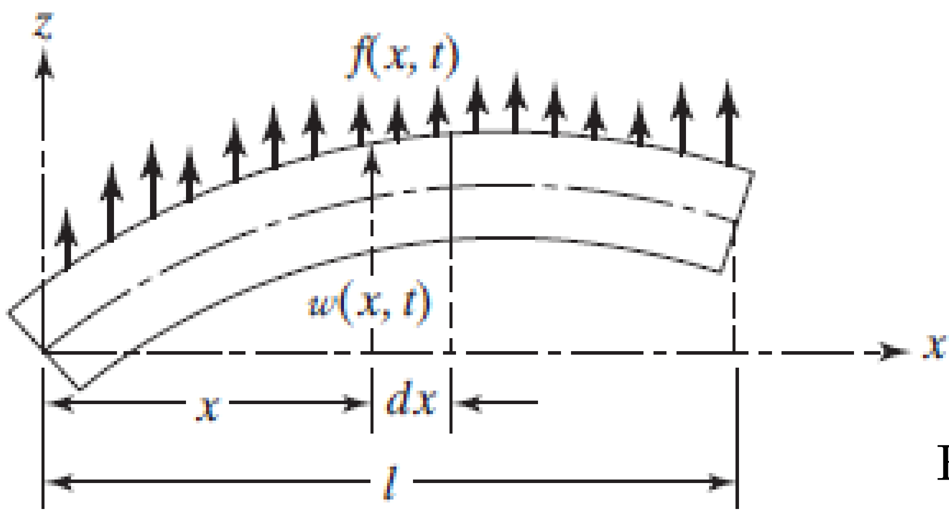


Fig.(1)

In order to apply Rayleigh s method, we need to derive expressions for the maximum kinetic and potential energies and Rayleigh s quotient. The kinetic energy of the beam can be expressed as:

$T = \frac{1}{2}\int_0^l \dot{w}^2\, dm = \frac{1}{2}\int_0^l \dot{w}^2\, \rho A(x) dx$ eq.1

The maximum kinetic energy can be found by assuming a harmonic variation $w(x,t) = W(x)\cos\omega t$:

$T_{max} = \frac{\omega^2}{2}\int_0^l W^2(x)\, \rho A(x) dx$................. eq.2

The potential energy of the beam V is the same as the work done in deforming the beam. By disregarding the work done by the shear forces, we have

$V = \frac{1}{2}\int_0^l M\, d\theta$................ eq.3

Where M is the bending moment given by $M = -EI\frac{d^2w}{dx^2}$

And θ is the slope of the deformed beam given by $\theta = \frac{\partial w}{\partial x}$

Thus Eq.(3) can be rewritten as:

$$V = \frac{1}{2}\int_0^l \left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 w}{\partial x^2}\,dx = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \quad \ldots\ldots\ldots\ldots\ldots \text{eq.4}$$

Since the maximum value of $w(x, t)$ is $W(x)$, the maximum value of V is given by

$$V_{max} = \frac{1}{2}\int_0^l EI\left(\frac{d^2W(x)}{dx^2}\right)^2 dx \quad \ldots\ldots\ldots\ldots\ldots \text{eq.5}$$

By equating T_{max} to V_{max} we obtain Rayleigh s quotient:

$$\omega^2 = \frac{EI}{\rho A}\frac{\int_0^l\left(\frac{d^2W(x)}{dx^2}\right)^2 dx}{\int_0^l W(x)^2 dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{eq.6}$$

Whereas the same formula for lumped rotor system is:

$$\omega = \sqrt{\frac{g\sum_{i=1}^n m_i W(x)_i}{\sum_{i=1}^n m_i {W(x)_i}^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{eq.7}$$

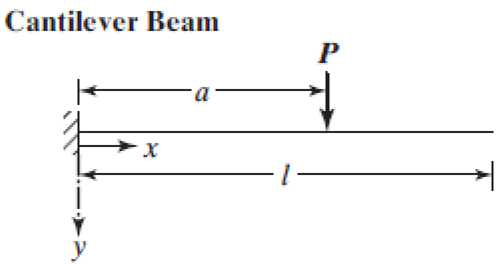
Where:

m_i is the mass or load applied at the i^{th} station

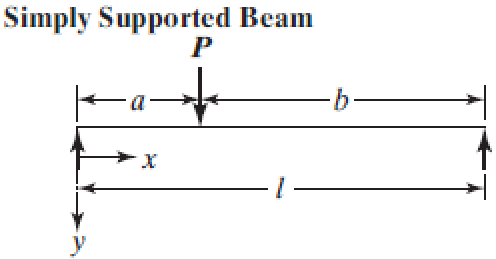
$W(x)_i$ is the deflection of the beam at the i^{th} station

g is the gravitational acceleration, its unit depends on $W(x)_i$ units.

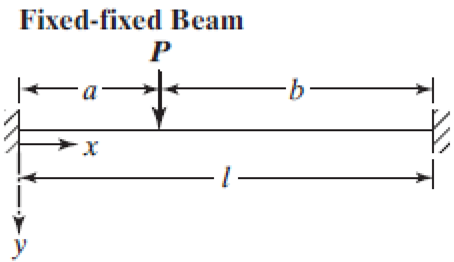
Useful formula for beam deflection noting that y(x) = W(x):



$$y(x) = \begin{cases} \dfrac{Px^2}{6EI}(3a - x); & 0 \le x \le a \\ \dfrac{Pa^2}{6EI}(3x - a); & a \le x \le l \end{cases}$$

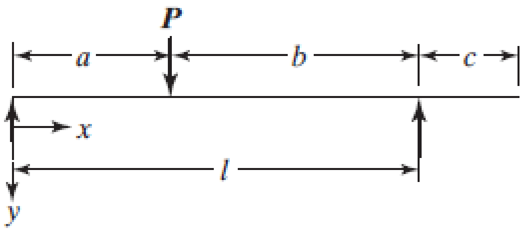


$$y(x) = \begin{cases} \dfrac{Pbx}{6EIl}(l^2 - x^2 - b^2); & 0 \le x \le a \\ \dfrac{Pa(l - x)}{6EIl}(2lx - x^2 - a^2); & a \le x \le l \end{cases}$$



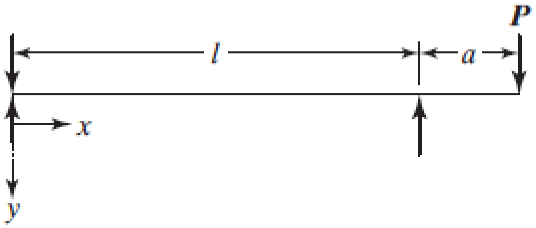
$$y(x) = \begin{cases} \dfrac{Pb^2x^2}{6EIl^3}[3al - x(3a + b)]; & 0 \le x \le a \\ \dfrac{Pa^2(l - x)^2}{6EIl^3}[3bl - (l - x)(3b + a)]; & a \le x \le l \end{cases}$$

Simply Supported Beam with Overhang

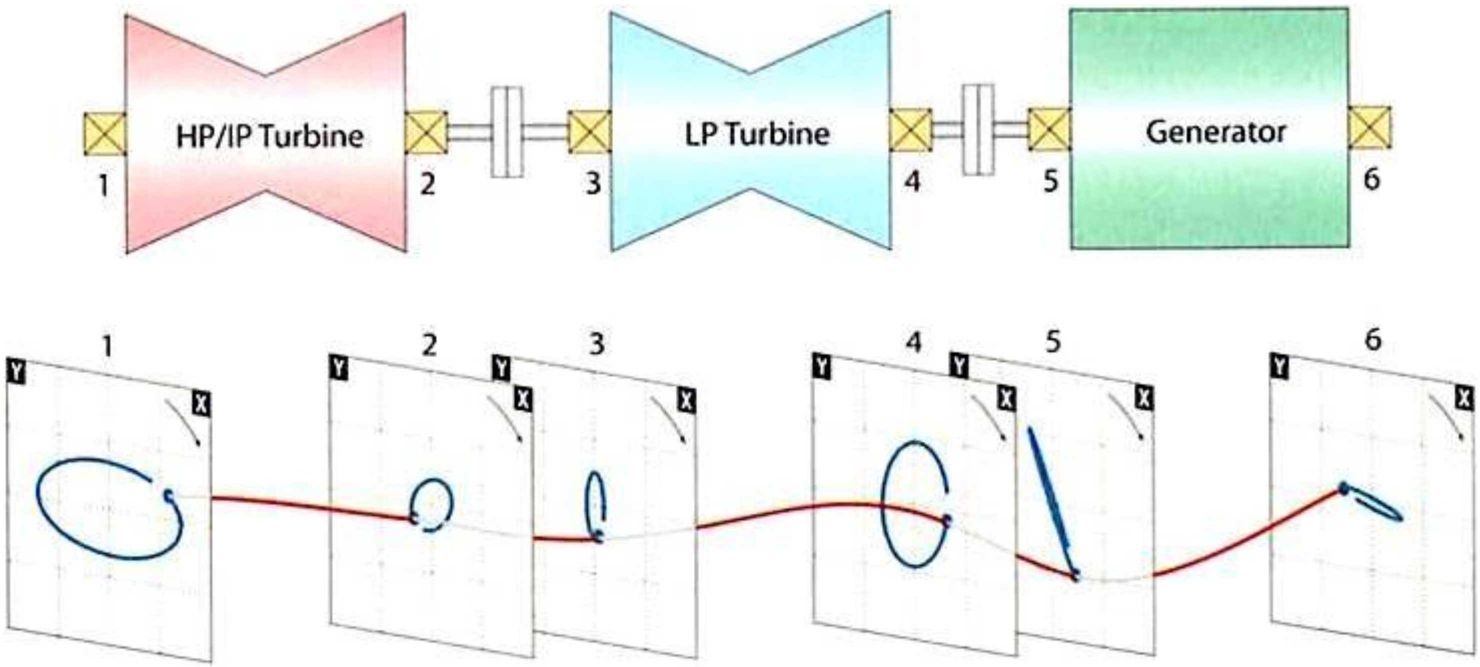


$$y(x) = \begin{cases} \text{Same as in case of simply supported beam} \\ \text{for } 0 \le x \le a \text{ and } a \le x \le l \\ \dfrac{Pa}{6EIl}(l^2 - a^2)(x - l); \quad l \le x \le l + c \end{cases}$$

Simply Supported Beam with Overhanging Load



$$y(x) = \begin{cases} \dfrac{Pax}{6EIl}(x^2 - l^2); \quad 0 \le x \le l \\ \dfrac{P(x - l)}{6EIl}[a(3x - l) - (x - l)^2]; \quad l \le x \le l + a \end{cases}$$



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5TH LECTURE

Flexibility Influence Coefficients Matrix and Stiffness Matrix

Development of the stiffness matrix using stiffness influence coefficients is straight-forward. For mechanical systems, the calculation of stiffness influence coefficients requires the application of the principles of statics and little algebra.

However, the calculation of a column of stiffness influence coefficients for a structural system modeled with (n) degrees of freedom requires the solution of (n) simultaneous equations.
This leads to significant computation time for systems with many degrees of freedom.

Flexibility influence coefficients provide a convenient alternative. They are easier to calculate than stiffness influence coefficients for structural systems and the knowledge of them is sufficient for solution of the free-vibration problem.

If the stiffness matrix, K, is nonsingular, then its inverse exists. The flexibility matrix, A, is defined by

$A = K^{-1}$ eq.1

The elements of K are determined by using stiffness influence coefficients. Analogously, flexibility influence coefficients can be used to determine A. The flexibility influence coefficient **(a_{ij})** is defined as the displacement of the particle whose displacement is represented by **(x_i)** when a **unit load** is applied to the particle whose displacement is represented by **(x_j)** and no other loading is applied to the system. If **(x_j)** represents an angular coordinate, then a unit moment is applied (see Figure.1).

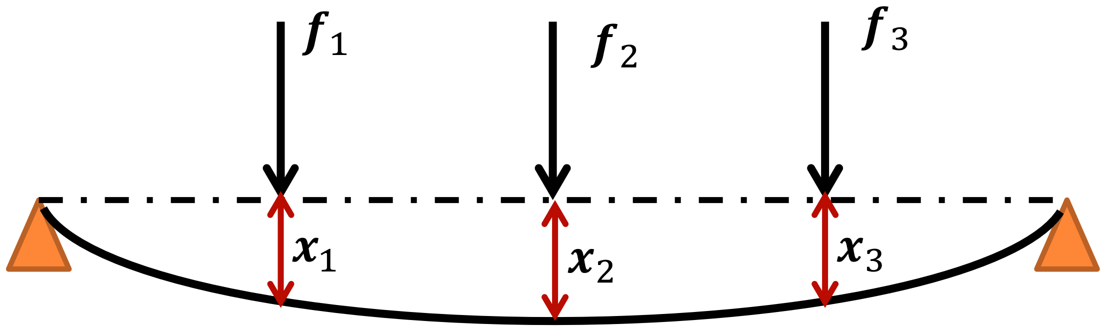


Fig.(1)

Suppose an arbitrary set of concentrated loads $\{f_1, f_2, \ldots\ldots, f_n\}$ is applied statically to an nDOF system. The load f_i is applied to the particle whose displacement is represented by **(x_i)** .

Using the definition of flexibility influence coefficients, **(x_j)** is calculated from:

$x_j = \sum_{i=1}^{n} a_{ji} f_i$ eq.2

So, for three d.o.f , eq.2 becomes:

$$x_1 = a_{11}f_1 + a_{12}f_2 + a_{13}f_3$$
$$x_2 = a_{21}f_1 + a_{22}f_2 + a_{23}f_3$$
$$x_3 = a_{31}f_1 + a_{32}f_2 + a_{33}f_3$$

And in matrix form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$\{x\} = [a]\{f\}$ eq.3

Where $[a]$ is the influence coefficient matrix of flexibility influence coefficient matrix. And $A = [a]$

Pre-multiply eq.3 by $[a]^{-1}$:

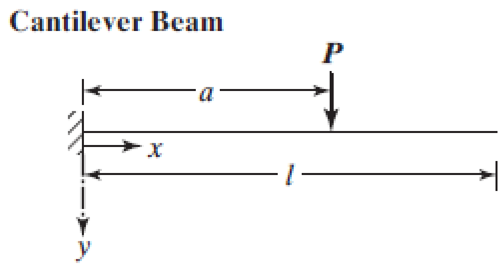
$[a]^{-1}\,\{x\} = [a]^{-1}[a]\{f\}$

$\{f\} = [a]^{-1}\ \{x\}$ eq.4

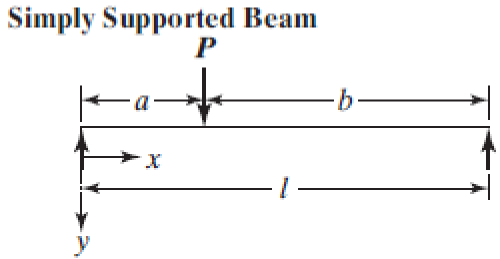
But $\{f\} = [k]\,\{x\}$ eq.5

comparison eq.4 and eq.5 gives: $[k] = [a]^{-1}$ or $[a] = [k]^{-1}$

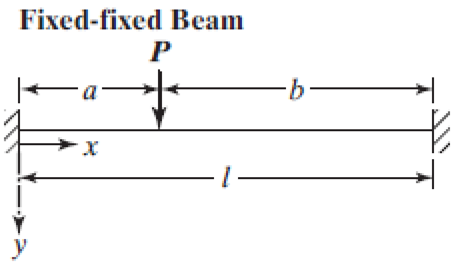
Useful formula for beam deflection noting that y(x) = W(x):



$$y(x) = \begin{cases} \dfrac{Px^2}{6EI}(3a - x); & 0 \le x \le a \\ \dfrac{Pa^2}{6EI}(3x - a); & a \le x \le l \end{cases}$$

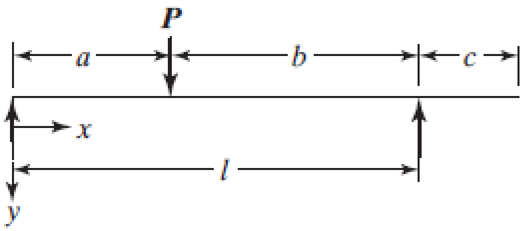


$$y(x) = \begin{cases} \dfrac{Pbx}{6EIl}(l^2 - x^2 - b^2); & 0 \le x \le a \\ \dfrac{Pa(l - x)}{6EIl}(2lx - x^2 - a^2); & a \le x \le l \end{cases}$$



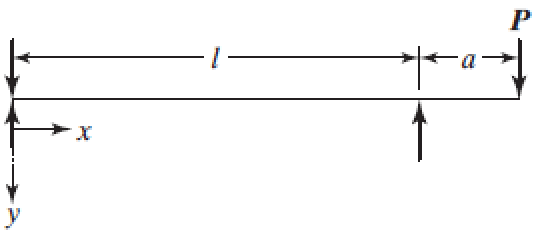
$$y(x) = \begin{cases} \dfrac{Pb^2x^2}{6EIl^3}[3al - x(3a + b)]; & 0 \le x \le a \\ \dfrac{Pa^2(l - x)^2}{6EIl^3}[3bl - (l - x)(3b + a)]; & a \le x \le l \end{cases}$$

Simply Supported Beam with Overhang

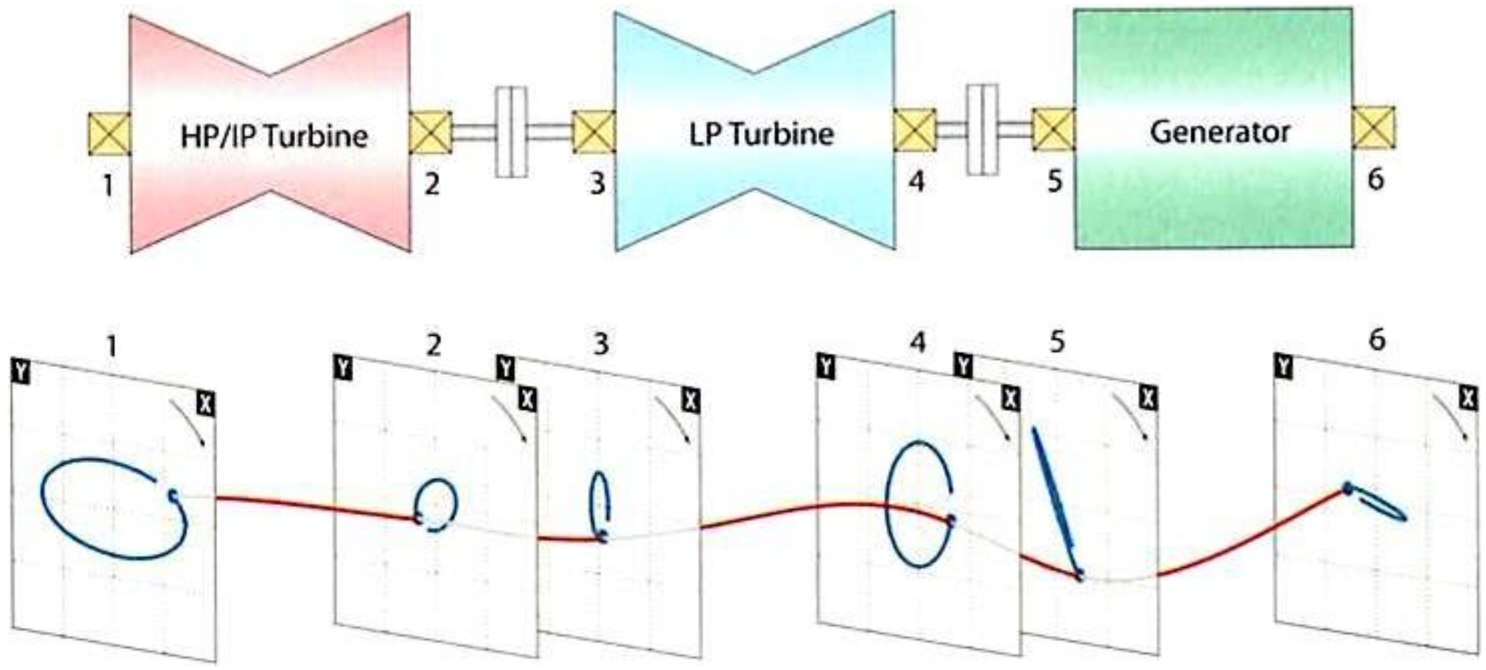


$$y(x) = \begin{cases} \text{Same as in case of simply supported beam} \\ \text{for } 0 \le x \le a \text{ and } a \le x \le l \\ \dfrac{Pa}{6EIl}(l^2 - a^2)(x - l); \quad l \le x \le l + c \end{cases}$$

Simply Supported Beam with Overhanging Load



$$y(x) = \begin{cases} \dfrac{Pax}{6EIl}(x^2 - l^2); \quad 0 \le x \le l \\ \dfrac{P(x - l)}{6EIl}[a(3x - l) - (x - l)^2]; \quad l \le x \le l + a \end{cases}$$



Rotor Dynamics

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6^{TH} LECTURE

Eigen Values and Eigen Vectors for the Rotor

for the free vibration of the undamped system of several degrees of freedom, the equations of motion expressed in matrix form become:

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{eq.1}$$

Where

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & \\ \cdot & & & \\ \cdot & & & \\ m_{n1} & m_{n2} & \ldots & m_{nn} \end{bmatrix} = mass\ matrix\ (a\ square\ matrix)$$

$$K = \begin{bmatrix} k_{11} & k_{12} & \ldots & \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} = stiffness\ matrix\ (a\ square\ matrix)$$

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \vdots \\ x_n \end{Bmatrix} = displacement\ vector\ (a\ column\ matrix)$$

If we pre-multiply eq.1 by M^{-1}, we obtain the following terms:

$$M^{-1}M = I \ (a\ unit\ matrix)$$

$$M^{-1}K = D \ (a\ system\ matrix)$$

and

$$I\ddot{X} + DX = 0 \quad \text{............... eq.2}$$

The matrix D is referred to as the *system matrix,* or the *dynamic matrix* since the dynamic properties of the system are defined by this matrix.

Assuming harmonic motion $\ddot{X} = -\lambda X$, where $\lambda = \omega^2$, eq.2 becomes:

$$[D - \lambda I]\{X\} = 0 \quad \text{............... eq.3}$$

The characteristic equation of the system is the determinant equated to zero, or

$$|D - \lambda I| = 0 \quad \text{.............. eq.4}$$

The roots λ_i of the characteristics equation are called ***eigenvalues,*** and the natural frequencies of the system are determined from them by the relationship:

$$\lambda_i = \omega^2{}_i \quad \text{............... eq.5}$$

By substituting λ_i into the matrix equation, eq.3 , we obtain the corresponding mode shape X_i which is called the ***eigenvector.***
Thus for an ***n***-degrees of freedom system, there will be ***n*** eigenvalues and ***n*** eigenvectors.

It is also possible to find the eigenvectors from the adjoint matrix of the system. If for conciseness, we make the abbreviation $\boldsymbol{B} = \boldsymbol{D} - \boldsymbol{\lambda I}$ and start with definition of the inverse

$$B^{-1} = \frac{1}{|B|} adj\ B \text{ .... eq.6}$$

We can pre-multiply by $|B|B$ to obtain: $|B|I = B\ adj\ B$

Or in terms of the original expression for B

$$|D - \lambda I|I = [D - \lambda I]adj[D - \lambda I] \text{ .... eq.7}$$

If now we let $\lambda = \lambda_i$, an eigenvalue, then the determinant on the left side of the equation is **zero** and we obtain:

$$[0] = [D - \lambda_i I]adj[D - \lambda_i I] \text{ .... eq.8}$$

The above equation is valid for all λ_i and represents $\boldsymbol{n}$ equations for the $\boldsymbol{n}$-degrees of freedom system. Comparing eq.8 with eq.3 for the ith mode

$$[D - \lambda_i I]\{X\}_i = 0$$

We recognize that the adjoint matrix, $adj[D - \lambda_i I]$, must consist of columns, each of which is the eigenvector X_i (multiplied by an arbitrary constant).

The chosen column which represents the eigenvector for the eigenvalue under consideration must satisfy the following conditions:

1- The chosen column must satisfy the shape of the mode shape under consideration.

2- The chosen column must contain the highest no. of unity.

Finding of Adj [A]

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

Solution: First find the cofactor of each element.

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

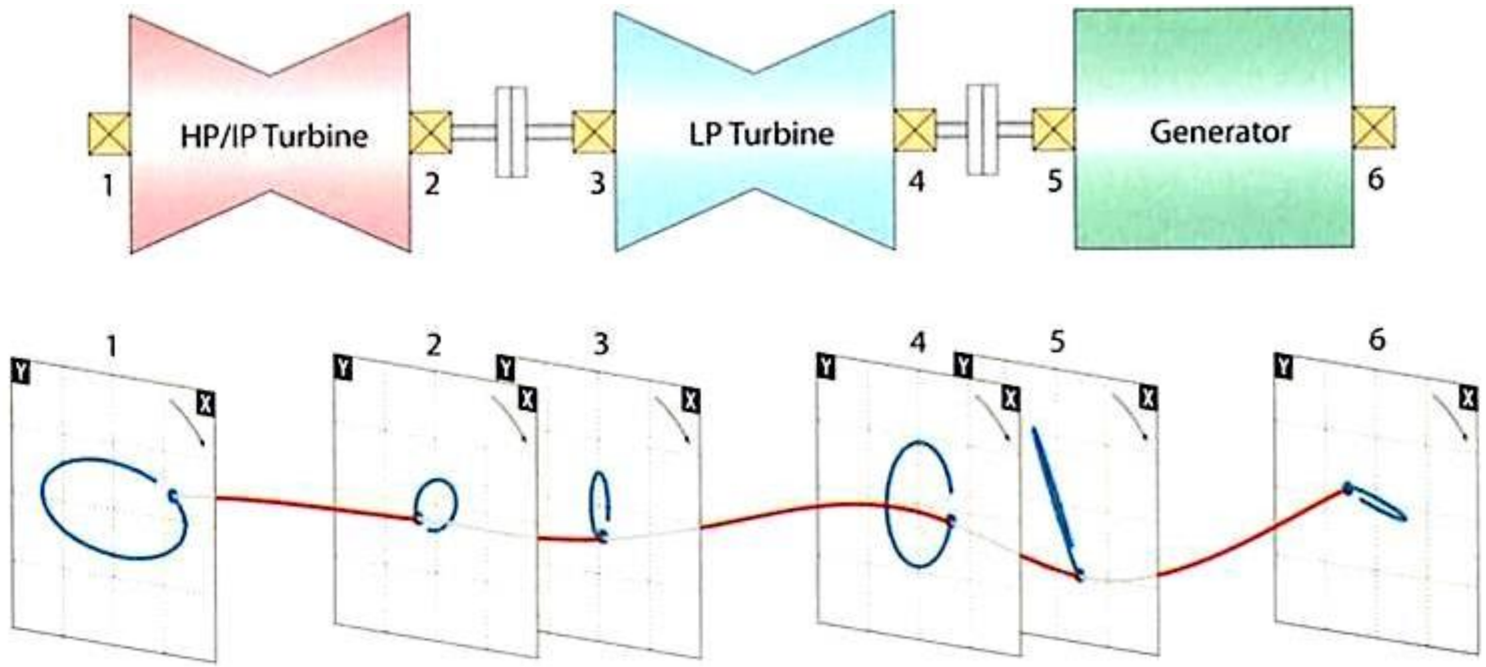
$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

As a result the cofactor matrix of A is

$$\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Finally the adjugate of A is the transpose of the cofactor matrix:

$$\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$



ROTOR DYNAMICS

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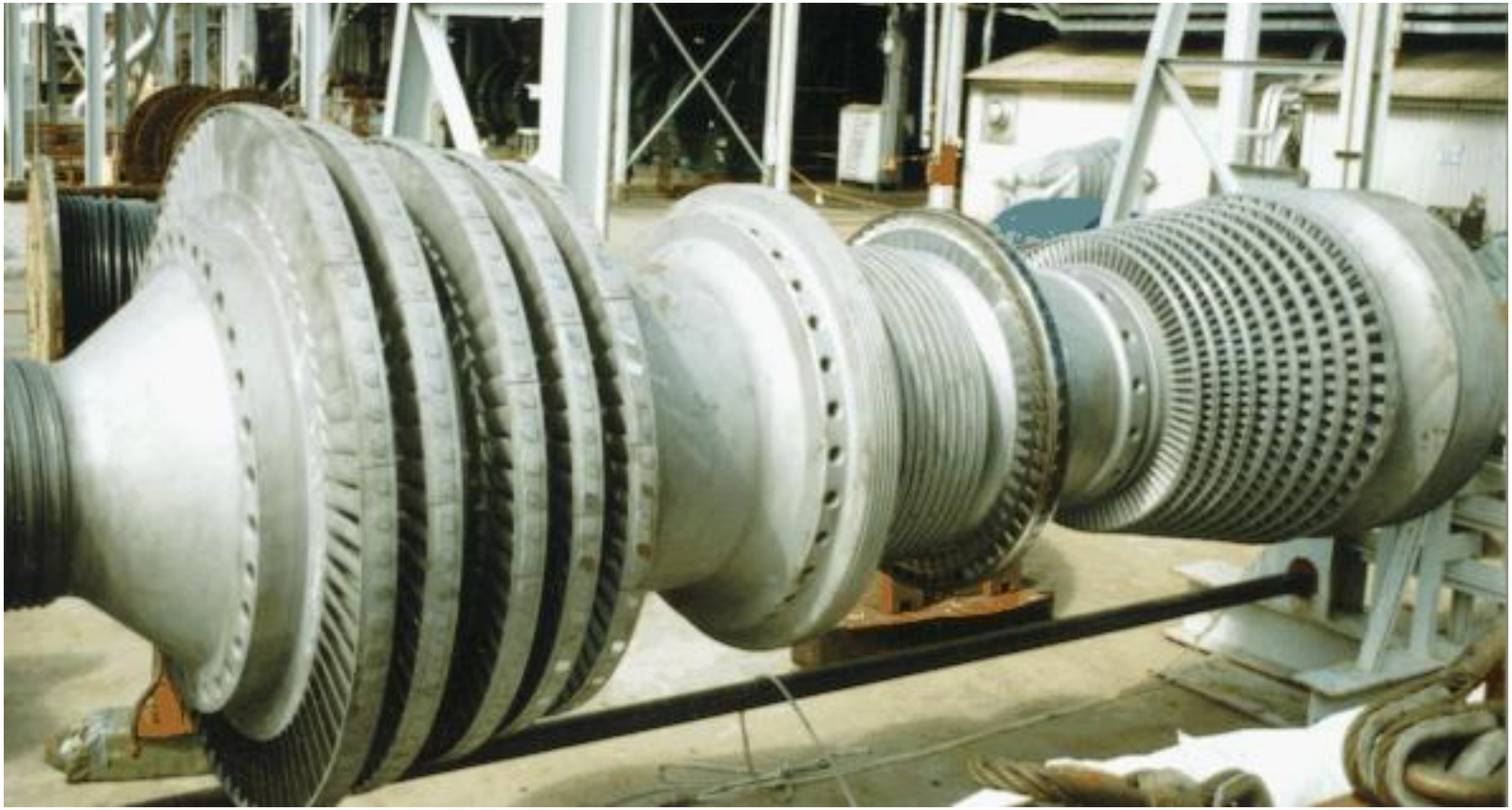
7th Lecture



Dunkerley's Method to Find the 1st Natural Frequency

Dunkerley's formula gives the approximate value of the fundamental frequency of a composite system in terms of the natural frequencies of its component parts. It is derived by making use of the fact that the higher natural frequencies of most vibratory systems are large compared to their fundamental frequencies.

To derive Dunkerley's formula, consider a general n-degree-of-freedom system whose eigenvalues can be determined by solving the frequency equation:

$$|-[K] + \omega^2[M]| = \{0\}$$

Or

$$\left|-\frac{1}{\omega^2}[I] + [a][M]\right| = \{0\} \text{ .... eq.1}$$

For a lumped-mass system with a diagonal mass matrix, Eq.1 becomes,

$$\left|-\frac{1}{\omega^2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{bmatrix}\right| = \{0\}$$

That is :-

$$\begin{vmatrix} (-\frac{1}{\omega^2}+a_{11}m_1) & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & (-\frac{1}{\omega^2}+a_{22}m_2) & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & (-\frac{1}{\omega^2}+a_{33}m_3) \end{vmatrix} = 0 \quad \text{......... eq.2}$$

The expansion of Eq.2 leads to:

$$\left(\frac{1}{\omega^2}\right)^3 - (a_{11}m_1 + a_{22}m_2 + a_{33}m_3)\left(\frac{1}{\omega^2}\right)^2 + \dots\dots\dots = 0 \quad \text{.............. eq.3}$$

This is a polynomial equation of n[th] degree in $(\frac{1}{\omega^2})$ Let the roots of Eq.3 be denoted $(\frac{1}{{\omega_1}^2}, \frac{1}{{\omega_2}^2}, \frac{1}{{\omega_3}^2})$ as:

$$\left(\frac{1}{\omega^2}-\frac{1}{{\omega_1}^2}\right)\left(\frac{1}{\omega^2}-\frac{1}{{\omega_2}^2}\right)\left(\frac{1}{\omega^2}-\frac{1}{{\omega_3}^2}\right) = \left(\frac{1}{\omega^2}\right)^3 - \left(\frac{1}{{\omega_1}^2}+\frac{1}{{\omega_2}^2}+\frac{1}{{\omega_3}^2}\right)\left(\frac{1}{\omega^2}\right)^2 \dots \text{eq.4}$$

Equating the coefficient of $\left(\frac{1}{\omega^2}\right)^2$ in Eqs.3 and 4 gives:

$$\frac{1}{{\omega_1}^2}+\frac{1}{{\omega_2}^2}+\frac{1}{{\omega_3}^2}=a_{11}m_1+a_{22}m_2+a_{33}m_3 \;\ldots\ldots\ldots\ldots\ldots \text{eq.5}$$

In most cases, the higher frequencies $\omega_2\,,\omega_3$ are considerably larger than the fundamental frequency ω_1 and so,

$$\frac{1}{{\omega_i}^2}<<\frac{1}{{\omega_1}^2}$$

Thus, Eq.5 can be approximately written as:

$$\frac{1}{{\omega_1}^2}\cong a_{11}m_1+a_{22}m_2+a_{33}m_3 \;\ldots\ldots\ldots\ldots\ldots\ldots \text{eq.6}$$

This equation is known as Dunkerley's formula. The fundamental frequency given by Eq.6 will always be smaller than the exact value. In some cases, it will be more convenient to rewrite Eq.6 as:

$$\frac{1}{{\omega_1}^2} \cong \frac{1}{{\omega_{11}}^2} + \frac{1}{{\omega_{22}}^2} + \frac{1}{{\omega_{33}}^2} \quad \ldots\ldots\ldots\ldots\ldots \text{eq.7}$$

where ${\omega_{11}}^2 = {}^{1}/_{a_{11} m_1} = {}^{k_{11}}/_{m_1}$ denotes the natural frequency of a single-degree of freedom system consisting of mass m_1 and spring of stiffness k_{11}.

Example:

Estimate the fundamental natural frequency of a simply supported beam carrying three identical equally spaced masses, as shown in Fig.

Solution: The flexibility influence coefficients

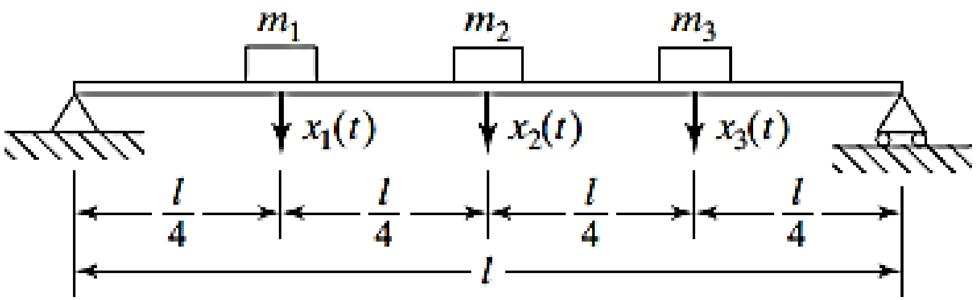
$$a_{11} = a_{33} = \frac{3}{256}\frac{l^3}{EI}, \qquad a_{22} = \frac{1}{48}\frac{l^3}{EI}$$

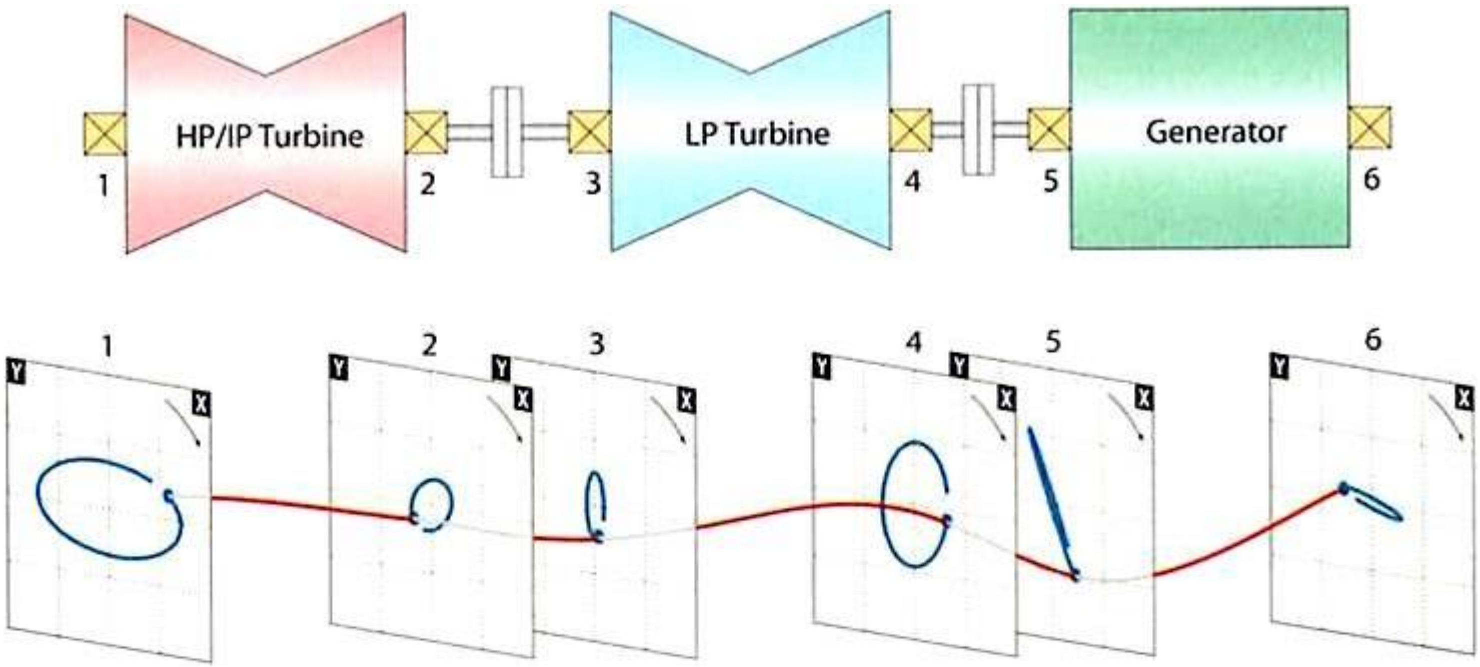
Using $m_1 = m_2 = m_3 = m$, Eq. **6** thus gives

$$\frac{1}{\omega_1^2} \simeq \left(\frac{3}{256} + \frac{1}{48} + \frac{3}{256}\right)\frac{ml^3}{EI} = 0.04427\,\frac{ml^3}{EI}$$

$$\omega_1 \simeq 4.75375\sqrt{\frac{EI}{ml^3}}$$

This value can be compared with the exact value of the fundamental frequency, $4.9326\sqrt{\frac{EI}{ml^3}}$





ROTOR DYNAMICS

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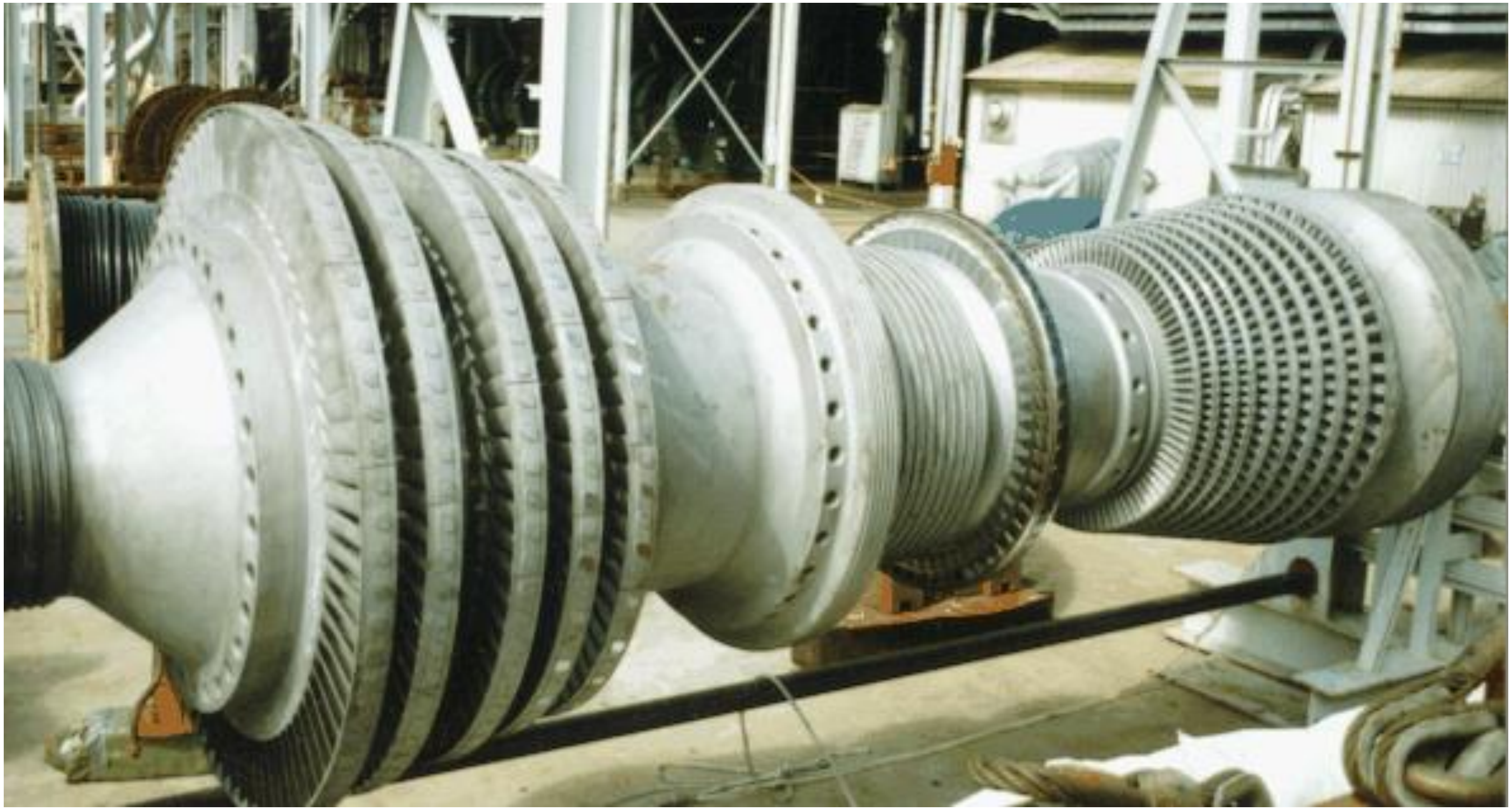
8TH & 9TH LECTURES



Iterative Technique to Find the Lower and Higher Natural Frequency

The iterative technique assumes that the natural frequencies are distinct and well separated such that $\omega_1 < \omega_2 < \cdots < \omega_n$.

The iteration process to find the fundamental natural frequency is stepped as follows:

1. Selecting a trial vector.
2. Pre-multiply the trial vector by the $[a][M]$.
3. Normalize the resulting column vector, usually by making one of its components equal to unity.
4. The normalized column vector is pre-multiplied by $[a][M]$ to obtain a third column vector.
5. Normalize the third vector in the same way as before and becomes still another trial column vector.

The process is repeated until the successive normalized column vectors converge to a common vector.

The convergence of the process can be explained as follows:

For multiple degrees of freedom with free vibration, the matrix equation of motion is:

$$[M]\{\ddot{x}\} + [K]\{x\} = 0 \text{ .... eq.1}$$

Pre-multiplying eq.1 by $[K]^{-1}$:

$$[K]^{-1}[M]\{\ddot{x}\} + [K]^{-1}[K]\{x\} = 0$$

$$[K]^{-1}[M]\{\ddot{x}\} + [I]\{x\} = 0$$

But $[K]^{-1} = [a]$ then:

$$[a][M]\{\ddot{x}\} + \{x\} = 0 \text{ .... eq.2}$$

If harmonic vibration occurs:

$$\{\ddot{x}\} = -\omega^2\{x\} \text{ .... eq.3}$$

By substituting eq.3 into eq.2:

$$-\omega^2[a][M]\{x\} + \{x\} = 0$$

Or

$$\{x\}_{i+1} = \omega^2[a][M]\{x\}_i \text{ .... eq.4}$$

Where (i) is denotes to the iteration no.

The eq.4 after iteration will converge to the fundamental natural frequency.

If we want to converge to the higher natural frequency, the iteration process is stepped as follows:

1. Selecting a trial vector.
2. Pre-multiply the trial vector by the dynamic matrix $[D]$.
3. Normalize the resulting column vector, usually by making one of its components equal to unity.
4. The normalized column vector is pre-multiplied by $[D]$ to obtain a third column vector.
5. Normalize the third vector in the same way as before and becomes still another trial column vector.

The convergence of the process can be explained as follows:

Pre-multiplying eq.1 by $[M]^{-1}$:

$$[M]^{-1}[M]\{\ddot{x}\} + [M]^{-1}[K]\{x\} = 0 \text{ .... eq.5}$$

But $[D] = [M]^{-1}[K]$

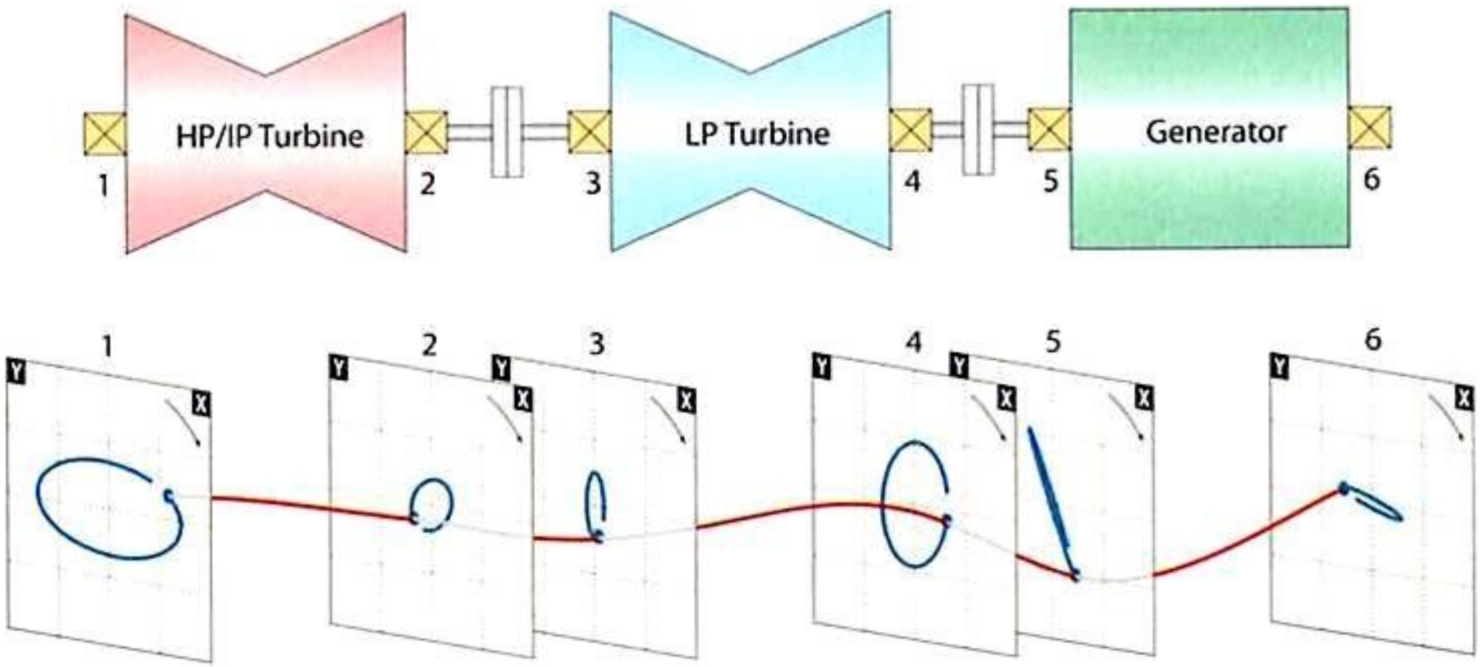
$$[I]\{\ddot{x}\} + [D]\{x\} = 0 \text{ .... eq.6}$$

by substituting eq.3 into eq.6:

$$-\omega^2\{x\} + [D]\{x\} = 0$$

Or

$$\{x\}_{i+1} = \frac{1}{\omega^2}[D]\{x\}_i \text{ .... eq.7}$$



Rotor Dynamics

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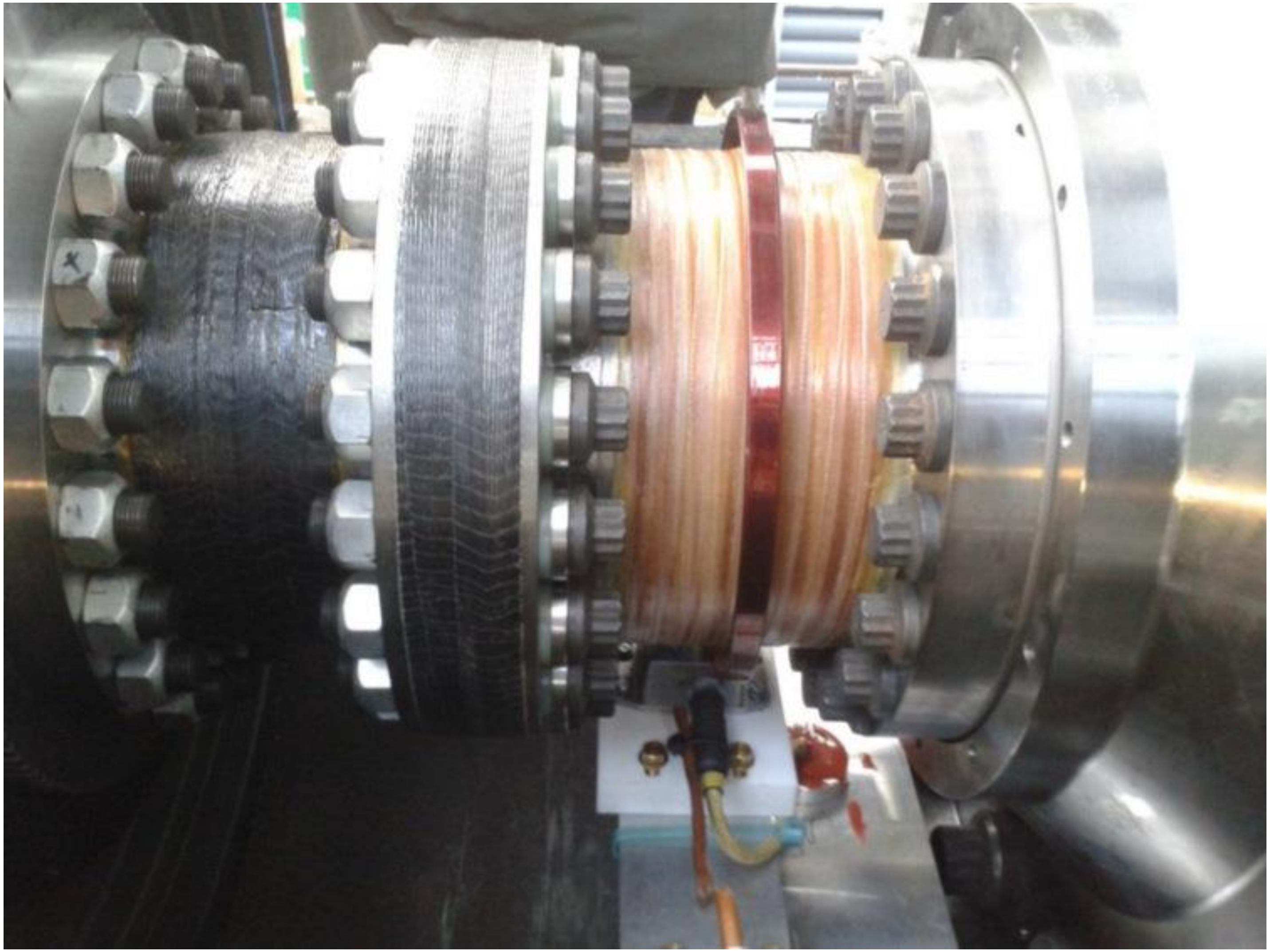
Fourth Year: First Semester

10TH , 11TH & 12TH LECTURES



Torsional Vibration

The study of torsional vibration of rotors is very important especially in applications where **high power transmission** and **high speed** are present. Torsional vibrations are predominant whenever there are **large discs** on relatively **thin shafts** (e.g., the flywheel of a punch press). Torsional vibrations may original from the following forces:

(i) Inertia forces of reciprocating mechanisms (e.g., due to pistons in IC engines),

(ii) Impulsive loads occurring during a normal machine cycle (e.g., during operations of a punch press),

(iii) Shock loads applied to electrical machinery (such as a generator line fault followed by fault removal and automatic closure),

(iv) Torques related to gear mesh frequencies, the turbine blade and compressor fan passing frequencies, etc.;

(v) A rotor rubs with the stator.

For machines having massive rotors and flexible shafts (where system natural frequencies of torsional vibrations may be close to, or within, the source frequency range during normal operation) torsional vibrations constitute a potential design problem area.

In such cases designers should ensure the accurate prediction of machine torsional frequencies, and frequencies of any torsional load fluctuations should not coincide with torsional natural frequencies. Hence, determination of torsional natural frequencies of the rotor system is very important and in the present lecture we shall deal with it in detail.

A Simple Rotor System with a Single Disc Mass

Consider a rotor system as shown Figure.1. The shaft is considered as **mass-less** and it provides **torsional stiffness**. The disc is considered as **rigid** and has **no flexibility**.

If an initial disturbance is given to the disc in the torsional mode (about its longitudinal or polar axis) and allow it to oscillate its own, it will execute free vibrations.

The free oscillation will be simple harmonic motion with a unique frequency, which is called the **torsional natural frequency** of the rotor system.

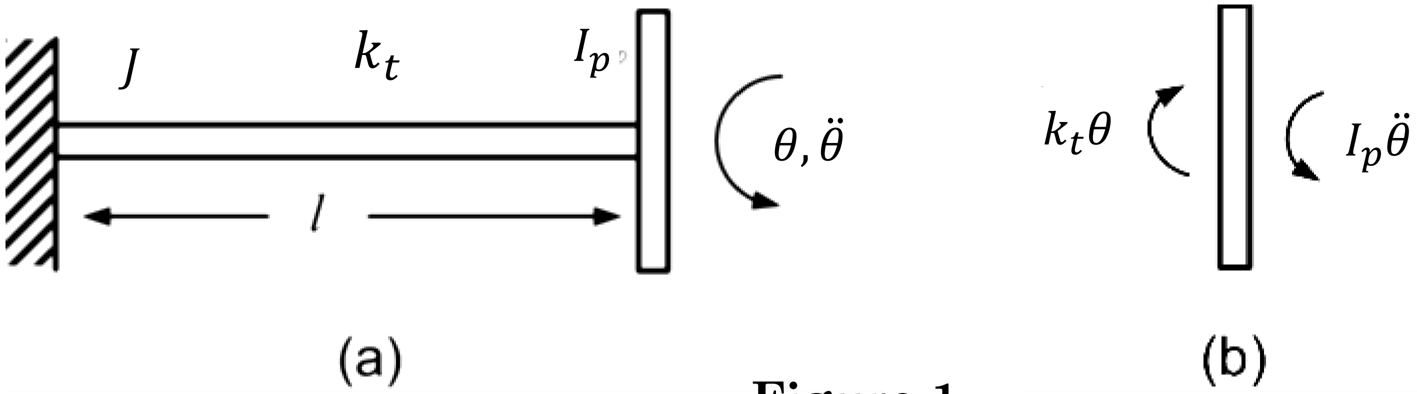


Figure.1

From the theory of torsion of the shaft (Timoshenko and Young, 1968), we have:

$$k_t = \frac{T}{\theta} = \frac{GJ}{l} \quad with\ J = \frac{\pi D^4}{32} \text{ .... eq.1}$$

Where

k_t torsional stiffness $({}^{N.m}/_{rad})$

T torque $(N.m)$

θ angular displacement (rad)

G modulus of rigidity (GPa)

l rotor length (m)

D rotor diameter (m)

J polar second moment of area of the shaft cross-section (m^4)

I_p polar mass moment of inertia of the disc $(kg.m^2)$

From the free body diagram of the disc as shown in Figure.1(b), we have

$$\sum External\ torque\ of\ disc = I_p\ddot{\theta}$$

$$-k_t\theta = I_p\ddot{\theta} \quad \text{or} \quad I_p\ddot{\theta} + k_t\theta = 0 \text{ .... eq.2}$$

Equation (6.2) is the equation of motion of the disc for free torsional vibrations. The free (or natural) vibration has a simple harmonic motion (SHM). For SHM of the disc, we have

$\theta = \emptyset \sin \omega_{nt}\, t$ so that $\ddot{\theta} = -\emptyset {\omega_{nt}}^2 \sin \omega_{nt}\, t$

and $\ddot{\theta} = -{\omega_{nt}}^2 \theta$............. eq.3

where $\emptyset$ is the amplitude of the torsional vibration, and ω_{nt} is the torsional natural frequency. On substituting eq.3 into eq.2, we get

$$-{\omega_{nt}}^2 I_p \theta + k_t \theta = 0$$

Or $\left(-{\omega_{nt}}^2 I_p + k_t\right)\theta = 0$

But $\theta \neq 0$, it gives

$$\omega_{nt} = \sqrt{\frac{k_t}{I_p}} = \sqrt{\frac{GJ}{lI_p}} \text{ .... eq.4}$$

which is similar to the case of single-DOF spring-mass system in where the polar mass moment of inertia and the torsional stiffness replace the mass and the spring stiffness, respectively.

A Two-Disc Torsional Rotor System

A two-disc torsional system is shown in Figure.2. In this case the whole of the rotor is free to rotate as the shaft is mounted on frictionless bearings. Hence, it is a free-free end condition, and the application of which can be found in an aircraft when it is flying and whole structure has torsional vibrations due to aerodynamic forces.

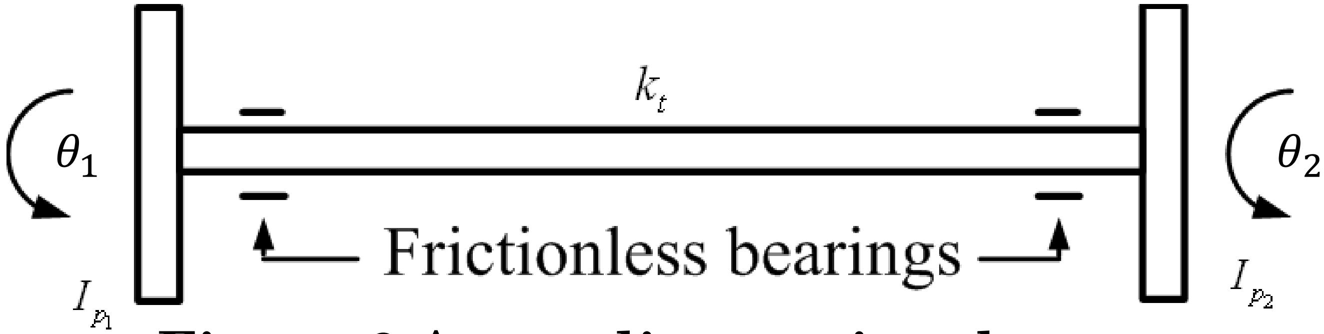


Figure.2 A two-disc torsional system

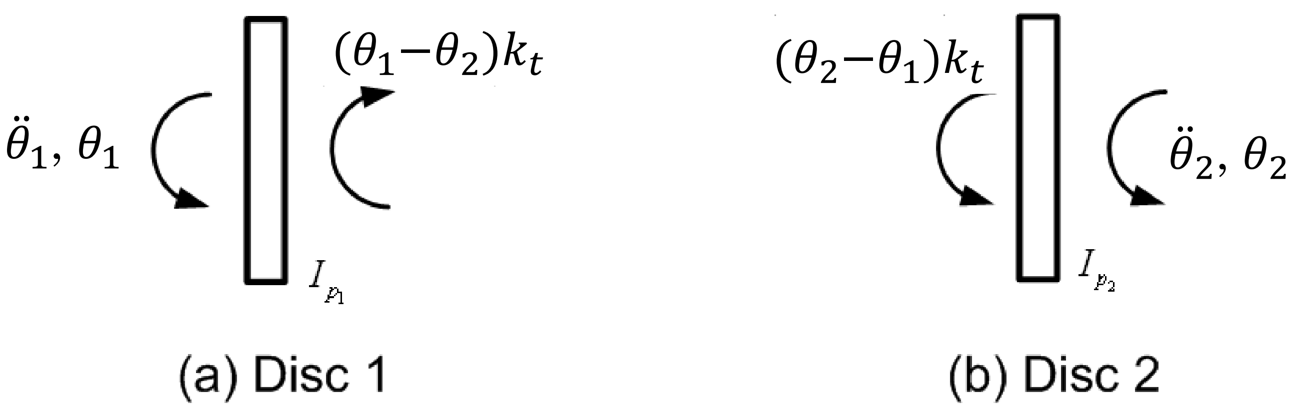


Figure.3 Free body diagrams of discs

From the free body diagram of discs as shown in Figure.3, we have

$$\sum External\ torque\ of\ disc = I_{p_1}\ddot{\theta}_1 \quad , \qquad -(\theta_1 - \theta_2)k_t = I_{p_1}\ddot{\theta}_1$$

$$I_{p_1}\ddot{\theta}_1 + (\theta_1 - \theta_2)k_t = 0 \text{ .... eq.5}$$

And

$$\sum External\ torque\ of\ disc = I_{p_2}\ddot{\theta}_2 \quad , \qquad -(\theta_2 - \theta_1)k_t = I_{p_2}\ddot{\theta}_2$$

$$I_{p_2}\ddot{\theta}_2 + (\theta_2 - \theta_1)k_t = 0 \text{ .... eq.6}$$

Noting eq.5, eq.6 can be assembled in a matrix form as

$$\begin{bmatrix} I_{p_1} & 0 \\ 0 & I_{p_2} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_t & -k_t \\ -k_t & k_t \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$$

By using eigen value method , we get:

$$\omega_{nt_1} = 0 \qquad and \qquad \omega_{nt_2} = \sqrt{\frac{(I_{p_1} + I_{p_2})k_t}{I_{p_1} I_{p_2}}}$$

Equivalent torsional system for stepped shaft

Figure.4 shows a stepped shaft with two large discs at ends with I_{p_1} and I_{p_2}. It is assumed that the rotor has free-free boundary conditions and the polar mass moment of inertia of shaft is negligible as compared to two discs at either ends of the shaft.

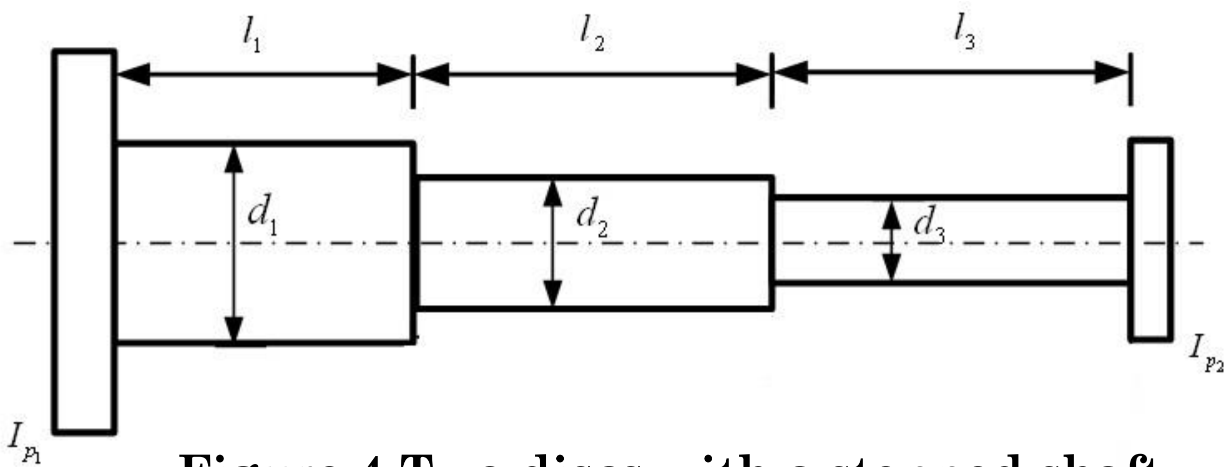


Figure.4 Two discs with a stepped shaft

In such cases the actual shaft should be replaced by an unstepped equivalent shaft for the purpose of the analysis as follows:

$$\frac{1}{k_{te}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}} \text{ .... eq.7}$$

Equivalent torsional geared system

In actual practice, it is rare that the rotor system has a single shaft (with either uniform or stepped cross sections) with multiple discs as we analyzed in previous sections. In some machine the shaft may not be continuous from one end of the machine to the other, but may have a gearbox installed at one or more locations. Hence, shafts will be having different angular velocities as shown in Figure.5. For the purpose of analysis the geared system must be reduced to system with a continuous shaft so that they may be analyzed for torsional vibrations by methods as described in preceding sections.

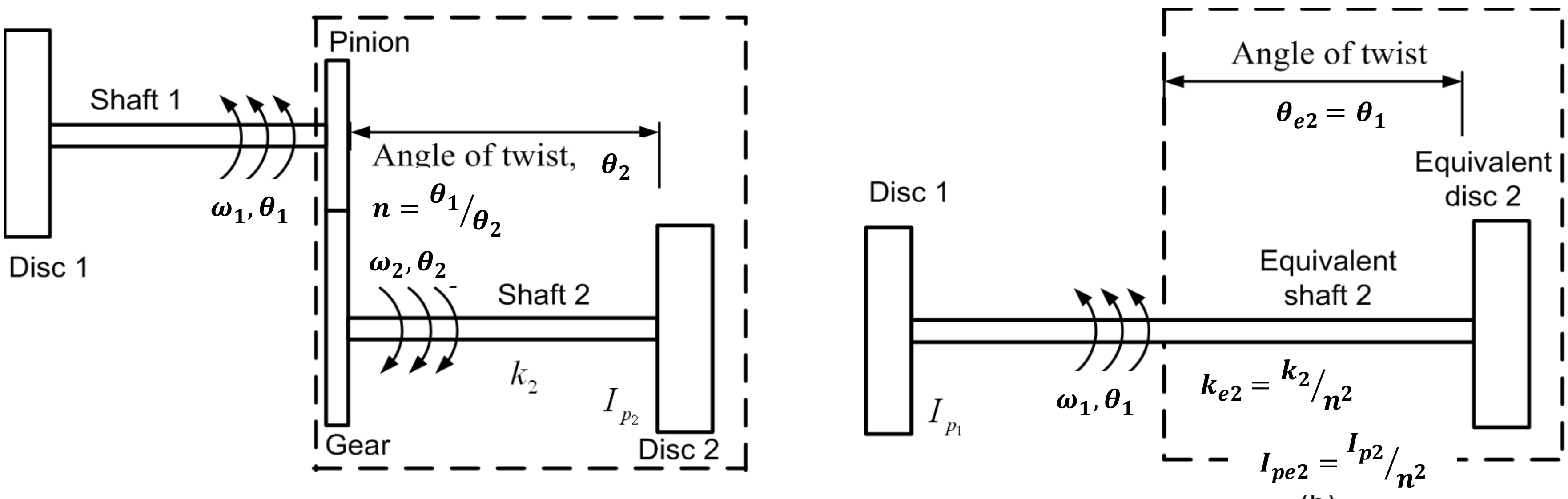
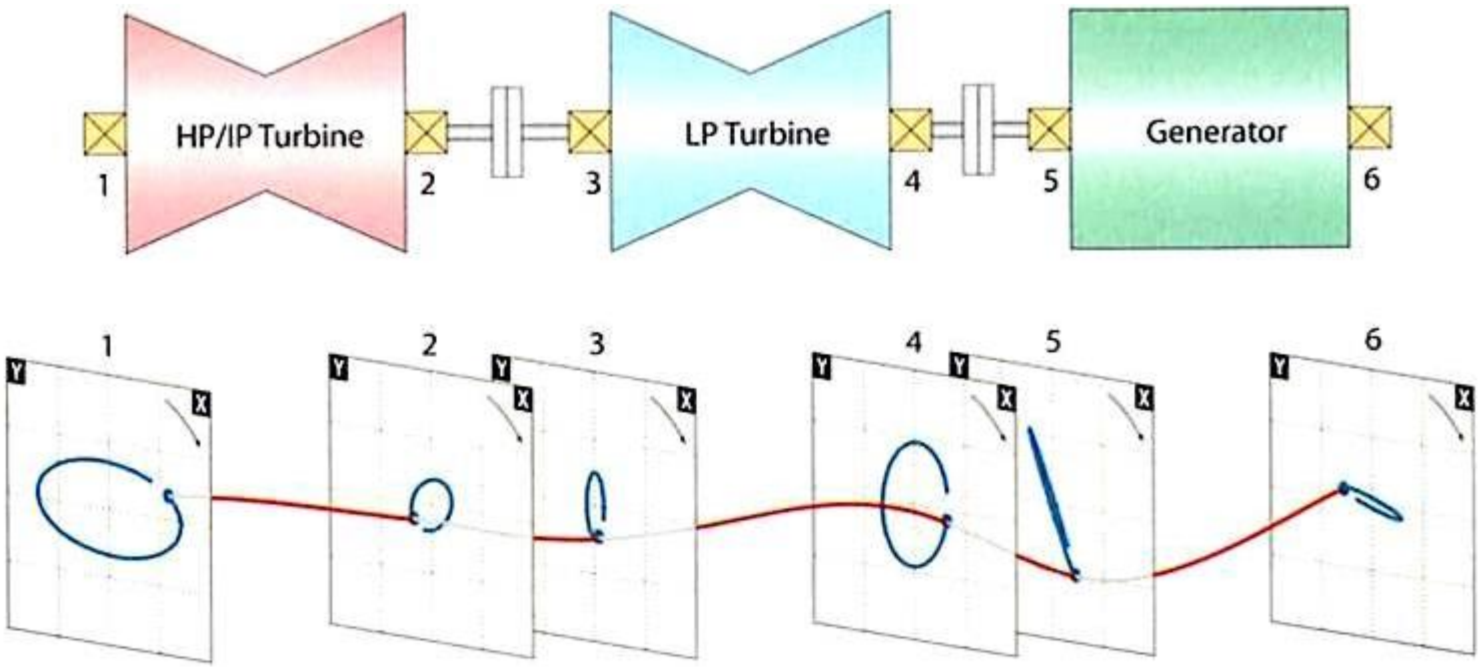


Fig. 5 (a) Actual geared system (b) An equivalent system without geared system



ROTOR DYNAMICS

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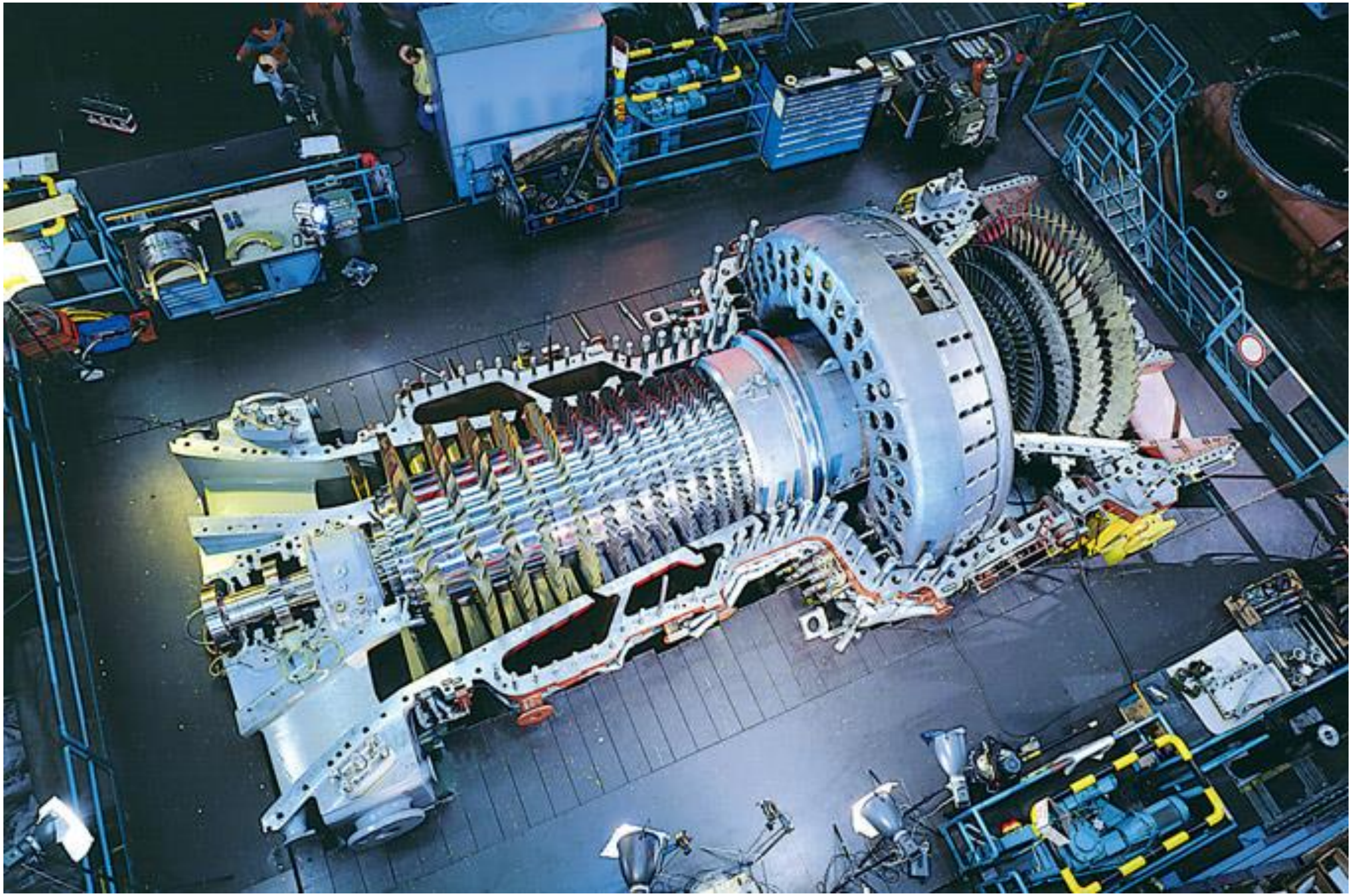
Fourth Year: First Semester

13TH , 14TH & 15TH Lectures



Holzer's Method

Holzer's method is essentially a **trial-and-error** scheme to find the natural frequencies of **undamped**, **damped**, **semidefinite**, **fixed**, or **branched** vibrating systems involving **linear** and **angular** displacements.

The method can also be programmed for computer applications. A **trial frequency** of the system is first **assumed**, and a solution is found when the assumed frequency **satisfies** the **constraints** of the system. This generally requires **several** trials. Depending on the trial frequency used, the fundamental as well as the higher frequencies of the system can be determined. The method also gives the **mode shapes**.

Consider the undamped torsional semidefinite system shown in Fig.1. The equations of motion of the discs can be derived as follows:

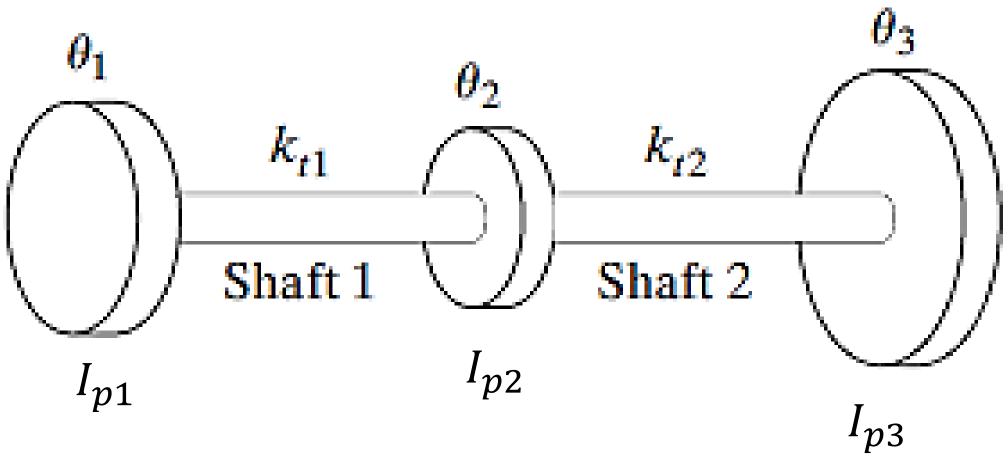


Figure.1

$I_{p1}\ddot{\theta}_1 + k_{t1}(\theta_1 - \theta_2) = 0$ …………… eq.1

$I_{p2}\ddot{\theta}_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) = 0$ ……………… eq.2

$I_{p3}\ddot{\theta}_3 + k_{t2}(\theta_3 - \theta_2) = 0$ ……………eq.3

Since the motion is harmonic in a natural mode of vibration, we assume that $\theta = \emptyset \sin \omega_{nt}\, t$ in Eqs. (1) to (3) and obtain:

$$\omega^2 I_{p1}\theta_1 = k_{t1}(\theta_1 - \theta_2) \ldots\ldots\ldots\ldots\ldots \text{eq.4}$$

$$\omega^2 I_{p2}\theta_2 = k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) \ldots\ldots\ldots\ldots\ldots\ldots \text{eq.5}$$

$$\omega^2 I_{p3}\theta_3 = k_{t2}(\theta_3 - \theta_2) \ldots\ldots\ldots\ldots\ldots \text{eq.6}$$

Summing these equations gives:

$$\sum_{i=1}^{3} \omega^2 I_{pi}\theta_i = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{eq.7}$$

Equation(7) states that the sum of the inertia torques of the semidefinite system must be zero. This equation can be treated as another form of the frequency equation, and the trial frequency must satisfy this requirement.

In Holzer's method, a trial frequency ω is assumed, and θ_1 is arbitrarily chosen as unity.
Next, θ_2 is computed from Eq.4, and then θ_3 is found from Eq.5. Thus we obtain:

$\theta_1 = 1$ eq.8

$\theta_2 = \theta_1 - \frac{\omega^2 I_{p1}}{k_{t1}}$ eq.9

$\theta_3 = \theta_2 - \frac{\omega^2 (I_{p1}+I_{p2})}{k_{t1}}$ eq.10

These values are substituted in Eq.7 to verify whether the constraint is satisfied. If Eq.7 is not satisfied, a new trial value of is assumed and the process repeated.

When the calculation is repeated with other values of the resulting graph appears as shown in Fig.2.

From this graph, the natural frequencies of the system can be identified as the values of $\boldsymbol{\omega}$ at which $\boldsymbol{M_t = 0}$.

The amplitudes θ_i (1, 2, … , n) corresponding to the natural frequencies are the mode shapes of the system.

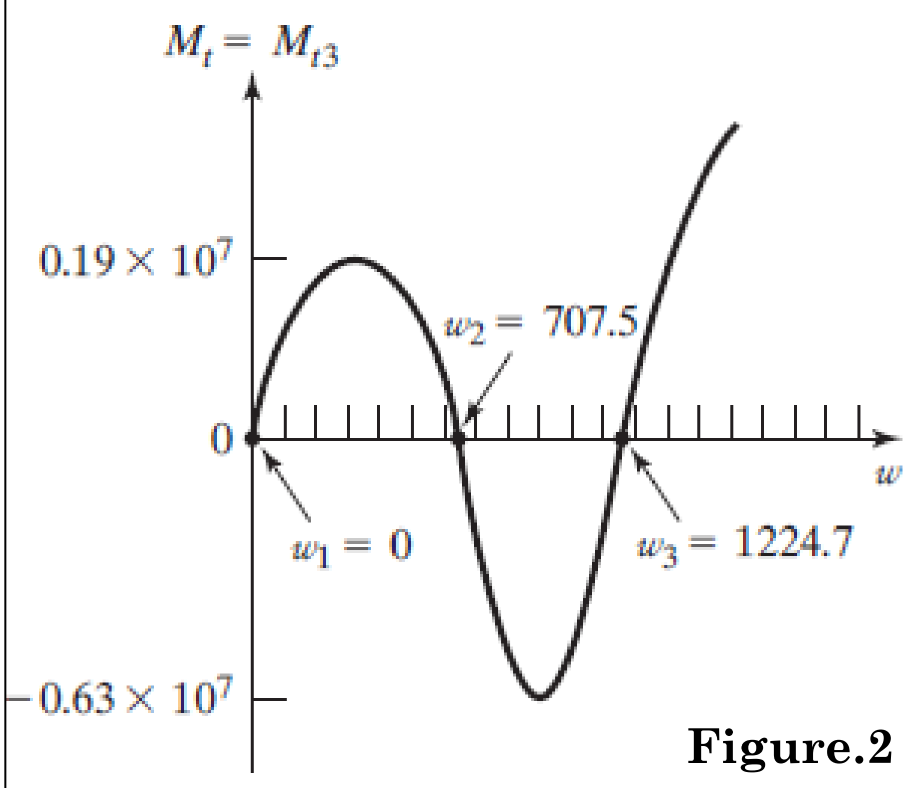


Figure.2

Example: The arrangement of the compressor, turbine, and generator in a thermal power plant is shown in Fig.3. Find the natural frequencies and mode shapes of the system.

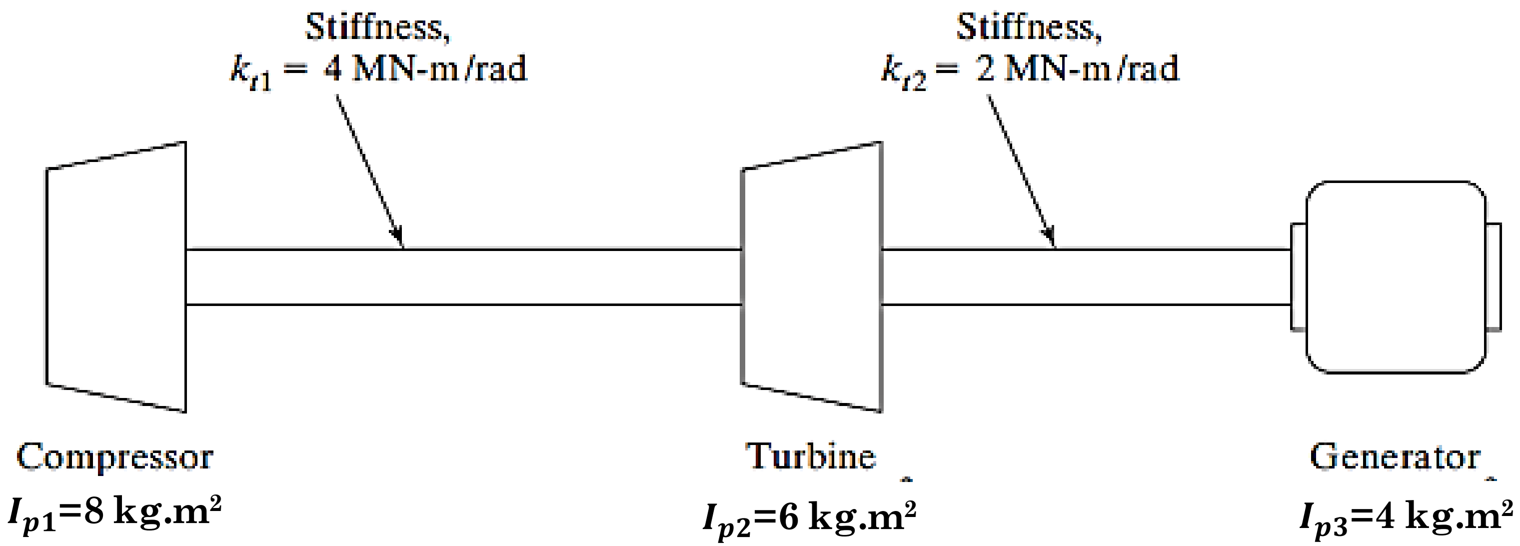


Figure.3

Solution: This system represents an unrestrained or free-free torsional system. **Table.1** shows its parameters and the sequence of computations. The calculations for the trial frequencies ω = 0, 10, 20, 700 and 710 are 20, 700, shown in this table.

Table.1

Parameters of the System	Quantity	Trial 1	2	3	. . .	71	72
		0	10	20		700	710
	ω^2	0	100	400		490000	504100
Station 1:							
$I_{p1} = 8$	Θ_1	1.0	1.0	1.0		1.0	1.0
$k_{t1} = 4 \times 10^6$	$M_{t1} = \omega^2 I_{p1}\theta_1$	0	800	3200		0.392*E*7	0.403*E*7
Station 2:							
$I_{p2} = 6$	$\Theta_2 = 1 - \frac{M_{t1}}{k_{t1}}$	1.0	0.9998	0.9992		0.0200	−0.0082
$k_{t2} = 2 \times 10^6$	$M_{t2} = M_{t1} + \omega^2 I_{p2}\theta_2$	0	1400	5598		0.398*E*7	0.401*E*7
Station 3:							
$I_{p3} = 4$	$\Theta_3 = \Theta_2 - \frac{M_{t2}}{k_{t2}}$	1.0	0.9991	0.9964		−1.9690	−2.0120
$K_{t3} = 0$	$M_{t3} = M_{t2} + \omega^2 I_{p3}\theta_3$	0	1800	7192		0.119*E*6	−0.494*E*5

The quantity M_{t3} denotes the torque to the right of Station.3 (generator), which must be zero at the natural frequencies. Figure.2 shows the graph of versus closely spaced trial values of are used in the vicinity of to obtain accurate values of the first two flexible mode shapes, shown in Fig.4. Note that the value $\omega = 0$ corresponds to the rigid body rotation.

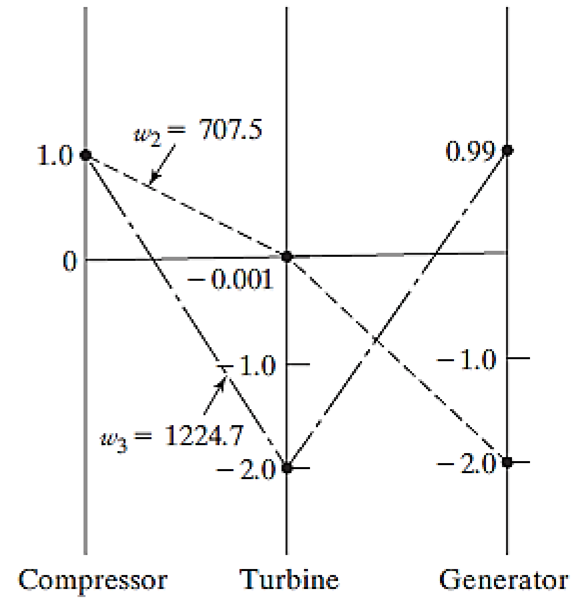


Figure.4