Subject : Mechanical Vibration	**موضوع: اهتزازات ميكانيكية**
Weekly Hours: Theoretical: 2	**الساعات الأسبوعية: نظري :2**
Tutorial: 1	**مناقشة: 1**
Experimental : 1	**عملي:1**
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Univesity of Technology
Mechanical Eng. Dept.
class: 4th year

Mechanical Vibration

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Contents:

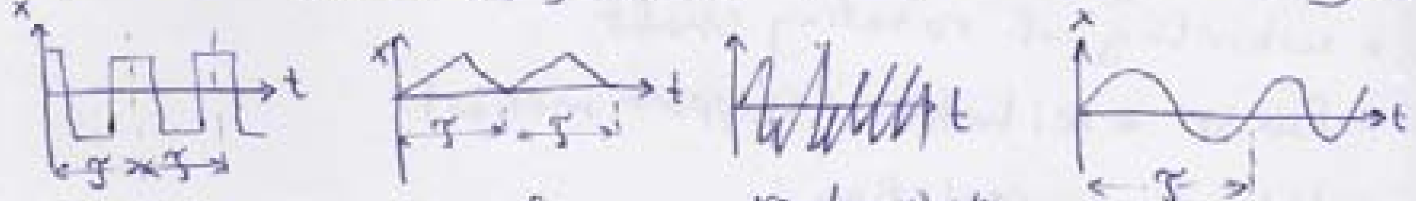
* Torsional Vibration, single rotor system
* vibration of continous systems.
* Rayleigh and Dunkerly method for determining natural frequency.

Basic concepts of vibration

1. Frequency ω (rad/s), F (Hz)
2. Natural frequency ω_n (rad/s)
3. Resonance
4. period τ $\quad \tau = \frac{1}{F}$

Simple harmonic motion:

periodic motion is said to be a periodic motion when it repeats itself after a constant interval of time called (τ) and its reciprocal called frequency (f)



step function
periodic motion

ramp function
periodic motion

random vibration
non-periodic

periodic + sinusoidal
Simple harmonic motion

equilbm position

$$X = A \sin \theta = A \sin \omega t$$

A = amplitude (mm)

ω = angular Velocity (rad/s)

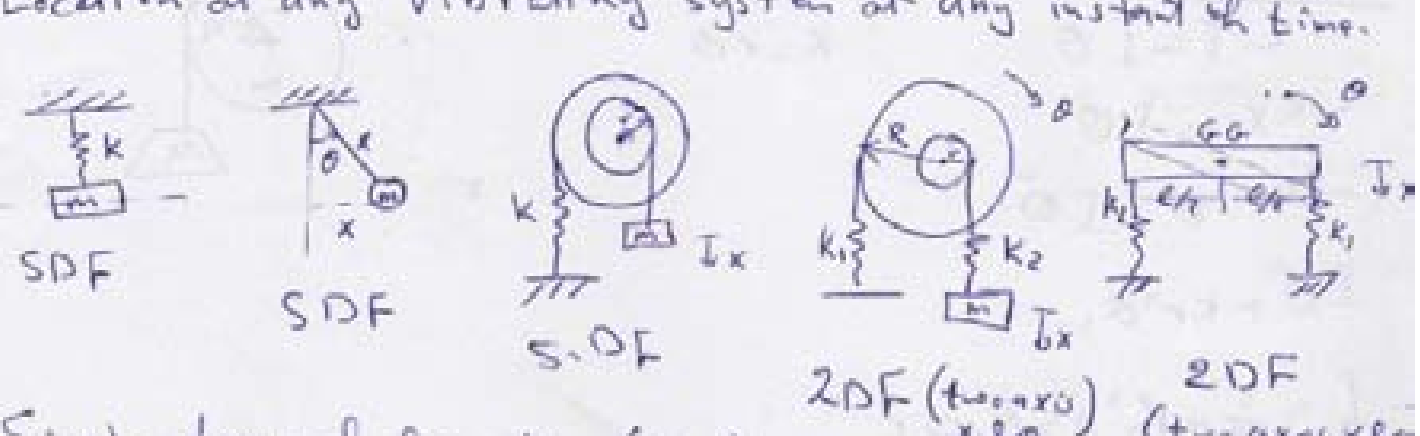
$$\therefore \dot{X} = A\omega \cos \omega t = A\omega \sin(\omega t + \tfrac{\pi}{2})$$

$$\ddot{X} = -A\omega^2 \sin \omega t = A\omega^2 \sin(\omega t + \pi)$$

$$f = \frac{\omega}{2\pi}$$

* **Degree of freedom:**

It is the number of undependent axes which specify the location at any vibrating system at any instant of time.



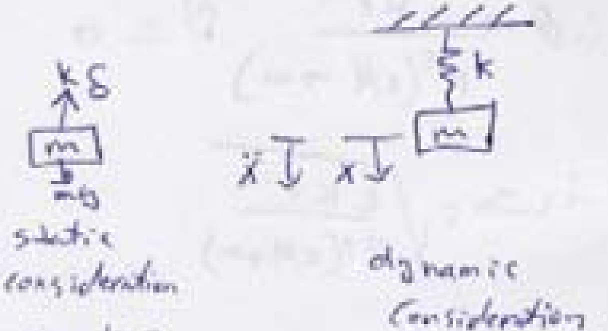
Single degree of freedom (SDF):

$\Sigma F = m\ddot{x}$

$mg - K(X+\delta) = m\ddot{x}$

$mg - KX - K\delta = m\ddot{x}$

$m\ddot{x} - KX = 0$ equation of motion

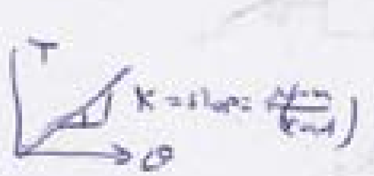


$mg = k\delta$

$X_{12} = \mp\sqrt{\frac{-k}{m}} = \mp j\omega_n$

$\therefore X = A\cos\omega_n t + B\sin\omega_n t$ general solution

Torsional Spring



$\frac{T}{J} = \frac{G\theta}{l} = \frac{\tau}{r}$

$\Sigma T = I\ddot{\theta}$

$-k\theta = I\ddot{\theta}$

$\ddot{\theta} + \frac{k}{I}\theta = 0$

$\omega_n = \sqrt{\frac{K}{I}} = \sqrt{\frac{GJ}{Il}}$ rad/s , $f_n = \frac{\omega_n}{2\pi}$ (Hz)

$I_{cylinder} = \frac{1}{2}mr^2$, $I_{ring} = mr^2$, $I_{bar} = \frac{1}{12}ml^2$ about centre

Ex: For the system shown in fig. Find the equation of motion and natural frequency.

Solution

$$\Sigma T = I\ddot{\theta} \qquad x = r\theta$$

$$-kxr = I\ddot{\theta}$$

$$-kr^2\theta = I\ddot{\theta}$$

$$I\ddot{\theta} + kr^2\theta = 0$$

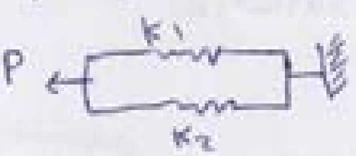
$$I = \frac{1}{2}mR^2 + MR^2$$

$$\therefore \ddot{\theta} + \frac{2kr^2}{R^2(2M+m)}\theta = 0$$

$$\therefore \omega_n = \sqrt{\frac{2kr^2}{R^2(2M+m)}}$$

⊛ Equivalent spring

spring in parallel



$$K_{eq} = k_1 + k_2$$

$$K_{eq} = \sum_{i=1}^{n} k_i$$

spring in series:

$$\frac{1}{K_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\frac{1}{K_{eq}} = \sum_{i=1}^{n} \frac{1}{K_i}$$

* <u>Simple Energy method</u>:

For the conservative system, the sum of the (K.E) system and (P.E) system at any instent of time must equal constant

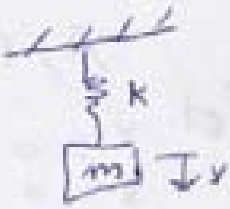
$$(K.E + P.E)_{system} = C$$

It is founed that the derivation above equation with respect to time yeild into equation of motion for the vibrating system:

$$\frac{d}{dt}(K.E + P.E)_{system} = 0$$

while equating the max. K.E & the max. P.E Leads to having the angular natural frequency of the system

<u>Ex:</u> for the system shown find the equation of motion and its natural frequency by using Simple Energy method.



<u>Sol:</u>

$K.E = \frac{1}{2} m \dot{x}^2$

$P.E = \frac{1}{2} k x^2$

$\frac{d}{dt}(P.E + K.E) = 0$

$m\dot{x}\ddot{x} + kx\dot{x} = 0$

$\dot{x}(m\ddot{x} + kx) = 0$

$\ddot{x} + \frac{k}{m}x = 0$ eq. of motion

$K.E_{max} = \frac{1}{2} m \dot{x}_{max}^2$ $\quad x = A \sin \omega_n t \quad x_{max} = A$

$P.E_{max} = \frac{1}{2} k x_{max}^2$ $\quad \dot{x} = A\omega_n \cos \omega_n t \quad \dot{x}_{max} = A\omega_n$

$\ddot{x} = -A\omega_n^2 \sin \omega_n t$

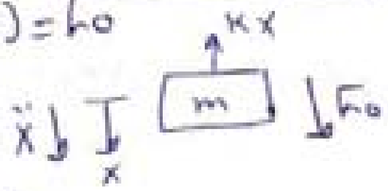
$K.E_{max} = P.E_{max}$

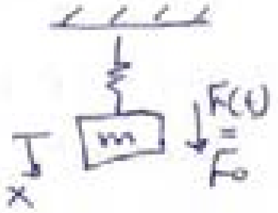
$\frac{1}{2} m A^2 \omega^2 = \frac{1}{2} k A^2$

$\omega_n^2 = \frac{k}{m} \Longrightarrow \omega_n = \sqrt{\frac{k}{m}}$ rad/s

Forced vibration without damping

$\Sigma F = m\ddot{x}$ at $F(t) = F_o$





$m\ddot{x} + Kx = F_o$ eq. of motion

$m\ddot{x} + Kx = 0$

$\therefore x_c = A\cos\omega_n t + B\sin\omega_n t$ [complementry solution]

$x = H$ constant

$\ddot{x} = 0$

$m(0) + K(H) = F_o$

$\therefore H = \frac{F_o}{K} \Rightarrow x_p = \frac{F_o}{K}$

$\therefore x_{general} = x_p + x_c$

$x = A\cos\omega_n t + B\sin\omega_n t + \frac{F_o}{K}$

A, B are constant

let $x = 0 \quad t = 0$

$\dot{x} = 0 \quad t = 0$

$\therefore A = -\frac{F_o}{K}, \quad B = 0$

$\therefore x = \frac{-F_o}{K}\cos\omega_n t + \frac{F_o}{K}$

$x = \frac{F_o}{K}(1 - \cos\omega_n t)$

at $F(t) = F_o \sin\omega t$

$\Sigma F = m\ddot{x}$

$m\ddot{x} + Kx = F_o \sin\omega t$

$x_p = H \sin\omega t \quad , \dot{x}_p = H\omega\cos\omega t$

$\ddot{x} = -H\omega^2 \sin\omega t$

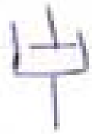
$\therefore H = \frac{F_o}{K - m\omega^2}$

$$\therefore \underset{\text{general}}{X} = X_c + X_p$$

$$= \underbrace{A\cos\omega_n t + B\sin\omega_n t}_{\text{transient solution}} + \underbrace{\frac{F_o/k}{1-\left(\frac{\omega}{\omega_n}\right)^2}\sin\omega t}_{\text{steady state Solution}}$$

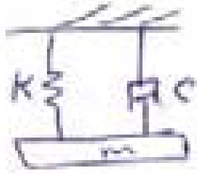
Damping

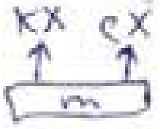
1. viscous damping
2. Columb damping
3. magnetic damping
4. specific damping



C = damping coeff. (N/m.s)

Damped Free Vibration:





$\Sigma F = m\ddot{x}$

$-(kx + c\dot{x}) = m\ddot{x}$

$m\ddot{x} + c\dot{x} + kx = 0$ eq. of motion

$x = Ae^{rt}$ $\quad \dot{x} = Are^{rt}$ $\quad \ddot{x} = Ar^2e^{rt}$

$mr^2 + cr + k = 0$

$$r_{12} = \frac{-\frac{c}{m} \mp \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2} = -\frac{c}{2m} \mp \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$x = Ae^{r_1 t} + Be^{r_2 t}$ general solution

Critical damping C_c

$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = 0 \qquad \frac{C_c}{2m} = \frac{k}{m} = \omega_n^2$

$C_c = 2m\omega_n^2 = 2\sqrt{km}$

$\frac{c}{C_c} = \xi$ = damping ratio

$\boxed{C = \xi C_c = 2m\omega_n\xi}$ I

$\xi = 0$ no damping

$\xi < 1$ under damping

$\xi > 1$ over damping

$\xi = 1$ critical damping

$$\therefore r_{12} = \frac{-c}{2m} \mp \sqrt{\left(\frac{c}{2m}\right)^2 - \omega_n^2}$$

$$r_{12} = \omega_n\left[-\xi \mp \sqrt{\xi^2 - 1}\right]$$

at $\xi = 1$ (critical damping)

$\therefore r_{12} = -\omega_n \quad \therefore r_1 = r_2$

$\therefore$ general solution $X = (A + Bt)e^{-\omega_n t}$

at $\xi < 1$

$$r_{12} = -\omega_n\left[-\xi \mp \sqrt{\xi^2 - 1}\right.$$

$$= -\xi\omega_n \mp i\sqrt{1-\xi^2}\,\omega_n$$

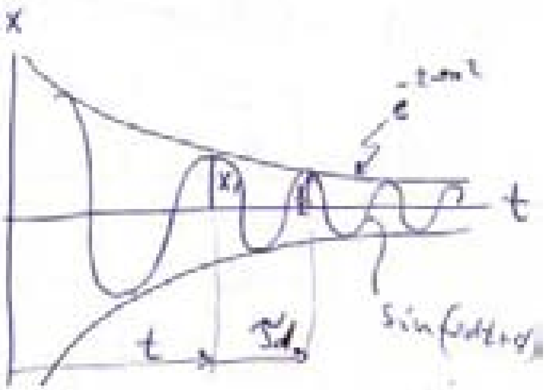
$$\therefore X = Ae^{-\xi\omega_n t}\left(B\cos\sqrt{1-\xi^2}\,\omega_n t + D\sin\sqrt{1-\xi^2}\,\omega_n t\right]$$

$\therefore \omega_d = \omega_n\sqrt{1-\xi^2}$ = damped natural frequency

$$\therefore x = Xe^{-\xi\omega_n t}\left(\sin \omega_d t + \phi\right)$$

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$

Logarithmic decrement



$$x_1 = X e^{-\xi\omega_n t}\left(\sin(\omega_d t+\phi)\right)$$

$$x_2 = X e^{-\xi\omega_n (t+\tau_d)} \sin\left(\omega_d(t+\tau_d)+\phi\right)$$

$$\frac{x_1}{x_2} = e^{\xi\omega_n \tau_d}$$

$$\ln\frac{x_1}{x_2} = \xi\omega_n\tau_d = \delta$$

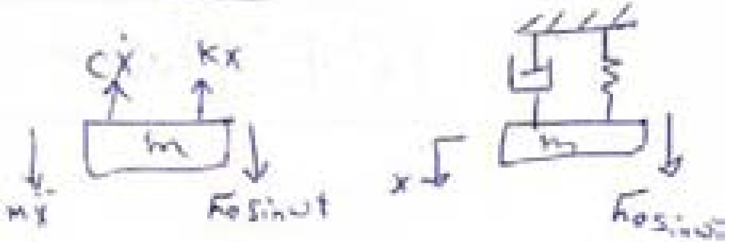
$$\delta = \xi\omega_n \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

at $\xi <<< 1$

$\therefore \sqrt{1-\xi^2} \simeq 1$

$$\boxed{\delta = 2\pi\xi}$$

Forced Vibration with damping



$\Sigma F = m\ddot{x}$

$m\ddot{x} + c\dot{x} + kx = F_o \sin\omega t$

$x = A\cos\omega t + B\sin\omega t$

$\dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$

$\ddot{x} = -A\omega^2\cos\omega t - B\omega^2\cos\omega t$

$-m\omega^2 B - c\omega A + kB = F_o$ → for $\sin\omega t$ coefficient

$-c\omega A + (k - m\omega^2)B = F_o$

$-m\omega^2 A + c\omega B + kA = 0$

$(k - m\omega^2)A + c\omega B = 0$ → for $\cos\omega t$ coefficient

$$-c\omega A + (k - m\omega^2) B = F_0$$

$$A = \frac{\begin{vmatrix} F_0 & k-m\omega^2 \\ 0 & c\omega \end{vmatrix}}{\begin{vmatrix} -c\omega & k-m\omega^2 \\ k-m\omega^2 & c\omega \end{vmatrix}} = \frac{F_0 c\omega}{-c^2\omega^2 - (k-m\omega^2)^2} = \frac{-F_0 c\omega}{(k-m\omega^2)^2 + c^2\omega^2}$$

$$B = \frac{\begin{vmatrix} -c\omega & F_0 \\ k-m\omega^2 & 0 \end{vmatrix}}{\begin{vmatrix} -c\omega & k-m\omega^2 \\ k-m\omega^2 & c\omega \end{vmatrix}} = \frac{F_0 (k-m\omega^2)}{(k-m\omega^2) + c^2\omega^2}$$

$$\therefore X = \frac{F_0}{k-m\omega^2 + c^2\omega^2} \left[(k-m\omega^2)\sin\omega t - c\omega \cos\omega t \right]$$

$$\therefore X = \frac{F_0}{\sqrt{(k-m\omega^2) + c^2\omega^2}} \sin(\omega t - \beta)$$

where $\beta = \tan^{-1} \frac{c\omega}{k-m\omega^2}$

$$\therefore \boxed{X = \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \sin(\omega t-\beta)}$$

$c = 2\xi m\omega_n$

$\omega_n^2 = k/m$

steady state solution

Transmissibility Ratio (TR)

It's the ratio of the max transmitted force to the floor to the max. impressed force.

$$x = \frac{F_o/K}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}\sin(\omega t-\beta)$$

let $A = \dfrac{F_o/k}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$ $\qquad x = A\sin(\omega t-\beta)$

$\qquad \dot{x} = A\omega\cos(\omega t-\beta)$

transmitted force to the floor

$$F_t = c\dot{x} + kx$$

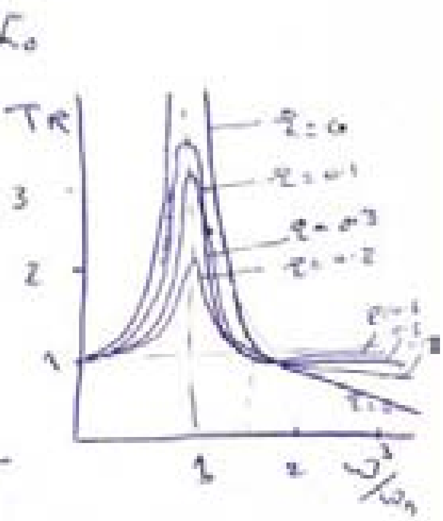
$$F_t = cA\omega\cos(\omega t-\beta) + kA\sin(\omega t-\beta)$$

$$F_t = A\left[c\omega\cos(\omega t-\beta) + k\sin(\omega t-\beta)\right]$$

$$F_{t_{max}} = \frac{F_o\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$TR = \frac{\text{Max forced transmitted to the floor } (F_{t_{max}})}{\text{max. impressed Force} = F_o}$$

$$TR = \sqrt{\frac{1+\left(2\xi\frac{\omega}{m_n}\right)^2}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$



at $TR = 1 \Rightarrow \dfrac{\omega}{\omega_n} = 0$

or $\dfrac{\omega}{\omega_n} = \sqrt{2}$

Lagrange eq.

For free undamped system the Lagrange eq. may be written as

$$\frac{d}{dt}\left(\frac{\partial K.E}{\partial \dot{q}_i}\right) - \frac{\partial K.E}{\partial q_i} + \frac{\partial P.E}{\partial q_i} = 0$$

wher q_i is the coordinate under consideration

For damped forced vibration Lagrange eq. become

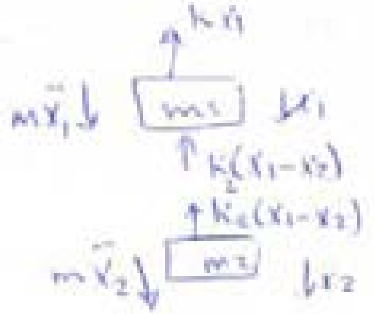
$$\frac{d}{dt}\left(\frac{\partial K.E}{\partial \dot{q}_i}\right) - \frac{\partial K.E}{\partial q_i} + \frac{\partial P.E}{\partial q_i} + \frac{\partial D.E}{\partial \dot{q}_i} = Q$$

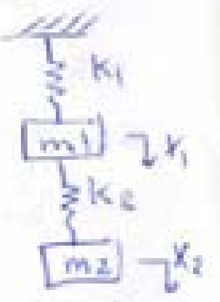
wher $D.E$ = damping energy $= \frac{1}{2} c \dot{x}^2$

Q = the generalized force (It's unit is unity of fo

It's applicable for single or multiple degree of freedom.

Two degree of freedom :—





$\sum F = m_1 \ddot{x}_1$

$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2)$

$$\boxed{m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0}$$

eq. ① of motion of mass m_1

$\sum F = m_2 \ddot{x}_2$

$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1)$

$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$ eq. ② of motion for mass m_2

assume :- $x_1 = A \sin(\omega t + \psi)$, $x_2 = B \sin(\omega t + \psi)$

$\ddot{x}_1 = -A\omega^2 \sin(\omega t + \psi)$ $\ddot{x}_2 = -B\omega^2 \sin(\omega t + \psi)$

substitute $x_1, \ddot{x}_1, x_2, \ddot{x}_2$ in eq. 1 & 2

$$\begin{bmatrix} k_1 + k_2 - m\omega^2 & -k_2 \\ -k_2 & k_2 - m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

for non-trival solution

$$\begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{vmatrix} = 0$$

$(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2 = 0$

The above eq is called frequency eq. or characteristic eq:

$$\omega^4 - \left\{\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right\}\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

$$\omega_{12}^2 = \frac{\left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right] \mp \sqrt{\left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right]^2 - 4 \frac{k_1 k_2}{m_1 m_2}}}{2}$$

The ~~absolute eq.~~

The previous eq. gives ω_1 & ω_2

so that

$$X_1 = A_1 \sin(\omega_1 t + \psi_1) + A_2 \sin(\omega_2 t + \psi_2)$$

$$X_2 = B_1 \sin(\omega_1 t + \psi_1) + B_2 \sin(\omega_2 t + \psi_2)$$

$$\therefore (K_1 + K_2 - m_1\omega^2)A - K_2 B = 0$$

$$\therefore \frac{A}{B} = \frac{K_2}{K_1 + K_2 - m_1\omega^2}$$

if $\omega = \omega_1$

$$\frac{B_1}{A_1} = \frac{k_1 + k_2 - m\omega_1^2}{K_2} = \lambda_1$$

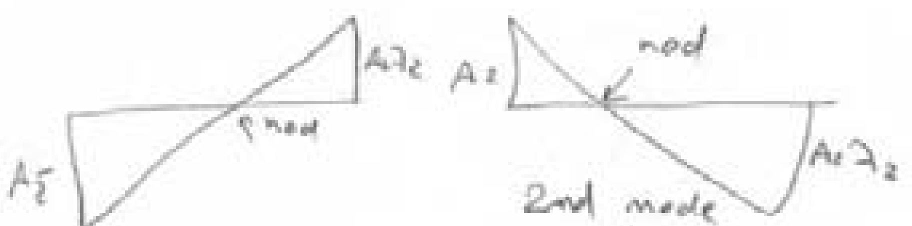
$$B_1 = A_1 \lambda_1$$

if $\omega = \omega_2$

$$\frac{B_2}{A_2} = \frac{k_1 + k_2 - m\omega_2^2}{K_2} = \lambda_2$$

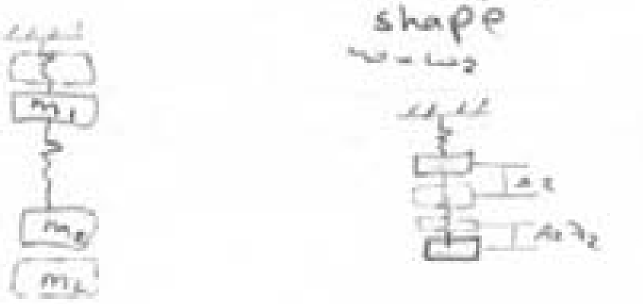
$$\therefore X_1 = A_1 \sin(\omega_1 t + \psi_1) + A_2 \sin(\omega_2 t + \psi_2)$$

$$X_2 = A_1\lambda_1 \sin(\omega_1 t + \psi_1) + A_2\lambda_2 \sin(\omega_2 t + \psi_2)$$

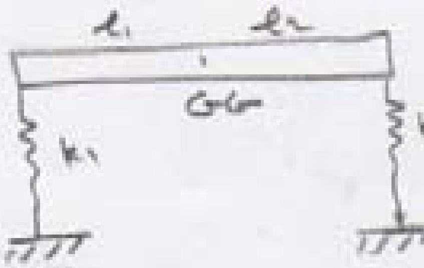


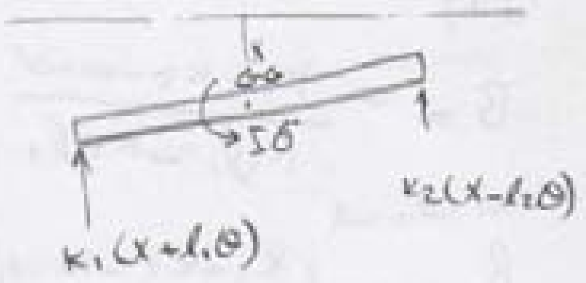
2nd mode shape
$\omega = \omega_2$

1st mode shape
$\omega = \omega_1$



Coordinate couplings:





$\Sigma F = m\ddot{x}$

$m\ddot{x} = -k_1(x+l_1\theta) - k_2(x-l_2\theta)$

$m\ddot{x} = -k_1x - k_1l_1\theta - k_2x + k_2l_2\theta$

$m\ddot{x} + (k_1+k_2)x + (k_1l_1 - k_2l_2)\theta = 0$ —— ①

$\Sigma T = I\ddot{\theta}$

$I\ddot{\theta} = -k_1(x+l_1\theta)l_1 + k_2(x-l_2\theta)l_2$

$I\ddot{\theta} + (k_1l_1^2 + k_2l_2^2)\theta + (k_1l_1 - k_2l_2)x = 0$ —— ②

$x = A\sin(\omega t + \psi)$

$\theta = B\sin(\omega t + \psi)$

$(k_1 + k_2 - m\omega^2)A + (k_1l_1 - k_2l_2)B = 0$ —— *

$(k_1l_1 - k_2l_2)A + (k_1l_1^2 + k_2l_2^2 - I\omega^2)B = 0$ —— **

$$\underbrace{\begin{bmatrix} k_1 + k_2 - m\omega^2 & k_1l_1 - k_2l_2 \\ k_1l_1 - k_2l_2 & k_1l_1^2 + k_2l_2^2 - I\omega^2 \end{bmatrix}}_{=0} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

$$\therefore \omega^4 - \left[\frac{k_1+k_2}{m} + \frac{k_1l_1^2 + k_2l_2^2}{I}\right]\omega^2 + \frac{(k_1+k_2)(k_1l_1^2+k_2l_2^2) - (k_1l_1 - k_2l_2)^2}{mI} = 0$$

this gives ω_1 & ω_2

Mode shapes:

$$(k_1 + k_2 - m\omega^2)A + (k_1 l_1 - k_2 l_2)B = 0$$

$$B = -\frac{k_1 + k_2 - m\omega^2}{k_1 l_1 - k_2 l_2} A$$

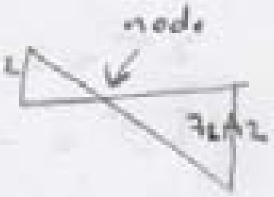
at $\omega = \omega_1$

$$B_1 = -\frac{(k_1 + k_2 - m\omega_1^2)}{(k_1 l_1 - k_2 l_2)} A_1 = \lambda_1 A_1$$

λ_1 should be posetive

if $\omega = \omega_2$

$$B_2 = \frac{-(k_1 + k_2 - m\omega_2^2)}{k_1 l_1 - k_2 l_2} A_2 = \lambda_2 A_2$$



Dynamic Absorber :-

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = F_o \sin\omega t \quad \text{— ①}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \quad \text{— ②}$$

$x_1 = A\sin\omega t$

$x_2 = B\sin\omega t$

$$-m_1\omega^2 A\sin\omega t + (k_1+k_2)A\sin\omega t - k_2 B\sin\omega t = F_o\sin\omega t$$

$$-m_1\omega^2 A + (k_1+k_2)A - k_2 B = F_o$$

$$(k_1+k_2-m_1\omega^2)A - k_2 B = F_o \quad \text{— *}$$

$$\therefore A = \frac{F_o(k_2 - m_2\omega^2)}{\Delta\omega} \qquad \Delta\omega = \begin{vmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{vmatrix}$$

$$B = \frac{F_o k_2}{\Delta\omega}$$

For dynamic absorber

$$\omega^2 = \frac{k_2}{m_2}$$

$$A = \frac{F_o\left(k_2 - m_2\frac{k_2}{m_2}\right)}{\Delta\omega} = 0$$

$$\therefore \Delta\omega = (k_1+k_2-m_1\omega^2)(k_2-m_2\omega^2) - k_2^2$$

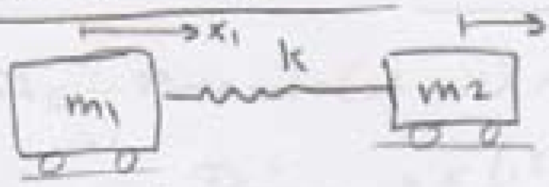
$$\therefore \Delta\omega = -k_2$$

$$B = -\frac{F_o}{k_2}$$

$$\therefore x_1 = 0 \qquad x_2 = -\frac{F_o}{k_2}\sin\omega t$$

For dynamic absorber $\frac{k_1}{m_1} = \frac{k_2}{m_2}$

Semi definite system:-



$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \quad \text{——} 1$$

$$m_2\ddot{x}_2 + k(x_2 - x_1) = 0 \quad \text{——} 2$$

$$x_1 = A\sin\omega t + \psi \qquad , \quad x_2 = B\sin(\omega t + \psi)$$

$$-m_1\omega^2 A + kA - kB = 0$$

$$(k - m_1\omega^2)A - kB = 0 \quad \text{——} *$$

$$(k - m_2\omega^2)B - kA = 0 \quad \text{——} **$$

Frequency determinte may be written as

$$\begin{vmatrix} k - m_1\omega^2 & -k \\ -k & k - m_2\omega^2 \end{vmatrix} = 0$$

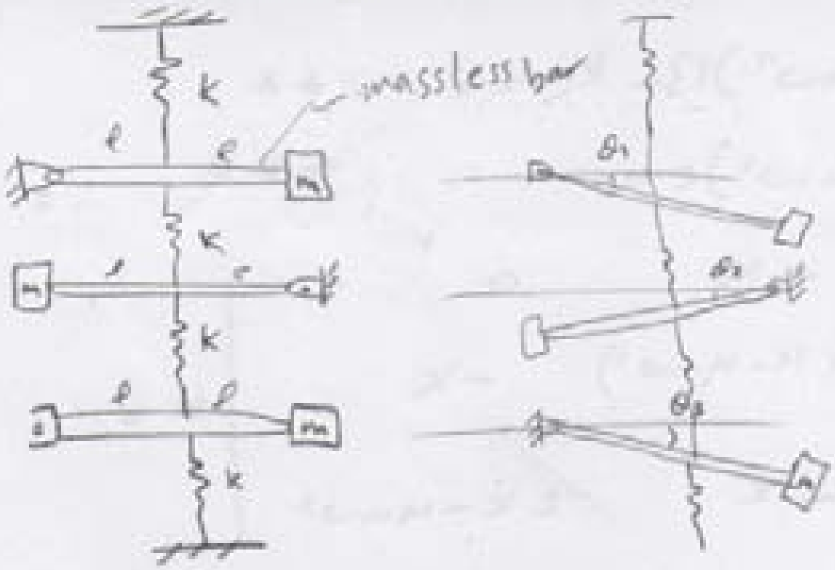
$$\omega^2\left[-k(m_1+m_2) + m_1 m_2\omega^2\right] = 0$$

eithe $\omega^2 = 0$

or $\omega_{12} = \mp\sqrt{\dfrac{k(m_1+m_2)}{m_1 m_2}}$

Even the system is two degree of freedum it behaves Like a single degree of freedum since it has one natural frequency The system Vibrates as one complete unite such system is called semi difinite system.

Multiple Degree of freedom systems:



$\Sigma T = I_1 \ddot{\theta}_1 \qquad I_1\ddot{\theta}_1 = -kl^2\theta_1 - k(l\theta_1 - l\theta_2)l$

$$I_1\ddot{\theta}_1 + 2kl^2\theta_1 - kl^2\theta_2 = 0 \quad \text{—①}$$

$\Sigma T = I_2\ddot{\theta}_2 \qquad I_2\ddot{\theta}_2 = -k(l\theta_2 - l\theta_1)l - k(l\theta_2 - l\theta_3)l$

$$\therefore I_2\ddot{\theta}_2 + 2kl^2\theta_2 - kl^2\theta_1 - kl^2\theta_3 = 0 \quad \text{—②}$$

$\Sigma T = I_3\ddot{\theta}_3 \qquad I_3\ddot{\theta}_3 = -k(l\theta_3 - l\theta_2)l - kl\theta_3 l$

$$I_3\ddot{\theta}_3 + 2kl^2\theta_3 - kl^2\theta_2 = 0 \quad \text{—③}$$

if $I_1 = I_2 = I_3 = m(2l)^2 = 4ml^2$

$4m\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = 0$

$4m\ddot{\theta}_2 + 2k\theta_2 - k\theta_1 - k\theta_3 = 0$

$4m\ddot{\theta}_3 + 2k\theta_3 - k\theta_2 = 0$

Assume $\theta_1 = A\sin(\omega t + \psi) \longrightarrow \ddot{\theta}_1$

$\theta_2 = B\sin(\omega t + \psi) \qquad \ddot{\theta}_2$

$\theta_3 = C\sin(\omega t + \psi) \qquad \ddot{\theta}_3$

$$(2K-4m\omega^2)A - KB = 0 \quad --- \; *$$

$$-KA + (2K-4m\omega^2)B - KC = 0 \quad **$$

$$-KB + (2K-4m\omega^2)C = 0 \quad ***$$

$$\begin{vmatrix} 2K-4m\omega^2 & -K & 0 \\ -K & (2K-4m\omega^2) & -K \\ 0 & -K & 2K-4m\omega^2 \end{vmatrix} = 0$$

$$(2K-4m\omega^2)\left[16m^2\omega^4 - 16Km\omega^2 + 2K\right] = 0$$

either $\quad 2K - 4m\omega^2 = 0$

$$\omega_2 = \frac{1}{\sqrt{2}}\sqrt{\frac{K}{m}} = 0.707\sqrt{\frac{K}{m}}$$

$$16m^2\omega^4 - 16Km\omega^2 + 2K^2 = 0$$

$$\omega^4 - \frac{K}{m}\omega^2 + \frac{K^2}{8m^2} = 0$$

either

$$\omega^2 = 0.1465\frac{K}{m} \implies \omega_1 = 0.3827\sqrt{\frac{K}{m}}$$

or

$$\omega^2 = 0.8535\frac{K}{m} \qquad \omega_3 = 0.9238\sqrt{\frac{K}{m}}$$

<u>Mode shape</u>

$$(2K-4m\omega^2)A - KB = 0 \quad *$$

$$-KA + (2K-4m\omega^2)B - KC = 0 \quad -- \; **$$

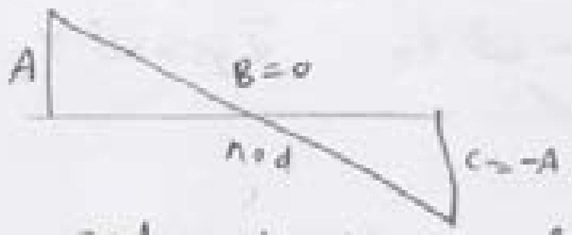
$$-KB + (2K-4m\omega^2)C = 0 \quad ***$$

$$B = \frac{2K-4m\omega^2}{K}A \qquad \text{if } \omega = \omega_1 \quad \therefore B = 1.414A$$

1st mode shape for $\omega_1 = 0.3827\sqrt{\frac{k}{m}}$

if $\omega = \omega_2 = 0.707\sqrt{\frac{k}{m}}$

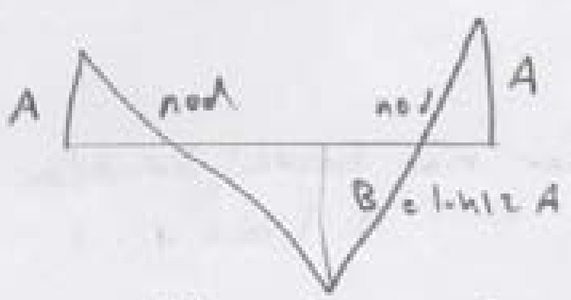
$\therefore C = -A \qquad B = 0$



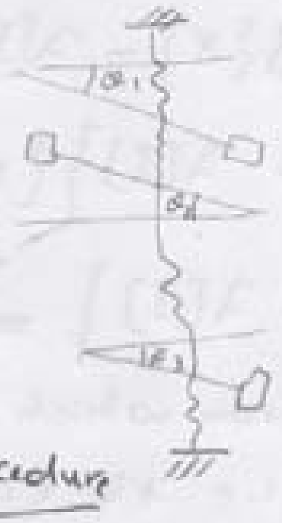
2nd mode shape for $\omega_2 = 0.707\sqrt{\frac{k}{m}}$

if $\omega = \omega_2 = 0.9238\sqrt{\frac{k}{m}}$

$\therefore C = A \qquad B = -1.412A$



3rd mod shape



Eigenvalues & Eigenvectors procedure

For multiple degree of freedom system without damping & with free vibration the eq. of motion governing such system in matrix notation

$$[M]\{\ddot{x}\} + [K]\{X\} = 0$$

Multiplying eq. by $[M]^{-1}$

$$[M]^{-1}[M]\{\ddot{X}\} + [M]^{-1}[K]\{x\} = 0$$

$$[I]\{\ddot{x}\} + [D]\{x\} = 0$$

$[D]$ is called the dynamic matrix

$$D = [M]^{-1}[k]$$

In general for harmonic motion

$$x = A\sin\omega t \qquad \ddot{x} = -\omega^2 \underbrace{(A\sin\omega t)}_{x} = -x\omega^2$$

$$\therefore \ddot{x}_1 = -\omega^2 x_1 \qquad \ddot{x}_2 = -\omega^2 x_2 \qquad \ddot{x}_3 = -\omega^2 x_3$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = -\omega^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\{\ddot{x}\} = -\omega^2\{x\}$$

$$\therefore [D]\{x\} - \omega^2[I]\{x\} = 0$$

let $\omega^2 = \lambda$

$$[D]\{x\} - \lambda[I]\{x\} = 0$$

$$\left[[D] - \lambda[I]\right]\{x\} = 0 \quad (*) \text{ for non trival Solution } |\ | = 0$$

$$\left|[D] - \lambda[I]\right| = 0$$

and from which we may deduced the values of λ_i i.e the eigen values.

The natural frequencies are calculated as follows

$$\omega_i = \sqrt{\lambda_i} \qquad i = 1, 2 \ldots n$$

<u>Eigen vectors (mode shape)</u>

let $[D] - \lambda[I] = [B]$

$$[B]^{-1} = \frac{Adj\,[B]}{|B|}$$

pre multiplying by $|B|\,[B]$

$$|B|\,[B][B]^{-1} = \frac{|B|\,[B]\,Adj\,[B]}{|B|}$$

$$|B|\,[I] = [B]\,Adj\,[B]$$

when $\lambda = \lambda_i$

$$0 = [[D]-\lambda[I]]\,Adj\,[[D]-\lambda_i[I]] \quad \text{--- } 2*$$

comparison of eq (1*) and (2*) after substituting $\lambda = \lambda_i$ gives that each column of the $Adj\,[D-\lambda I]$ must equal $\{x\}_i$ or multiplyed by an arbitrary construt which represent the eigenvector (mode shape) at any value of λ_i

Ex: Solve the prevouse example using the eigen values procedure find the natural frequences & their associated mode shape.

$$4m\ddot{\theta}_1 + 2K\theta_1 - K\theta_2 = 0$$
$$4m\ddot{\theta}_2 + 2K\theta_2 - K\theta_1 - K\theta_3 = 0$$
$$4m\ddot{\theta}_3 + 2K\theta_3 - K\theta_2 = 0$$

$$\begin{bmatrix} 4m & 0 & 0 \\ 0 & 4m & 0 \\ 0 & 0 & 4m \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 2K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & 2K \end{bmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = 0$$

$$[M]\,\{\ddot{\theta}\} + [K]\,\{\theta\} = 0$$

$$[M]^{-1} = \begin{pmatrix} \frac{1}{4m} & 0 & 0 \\ 0 & \frac{1}{4m} & 0 \\ 0 & 0 & \frac{1}{4m} \end{pmatrix}$$

$$\therefore D = [M]^{-1}[K] = \begin{bmatrix} \frac{2K}{4m} & \frac{-K}{4m} & 0 \\ \frac{-K}{4m} & \frac{2K}{4m} & \frac{-K}{4m} \\ 0 & \frac{-K}{4m} & \frac{2K}{4m} \end{bmatrix}$$

For non trival solu:

$$[D]-\lambda[I]=0$$

$$\left(\begin{bmatrix} \frac{2k}{4m} & \frac{-k}{4m} & 0 \\ \frac{-k}{4m} & \frac{2k}{4m} & \frac{-k}{4m} \\ 0 & \frac{-k}{4m} & \frac{2k}{4m} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} \frac{2k}{4m}-\lambda & \frac{-k}{4m} & 0 \\ \frac{-k}{4m} & \frac{2k}{4m}-\lambda & \frac{-k}{4m} \\ 0 & \frac{-k}{4m} & \frac{2k}{4m}-\lambda \end{vmatrix} = 0$$

$$\therefore \left(\frac{2k}{4m}-\lambda\right)\left[\lambda^2 - \frac{k}{m}\lambda + \frac{k^2}{8m^2}\right] = 0$$

eithe $\frac{2k}{4m}-\lambda = 0 \Rightarrow \lambda_2 = \frac{k}{2m}$ $\quad \omega_2 = 0.707\sqrt{\frac{k}{m}}$

or $\lambda^2 - \frac{k}{m}\lambda + \frac{k^2}{8m^2} = 0$

$\therefore \lambda_1 = 0.1468\frac{k}{m} \Rightarrow \omega_1 = 0.3927\sqrt{\frac{k}{m}}$

$\lambda_3 = 0.853\frac{k}{m} \quad \omega_3 = 0.923\sqrt{\frac{k}{m}}$

Let $[[D]-\lambda[I]] = [B]$

$Adj\,[B] = [B]_c^T$

$$[B]_{coeff} = (-1)^n \Big| \quad \Big|$$

$n = i + j$ $\quad$ i = No. of raw
$\quad\quad\quad\quad\quad$ j = " column

$$B = \begin{bmatrix} \frac{2k}{4m} - \lambda & \frac{-k}{4m} & 0 \\ \frac{-k}{4m} & \frac{2k}{4m} - \lambda & \frac{-k}{4m} \\ 0 & -\frac{k}{4m} & \frac{2k}{4m} - \lambda \end{bmatrix}$$

$$B_{coef} = \begin{pmatrix} \left(\frac{2k}{4m} - \lambda\right)^2 - \left(\frac{k}{4m}\right)^2 & \frac{k}{4m}\left(\frac{2k}{4m} - \lambda\right) & \left(\frac{k}{4m}\right)^2 \\ \frac{k}{4m}\left(\frac{2k}{4m} - \lambda\right) & \left(\frac{2k}{4m} - \lambda\right)^2 & \frac{k}{4m}\left(\frac{2k}{4m} - \lambda\right) \\ \left(\frac{k}{4m}\right)^2 & \frac{k}{4m}\left(\frac{2k}{4m} - \lambda\right) & \left(\frac{2k}{4m} - \lambda\right)^2 - \left(\frac{k}{4m}\right)^2 \end{pmatrix}$$

for this problem $[B]_{coeff}$ is symetric matrix

$$\text{Adj}\,[B] = [B]^T_{coef} = [B]_{coeff}$$

Eigenvectors (mode shape)

Subsituting $\lambda_1 = 0.1468 \frac{k}{m}$ in to Adj $[B]$

$$\frac{k^2}{m^2}\begin{pmatrix} 0.062109 & 0.08825 & 0.0625 \\ 0.08825 & 0.124609 & 0.08825 \\ 0.0625 & 0.08825 & 0.062109 \end{pmatrix}$$

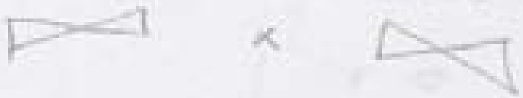
$$0.06209 \begin{bmatrix} 1 \\ 1.42 \\ 1.006 \end{bmatrix} \quad 0.08825 \begin{bmatrix} 1 \\ 1.412 \\ 1 \end{bmatrix} \quad 0.0625 \begin{bmatrix} 1 \\ 1.412 \\ 0.9937 \end{bmatrix}$$

Choose $\begin{bmatrix} 1 \\ 1.412 \\ 1 \end{bmatrix}$

1st mode shape
1st eigenvector

where $\lambda = \lambda_2 = 0.5\frac{k}{m}$

$$\frac{k^2}{m^2}\begin{bmatrix} -0.0625 & 0 & 0.0625 \\ 0 & 0 & 0 \\ 0.0625 & 0 & -0.0625 \end{bmatrix}$$



$$-0.0625\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad 0.0625\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Choose $\begin{bmatrix}1\\0\\-1\end{bmatrix}$

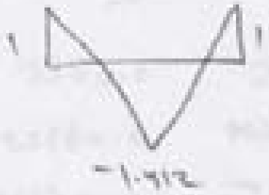
2nd mode shape for $\lambda_2 = 0.5\frac{k}{m}$

put $\lambda = \lambda_3 = 0.853\frac{k}{m}$ in to Adj[B]

$$\frac{k^2}{m^2}\begin{bmatrix} 0.06219 & -0.08825 & -0.0625 \\ -0.08825 & 0.124609 & -0.08825 \\ 0.0625 & -0.08825 & 0.062109 \end{bmatrix}$$

$$0.062109\begin{bmatrix}1\\-1.42\\1.006\end{bmatrix} \quad -0.08825\begin{bmatrix}1\\-1.412\\1\end{bmatrix} \quad 0.0625\begin{bmatrix}1\\-1.412\\0.9937\end{bmatrix}$$

Choose $\begin{bmatrix}1\\-1.412\\1\end{bmatrix}$



3rd mode shape from $\lambda = 0.853\frac{k}{m}$