Subject : Mathematics II
Weekly Hours : Theoretical: 2 UNITS: 4
Tutorial: 1
Experimental :

موضوع: رياضيات II
الساعات الأسبوعية: نظري :2 الوحدات : 4
مناقشة: 1
عملي:

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Differential Equations

Partial Differential Equations

هي تلك المعادلات التي تحتوي على المشتقة لأكثر من متغير

$$\frac{\partial^2 V}{\partial P^2} + \frac{\partial^2 V}{\partial t^2} + \cdots$$

Ordinary Differential Equations

هي تلك المعادلات التي تحتوي على المشتقة لمتغير واحد فقط

$$\left(\frac{d^2 y}{d x^2}\right)^3 + x \frac{dy}{dx} \cdots\cdots$$

degree = 3

order = 2

Ordinary Differential Equations

المعادلات التفاضلية الاعتيادية

non-linear لاخطية

linear خطية

تسمى المعادلة التفاضلية الاعتيادية خطية عند الحالات التالية :-

1. عدم ضرب المشتقة في نفسها او في مشتقة ثانية ، مثل :

$$\bar{y}\,\bar{\bar{y}} + 2y = 0 \qquad \& \qquad (\bar{y})^2 + 2xy = \bar{\bar{y}}$$

2. عدم ضرب المتغير المعتمد dependent variable (y) بالمشتقة

$$y\,\bar{\bar{y}} + 2xy = 0 \qquad \& \qquad y\,\bar{y} + \bar{\bar{y}} = x$$

3. عدم ضرب المشتقة الثانية مأخوذة في دالة لـ x ، مثل :-

$$\cos x \cdot \bar{\bar{y}} + 2\bar{y} = 0$$

Linear Differential Equations

- 1^{st} order المرتبة الأولى
  1. separable منفصلة
  2. homogeneous متجانسة
  3. exact تامة
  4. linear خطية
  5. Bernoulli's eq. معادلة برنولي
     (non-linear)
- 2^{nd} and higher order

2^{nd} order Differential Eqns.

- Reducible to 1^{st} order
  قابلة للاختزال إلى الرتبة الأولى
- homogeneous
  متجانسة
  $a\ddot{y}+b\dot{y}+cy=0$
- non-homogeneous
  غير متجانسة
  $a\ddot{y}+b\dot{y}+cy=f(x)$

higher order D. E.

- homogeneous
- non-homogeneous

Differential Equations :

المعادلة التفاضلية : هي تلك المعادلة التي تحتوي على المشتقة وحل المعادلة التفاضلية هو التخلص من المشتقة .

1^{st} order Differential Equations :

① Separable : (منفصلة)

هي تلك المعادلة التي يمكن عبرها فصل متغيرات x على حدة ومتغيرات y على حدة بحيث تكون مكتوبة بالشكل التالي :-

$$f(x)\, dx = g(y)\, dy$$

ex-1 : Solve $\dfrac{dy}{dx} = \dfrac{x\sqrt{1+y^2}}{2-3x^2}$

Sol<u>n</u>

$$\int \frac{dy}{\sqrt{1+y^2}} = \int \frac{x\, dx}{2-3x^2}$$

$$\sinh^{-1} y = -\frac{1}{6} \ln|2-3x^2| + c$$

② Homogeneous متجانسة

أي معادلة اذا جدّدنا فيها x ب (λx) و y ب (λy) تبقى المعادلة دون تغيير . بحيث يمكن كتابة تلك المعادلة بالشكل التالي

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

تسمى متجانسة ولكي تختزل الى منفصلة نفرض أن

$$\frac{y}{x} = v$$

ex.2: Solve $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

soln

$x \rightarrow \lambda x \quad \& \quad y \rightarrow \lambda y$

$$\frac{d(\lambda y)}{d(\lambda x)} = \frac{\lambda x + \lambda y}{\lambda x - \lambda y}$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{x+y}{x-y} \Longrightarrow$ بقت المعادلة دون تغيير

$\therefore$ It is homogeneous

نقسم المعادلة على x ، فنحصل :

$$\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = f\left(\frac{y}{x}\right) \qquad \text{—— (1)}$$

let $v = \dfrac{y}{x} \Longrightarrow y = x\,v \Longrightarrow dy = x\,dv + v\,dx$

بالقسمة على dx $\Longleftarrow$

$$\therefore \quad \frac{dy}{dx} = x\frac{dv}{dx} + v \qquad \text{—— (2)}$$

by substituting Eq.(2) into Eq.(1), gets :-

$$x\frac{dv}{dx} + v = \frac{1+v}{1-v} \Longrightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{1+v-v+v^2}{1-v} \Longrightarrow x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

It is separable

$$\therefore \int\frac{dx}{x} = \int\frac{1-v}{1+v^2}\,dv \Longrightarrow \int\frac{dx}{x} = \int\frac{dv}{1+v^2} - \int\frac{v\,dv}{1+v^2}$$

$$\ln|x| = \tan^{-1} v - \frac{1}{2}\ln|1+v^2| + c$$

$$\ln|x| = \tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| + c$$

ex. 3: Solve $\dfrac{dy}{dx} = \dfrac{x-2y+1}{3x-6y+4}$

Soln: $\dfrac{dy}{dx} = \dfrac{(x-2y)+1}{3(x-2y)+4}$

let $x-2y=u \Rightarrow dx-2dy=du$ $\div dx$

$1-2\dfrac{dy}{dx} = \dfrac{du}{dx} \Rightarrow \dfrac{dy}{dx} = \frac{1}{2}\left(1-\dfrac{du}{dx}\right)$

$\therefore \frac{1}{2}\left(1-\dfrac{du}{dx}\right) = \dfrac{u+1}{3u+4} \Rightarrow 1-\dfrac{du}{dx} = \dfrac{2u+2}{3u+4}$

$\dfrac{du}{dx} = 1-\dfrac{2u+2}{3u+4} \Rightarrow \dfrac{du}{dx} = \dfrac{3u+4-2u-2}{3u+4}$

$\therefore \dfrac{du}{dx} = \dfrac{u+2}{3u+4}$ separable

$\therefore \int \dfrac{(3u+4)}{u+2}\,du = \int dx$

let $u+2=t \Rightarrow du=dt$ & $u=t-2$

$\therefore \int \dfrac{3(t-2)+4}{t}\,dt = \int dx$

$\int \dfrac{3t-2}{t}\,dt = \int dx \Rightarrow \int 3\,dt - 2\int \dfrac{dt}{t} = \int dx$

$\therefore 3t - 2\ln|t| = x+c$

$\therefore 3(u+2) - 2\ln|u+2| = x+c$

$3((x-2y)+2) - 2\ln|(x-2y)+2| = x+c$

ex.4 : Solve $\frac{dy}{dx} = \frac{x-y-2}{x+y+4}$

sol$\underline{n}$: let

$$\left.\begin{aligned} x &= X+h \\ y &= Y+k \end{aligned}\right\}$$ h & K are constants

$$\frac{dy}{dx} = \frac{dY}{dX}$$

$$\therefore \quad \frac{dY}{dX} = \frac{(X+h)-(Y+k)-2}{(X+h)+(Y+k)+4}$$

$$= \frac{X-Y+(h-k-2)}{X+Y+(h+k+4)} \quad \text{———} \circledast$$

المعادلة ⊛ ليست متجانسة لوجود الثوابت ولكنها تصبح متجانسة

إذا فرضنا أنّ :

$$h-k-2=0$$

$$h+k+4=0$$

بالجمع

$$2h+2=0 \Rightarrow h=-1 \quad \therefore \quad k=-3$$

$$\therefore \quad x = X-1 \quad \& \quad y = Y-3$$

وعليه فإنّ المعادلة ⊛ تصبح كما يلي :-

$$\frac{dY}{dX} = \frac{X-Y}{X+Y} \quad \text{———} \circledast\circledast$$ It is homogeneous

$$\therefore \quad \frac{dY}{dX} = \frac{1-\frac{Y}{X}}{1+\frac{Y}{X}} \quad \text{———} (1)$$

let $\frac{Y}{X} = v \Rightarrow \frac{dY}{dX} = X\frac{dv}{dX} + v \quad \text{———}(2)$

by substituting Eq.(2) into Eq.(1), gets :

$$X\frac{dv}{dX} + v = \frac{1-v}{1+v} \Rightarrow X\frac{dv}{dX} = \frac{1-v}{1+v} - v \Rightarrow$$

$$X\frac{dv}{dX} = \frac{1-v-v-v^2}{1+v} \Rightarrow X\frac{dv}{dX} = \frac{1-2v-v^2}{1+v}$$

$$\therefore \int \frac{(1+v)\,dv}{1-2v-v^2} = \int \frac{dx}{x} \Rightarrow -\frac{1}{2}\ln|1-2v-v^2| = \ln x + c$$

$$\therefore -\frac{1}{2}\ln\left|1-2\left(\frac{Y}{X}\right)-\left(\frac{Y}{X}\right)^2\right| = \ln|X| + c$$

$$-\frac{1}{2}\ln\left|1-2\left(\frac{y+3}{x+1}\right)-\left(\frac{y+3}{x+1}\right)^2\right| = \ln|x+1| + c$$

ملاحظة مهمة : أي حل المعادلة $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

نستخرج قيمة M حيث أن :-

$$M = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

فاذا كانت قيمة $M = 0$ نستعمل طريقة حل ex.3

واذا كانت $M \neq 0$ فنستعمل طريقة ex.1

ex.3 $\quad \frac{dy}{dx} = \frac{x-2y+1}{3x-6y+1} \Rightarrow M = \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = -6+6 = 0$

ex.1 $\quad \frac{dy}{dx} = \frac{x-y-2}{x+y+1} \Rightarrow M = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1 = 2 \neq 0$

③ Exact تامة

اية معادلة تكتب بالشكل التالي :-

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

دائما موجبة

بحيث أن :-

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

فأن المعادلة تسمى تامة

ملاحظة : اية معادلة تحتوي على دالة مثلثية او دالة زائدية او لوغاريتمية او أسية فانها ليست متجانسة .

ex. 5 : Solve $\dfrac{dy}{dx} = \dfrac{x^3 - 5x^4y^3}{3x^5y^2 - \sin y}$

soln :

المعادلة ليست منفصلة لانه لا يمكن فصل متغيرات x على حدة ومتغيرات y على حدة وليست متجانسة لاحتوائها على دالة مثلثية .

$\therefore \ (3x^5y^2 - \sin y)\,dy = (x^3 - 5x^4y^3)\,dx$

$\therefore \ (3x^5y^2 - \sin y)\,dy + (-x^3 + 5x^4y^3)\,dx = 0$

$\left.\begin{array}{l} \dfrac{\partial}{\partial x}(3x^5y^2 - \sin y) = 15x^4y^2 \\ \dfrac{\partial}{\partial y}(-x^3 + 5x^4y^3) = 15x^4y^2 \end{array}\right\}$ متساويان

$\therefore$ It is exact

الآن، نفتح الأقواس

$\underbrace{3x^5y^2\,dy}_{\text{صعب التكامل}} - \sin y\,dy - x^3\,dx + \underbrace{5x^4y^3\,dx}_{\text{صعب التكامل}} = 0$

صعب التكامل ⟸ لاحتوائهما على x و y معاً ⟹ صعب التكامل

الحدود التي لا نستطيع ان نكاملها نحصرها بين أقواس

$(3x^5y^2\,dy + 5x^4y^3\,dx) - \sin y\,dy - x^3\,dx = 0$

$\int d(x^5y^3) - \int \sin y\,dy - \int x^3\,dx = 0$

دائماً نجده اذا كانت المعادلة تامة وهو عبارة عن مشتقة حاصل ضرب دالتين

$x^5y^3 + \cos y - \dfrac{x^4}{4} = C$

Integrating Factor العامل التكاملي

بعض المعادلات ليست تامة ولكنها تصبح تامة بعد ضربها بعامل تكاملي مناسب

* مشتقة حاصل ضرب دالتين $x\,dy + y\,dx = d(xy)$

$x\,dy - y\,dx = d(\,?\,)$

احتمالات العامل التكاملي عديدة جداً

$x\,dy - y\,dx = d(\,?\,)$

* نقسم على x^2 $\quad \dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$ * مشتقة قسمة دالتين

* نقسم على y^2 $\quad \dfrac{x\,dy - y\,dx}{y^2} = \dfrac{-(y\,dx - x\,dy)}{y^2} = -d\left(\dfrac{x}{y}\right)$

* نقسم على xy $\quad \dfrac{x\,dy - y\,dx}{xy} = \dfrac{dy}{y} - \dfrac{dx}{x} = d(\ln y) - d(\ln x)$

$= d(\ln y - \ln x) = d\left(\ln \dfrac{y}{x}\right)$

ex. 6: solve $\dfrac{dy}{dx} = \dfrac{y - 5x^4 y^7}{x}$

Sol^{n}

هذه المعادلة ليست متجانسة ولا منفصلة

$x\,dy = (y - 5x^4y^7)\,dx$

$x\,dy + (-y + 5x^4y^7)\,dx = 0$

$\left.\begin{array}{l} \dfrac{\partial}{\partial x}(x) = 1 \\ \dfrac{\partial}{\partial y}(-y + 5x^4y^7) = -1 + 35x^4y^6 \end{array}\right\}$ غير متساويين

$\therefore\; x\,dy - y\,dx = -5x^4y^7\,dx$ not exact

الشيء الذي يجعلنا نستخدم العامل التكاملي هو قسمة المعادلة السابقة على x^2

$$\therefore \quad \frac{x\,dy - y\,dx}{x^2} = -5x^2 y^7\,dx$$

يجب التخلص من y بالضرب ب $\left(\frac{x^7}{x^7}\right)$

$$\therefore \quad d\left(\frac{y}{x}\right) = -5x^9\left(\frac{y}{x}\right)^7 dx$$

$$\frac{d\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^7} = -5x^9\,dx$$

$$\therefore \quad \int \left(\frac{y}{x}\right)^{-7} d\left(\frac{y}{x}\right) = -5\int x^9\,dx$$

$$-\frac{1}{6}\left(\frac{y}{x}\right)^{-6} = -5\,\frac{x^{10}}{10} + c$$

④ Linear الخطية

أية معادلة تُكتب بالشكل التالي

$$\boxed{\frac{dy}{dx} + p(x)\,\overset{1}{y} = Q(x)}$$

تُسمى خطية ب y (linear in y) ولكي تكون تامة تُضرب بالعامل التكاملي

$$I.F. = e^{\int p(x)\,dx}$$

أو تكون مكتوبة بالشكل التالي

$$\frac{dx}{dy} + P(y)\,\overset{1}{x} = Q(y) \qquad \text{linear in } x$$

$$I.F. = e^{\int P(y)\,dy}$$

ex-4 : solve $\dfrac{dy}{dx} = \dfrac{3y - 4x^5}{x}$

sol^n :

المعادلة ليست منفصلة ولا متجانسة ولا تامة

$$\therefore \quad \frac{dy}{dx} = \frac{3}{x}\,y - 4x^4$$

or $$\frac{dy}{dx} + \left(-\frac{3}{x}\right) y^1 = -4x^4 \quad \text{———} (*)$$

linear in y , $p(x) = -\dfrac{3}{x}$

$$\therefore \quad I.F. = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$$

Eq.$(*)$ is multiplied by $(x^{-3}\,dx)$

$$x^{-3}\,dy - 3x^{-4}\,y\,dx = -4x\,dx$$

It is exact

$$\therefore \quad \int d(x^{-3}y) = \int -4x\,dx$$

$$\therefore \quad x^{-3}y = -2x^2 + c$$

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⑤ **Bernouli Equation** معادلة برنولي

أية معادلة تكتب بالشكل التالي

$$\boxed{\frac{dy}{dx} + p(x)\,y^1 = Q(x)\,y^n} \qquad (\text{Bernouli in } y)$$

ولكي تكون خطية بـ z نفرض ان

$$y^{1-n} = z$$

or $$\frac{dx}{dy} + p(y)\,x^1 = Q(y)\,x^n \qquad (\text{Bernouli in } x)$$

$$x^{1-n} = z$$

ex. 8: Solve $\dfrac{dy}{dx} = \dfrac{2x^5y^3 - 4y}{x}$

soln:

ليست منفصلة ، غير تامة ، غير متجانسة

$$\frac{dy}{dx} = 2x^4y^3 - \frac{4}{x}y \implies \frac{dy}{dx} + \left(\frac{4}{x}\right)y = 2x^4y^3 \quad \text{---} (*)$$

Bernouli in y $\left(\dfrac{dy}{dx} + p(x)\,y = Q(x)\,y^n \right)$

let $y^{1-3} = z$

$$\therefore \quad z = y^{-2} \implies \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$$

Eq. $(*)$ is multiplied by $(2y^{-3})$, gets

$$-2y^{-3}\frac{dy}{dx} - \frac{8}{x}y^{-2} = -4x^4$$

or

$$\frac{dz}{dx} + \left(-\frac{8}{x}\right)z = -4x^4 \quad \text{---} (**)$$

(linear in z, $p(x) = -\frac{8}{x}$)

$$\text{I.F.} = e^{\int -\frac{8}{x}dx}$$

$$= e^{-8\ln x} = e^{\ln x^{-8}} = x^{-8}$$

Eq. $(**)$ is multiplied by $(x^{-8}\,dx)$, gets:

$$(x^{-8}\,dz - 8x^{-9}z\,dx) = -4x^{-4}\,dx$$

$$\int d(x^{-8}\cdot z) = \int -4x^{-4}\,dx$$

$$x^{-8}z = -\frac{4}{-3}x^{-3} + c$$

$$x^{-8}y^{-2} = \frac{4}{3}x^{-3} + c$$

Second Order Differential Equations :

1- Reducible to 1^{st} order قابلة للاختزال الى الرتبة الأولى

2. Homogeneous $a\ddot{y} + b\dot{y} + cy = 0$ متجانسة

3. Non-homogeneous غير متجانسة

① Reducible to 1^{st} order :

وهي حالة خاصة من المعادلات التي عندها نفرض أن $\dot{y} = P$

ثم نجري عملية التكامل ان أمكن

ex-9 : Solve $\ddot{y} - x(\dot{y})^2 = 0$.

Soln : put $\dot{y} = P \Rightarrow \ddot{y} = \dfrac{dP}{dx}$

$$\therefore \frac{dP}{dx} - xP^2 = 0 \qquad \text{separable}$$

$$\therefore \int \frac{dP}{P^2} = \int x\,dx \Rightarrow -\frac{1}{P} = \frac{x^2}{2} + c$$

$$\therefore P = \frac{-1}{\frac{x^2}{2} + c} \Rightarrow \frac{dy}{dx} = \frac{-1}{\frac{x^2}{2} + c} \qquad \text{separable}$$

$$\therefore \int dy = -\frac{1}{c} \int \frac{\sqrt{2c} * \frac{dx}{\sqrt{2c}}}{1 + \left(\frac{x}{\sqrt{2c}}\right)^2}$$

$$\therefore y = -\sqrt{\frac{2}{c}}\, \tan^{-1}\left(\frac{x}{\sqrt{2c}}\right) + k$$

2. Homogeneous :

أية معادلة تكتب بالشكل التالي

$$a\ddot{y} + b\dot{y} + cy = 0 \quad \text{——} (1)$$

or $$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

or $$a\,D^2y + b\,Dy + cy = 0 \qquad \text{where } D = \frac{d}{dx}$$

let $y = e^{mx}$, $\dot{y} = m e^{mx}$, $\ddot{y} = m^2 e^{mx}$

نعوض في معادلة (1) فنحصل

$$a m^2 e^{mx} + b m e^{mx} + c e^{mx} = 0$$

$$(a m^2 + a m + c)\, e^{mx} = 0$$

$e^{mx} \neq 0 \quad \therefore \quad am^2 + bm + c = 0 \quad \text{——} (2)$

معادلة (2) تسمى المعادلة المميزة (characteristic eq.)

والمعادلة المميزة جذران هما m_1 و m_2 والجذرين m_1, m_2

لها ثلاث حالات :-

① If $m_1 \neq m_2 \Rightarrow y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

② If $m_1 = m_2 = m \Rightarrow y = c_1 e^{mx} + c_2 x e^{mx}$

③ If $\left.\begin{matrix} m_1 = \alpha + i\beta \\ m_2 = \alpha - i\beta \end{matrix}\right\} \Rightarrow y = e^{\alpha x}\{c_1 \sin\beta x + c_2 \cos\beta x\}$

where $i = \sqrt{-1}$

ex.10: solve $\ddot{y} + 4\dot{y} + 3y = 0$

sol$\underline{n}$ let $y = e^{mx} \Rightarrow m^2 + 4m + 3 = 0$

$(m+3)(m+1) = 0 \Rightarrow m_1 = -3$ & $m_2 = -1$

$\therefore \quad y = c_1 e^{-3x} + c_2 e^{-x}$

ex-11: Solve $\ddot{y} - 4\dot{y} + 4y = 0$

soln: let $y = e^{mx}$ $\Longrightarrow$ $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$ $\Longrightarrow$ $m_1 = m_2 = 2$

$y = c_1 e^{2x} + c_2 x e^{2x}$

ex-12: Solve $\ddot{y} + 2\dot{y} + 5y = 0$

characteristic eq. $m^2 + 2m + 5 = 0$

$$m_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \mp \sqrt{4 - 4 * 1 * 5}}{2}$$

$= -1 \mp 2i$ $\alpha = -1$, $\beta = 2$

$\therefore$ $y = e^{(-1)x} \{ c_1 \sin 2x + c_2 \cos 2x \}$

ex-13: solve $\ddot{y} - 6\dot{y} + 9y = 0$

soln

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

or $D^2y - 6Dy + 9y = 0$

or $(D^2 - 6D + 9)y = 0$

$(D - 3)^2 y = 0$

نفس الحل

نكتب m بدل D

let $y = e^{mx}$ $\Longrightarrow$ $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$ $\Longrightarrow$ $m_1 = m_2 = 3$

$\therefore$ $y = c_1 e^{3x} + c_2 x e^{3x}$

3- Non-homogeneous :

أي معادلة تكتب بالشكل التالي :

$$a\ddot{y} + b\dot{y} + cy = f(x)$$

لهذه المعادلة حلان :-

① الحل التجانس (y_h) homogeneous solution

let $a\ddot{y} + b\dot{y} + cy = 0$

② الحل الخاص (y_p) particular solution

$$y = y_h + y_p$$

طريقة ايجاد الحل الخاص

1. Undetermined Coefficients Method

* طريقة المعاملات غير المعينة :

اذا احتوت المعادلة على احد هذه الدوال في الطرف الثاني فأن y_p تكون كما موضح في الجدول

$f(x)$	y_p
① e^{ax}	Ke^{ax} $\Rightarrow$ اذا كانت e^{ax} غير موجودة في y_h
	Kxe^{ax} $\Rightarrow$ اذا كانت e^{ax} موجودة في y_h مرة واحدة
	Kx^2e^{ax} $\Rightarrow$ اذا كانت e^{ax} موجودة في y_h مرتان
② x^n	$Ax^n + Bx^{n-1} + \dots + K$
$2x^2+1$	$Ax^2 + Bx + C$
③ $\sin ax$, $\cos ax$, $2\sin ax \mp 3\cos ax$	$A\sin ax + B\cos ax$
④	$x\{A\sin ax + B\cos ax\}$ $\Rightarrow$ اذا كانت دالة الـ $\sin ax$ او الـ $\cos ax$ موجودة في الـ y_h

ex.14: solve $y'' - 2y' + y = 3e^{2X} - 5e^{4X}$

soln: characteristics eq. is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0 \implies m_1 = m_2 = 1$$

$$y_h = c_1 e^x + c_2 x e^x$$

let

$$y_p = K e^{2X} + H e^{4X}$$
$$y'_p = 2K e^{2X} + 4H e^{4X}$$
$$y''_p = 4K e^{2X} + 16H e^{4X}$$

نعوض في المعادلة الأصلية

$(4Ke^{2X} + 16He^{4X}) - 2(2Ke^{2X} + 4He^{4X}) + (Ke^{2X} + He^{4X}) = 3e^{2X} - 5e^{4X}$

$\therefore\ 4Ke^{2X} + 16He^{4X} - 4Ke^{2X} - 8He^{4X} + Ke^{2X} + He^{4X} = 3e^{2X} - 5e^{4X}$

$$\implies 9He^{4X} + Ke^{2X} = 3e^{2X} - 5e^{4X}$$

$$\therefore\ 9H = -5 \implies H = -\frac{5}{9}$$

$$\&\quad K = 3$$

$$\therefore\ y_p = 3e^{2X} - \frac{5}{9}e^{4X}$$

$$\therefore\ y = c_1 e^x + c_2 x e^x + 3e^{2X} - \frac{5}{9}e^{4X}$$

ex-15: Solve $\bar{\bar{y}} - 6\bar{y} + 9y = 2e^{3x}$

Sol$\underline{\underline{n}}$

$$\bar{\bar{y}}_h - 6\bar{y}_h + 9y_h = 0$$

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0 \Rightarrow m_1 = m_2 = 3$$

$$\therefore\ y_h = c_1 e^{3x} + c_2 x e^{3x}$$

$$f(x) = 2e^{3x} \longrightarrow$$ (مكرر مرتين في y_h)

$$\therefore \text{ let } \ y_p = K x^2 e^{3x}$$

$$\bar{y}_p = 2Kx e^{3x} + 3Kx^2 e^{3x}$$

$$\bar{\bar{y}}_p = 2K(3xe^{3x} + e^{3x}) + 3K(3x^2e^{3x} + 2xe^{3x})$$

نعوض عن y_p ، $\bar{y}_p$ ، $\bar{\bar{y}}_p$ في المعادلة الأصلية

$$(9Kx^2e^{3x} + 12Kxe^{3x} + 2Ke^{3x}) - 6(2Kxe^{3x} + 3Kx^2e^{3x}) + 9(Kx^2e^{3x}) = 2e^{3x}$$

$$2Ke^{3x} = 2e^{3x} \Rightarrow K = 1$$

$$\therefore\ y_p = x^2 e^{3x}$$

$$\therefore\ y = c_1 e^{3x} + c_2 x e^{3x} + x^2 e^{3x}$$

H.W.

ex-16: Solve $\bar{\bar{y}} - 4\bar{y} + 3y = 5e^{3x}$

Ans.: $y = c_1 e^{3x} + c_2 e^{x} + \frac{5}{2} x e^{3x}$

ex.17: Solve $y'' - 4y' - 5y = 2\sin 2x$

Soln: $y'' - 4y' - 5y = 0$

ch. eq. $m^2 - 4m - 5 = 0$

$(m-5)(m+1) = 0 \Rightarrow m_1 = 5 \ \& \ m_2 = -$

$\therefore \ y_h = c_1 e^{5x} + c_2 e^{-x}$

let

$$\left.\begin{aligned} y_p &= A \sin 2x + B \cos 2x \\ y_p' &= 2A \cos 2x - 2B \sin 2x \\ y_p'' &= -4A \sin 2x - 4B \cos 2x \end{aligned}\right\}$$ نعوضها في المعادلة الأصلية

$(-4A \sin 2x - 4B \cos 2x) - 4(2A \cos 2x - 2B \sin 2x)$

$-5(A \sin 2x + B \cos 2x) = 2 \sin 2x$

$\Rightarrow (-4A + 8B - 5A)\sin 2x + (-4B - 8A - 5B)\cos 2x$

$= 2 \sin 2x$

نقارن المعاملات

$$\left.\begin{aligned} \sin 2x &: \quad 8B - 9A = 2 \\ \cos 2x &: \quad -9B - 8A = 0 \end{aligned}\right\} \Rightarrow A = -\frac{18}{145} \ \& \ B = \frac{16}{145}$$

$\therefore \ y = y_h + y_p$

$= c_1 e^{5x} + c_2 e^{-x} - \frac{18}{145} \sin 2x + \frac{16}{145} \cos 2x$

ex.18: Solve $y'' + y' - 2y = 2x - 5x^3$

Soln $y'' + y' - 2y = 0$

ch. eq. $m^2 + m - 2 = 0 \Rightarrow (m+2)(m-1) = 0$

$\therefore \ m_1 = 1 \ \& \ m_2 = -2$

$\therefore \ y_h = c_1 e^{x} + c_2 e^{-2x}$

let
$$\left.\begin{aligned} y_p &= Ax^3 + Bx^2 + Cx + D \\ \bar{\bar{y}}_p &= 3Ax^2 + 2Bx + C \\ \bar{\bar{y}}_p &= 6Ax + 2B \end{aligned}\right\}$$ نعوض في المعادلة الأصلية

$$(6Ax + 2B) + (3Ax^2 + 2Bx + C) - 2(Ax^3 + Bx^2 + Cx + D) = 2x - 5x^3$$

نقارن المعاملات

x^3 : $\quad -2A = -5 \quad \Rightarrow A = \frac{5}{2}$

x^2 : $\quad 3A - 2B = 0 \quad \Rightarrow B = \frac{15}{4}$

x : $\quad 6A + 2B - 2C = 2 \quad \Rightarrow C = \frac{41}{4}$

x^0 : $\quad 2B + C + 2D = 0 \quad \Rightarrow D = \frac{71}{8}$

$$\therefore \quad y_p = \frac{5}{2}x^3 + \frac{15}{4}x^2 + \frac{41}{4}x + \frac{71}{8}$$

$$\therefore \quad y = y_h + y_p$$
$$= c_1 e^{x} + c_2 e^{-2x} + \frac{5}{2}x^3 + \frac{15}{4}x^2 + \frac{31}{4}x + \frac{71}{8}$$

2. Variation of Parameters : طريقة تغيير الثوابت

وهي طريقة عامة إلا أنها تكون في بعض الأحيان صعبة بسبب صعوبة التكامل .

$$a\bar{\bar{y}} + b\bar{y} + cy = f(x)$$

$$y_h = c_1 y_1 + c_2 y_2$$

let
$$y_p = v_1(x)\, y_1 + v_2(x)\, y_2$$

ثم

معادلات أساسية لإيجاد v_2 & v_1

$$\boxed{\begin{aligned} \bar{v}_1 y_1 + \bar{v}_2 y_2 &= 0 \\ \bar{v}_1 \bar{y}_1 + \bar{v}_2 \bar{y}_2 &= f(x) \end{aligned}}$$

A

B

نستخدم قاعدة كرامر (Cramer Rule) لإيجاد $V_1(x)$ & $V_2(x)$ من معادلتي Ⓐ و Ⓑ

$$V_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \& \quad V_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

ex. 19 : solve $y'' + y = \sec x$

Sol$\underline{n}$:

لهذه المعادلة لا يمكن إيجاد الحل باستخدام طريقة المعاملات غير المعينة.

نجد الحل المتجانس $\Rightarrow$ $y'' + y = 0 \Rightarrow$ ch. eq. $m^2 + 1 = 0$

$\therefore \quad m_1 = 0 + i \quad , \quad m_2 = 0 - i$

$\alpha = 0 \quad , \quad \beta = 1$

$\therefore \quad y_h = e^{0 \cdot x} \{ c_1 \sin x + c_2 \cos x \} \quad \therefore$

$y_h = c_1 \sin x + c_2 \cos x$

i.e. $y_h = c_1 y_1 + c_2 y_2$

let $y_p = V_1 \sin x + V_2 \cos x$

$\therefore \quad V_1' \sin x + V_2' \cos x = 0$ —— Ⓐ

$V_1' \cos x + V_2' (-\sin x) = \sec x$ —— Ⓑ

$$V_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \cdot \sec x}{-\sin^2 x - \cos^2 x} = \frac{-1}{-1} = 1$$

$\therefore \quad V_1' = 1 \quad \therefore \quad V_1 = x$

$$\acute{V}_2 = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \cdot \sec x}{-1} = -\sin x \cdot \frac{1}{\cos x}$$

$$\therefore V_2 = \int \frac{-\sin x}{\cos x}\, dx = \ln|\cos x| + c$$

$$\therefore \quad y_p = V_1 \sin x + V_2 \cos x$$

$$= x \sin x + (\ln|\cos x|)\cos x$$

$$\therefore \quad y = y_h + y_p$$

** -- **

<u>ex</u>. 20 : Solve $y'' - 6y' + 9y = 2e^{3x}$

Sol<u>n</u> : $y_h \Rightarrow y'' - 6y' + 9y = 0$

ch. eq. $\Rightarrow m^2 - 6m + 9 = 0$

$$(m-3)(m-3) = 0 \Rightarrow m_1 = m_2 = 3$$

$$\therefore \quad y_h = c_1 e^{3x} + c_2 x e^{3x}$$

let $y_p = V_1 e^{3x} + V_2 x e^{3x}$

$$\therefore \quad \acute{V}_1 e^{3x} + \acute{V}_2 x e^{3x} = 0 \quad \text{—— (1)}$$

$$\acute{V}_1 (3e^{3x}) + \acute{V}_2 (3xe^{3x} + e^{3x}) = 2e^{3x} \quad \text{—— (2)}$$

بقسمة معادلتي (1) و (2) على e^{3x} نحصل :

$$\acute{V}_1 + x\acute{V}_2 = 0 \quad \text{—— (A)}$$

$$\acute{V}_1 3 + (3x+1)\acute{V}_2 = 2 \quad \text{—— (B)}$$

$$\bar{V}_1 = \frac{\begin{vmatrix} 0 & x \\ 2 & 3x+1 \end{vmatrix}}{\begin{vmatrix} 1 & x \\ 3 & 3x+1 \end{vmatrix}} = \frac{0-2x}{3x+1-3x} = -2x$$

$$\therefore V_1 = \int -2x\,dx \implies V_1 = -x^2$$

$$\bar{V}_2 = \frac{\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & x \\ 3 & 3x+1 \end{vmatrix}} = \frac{2-0}{1} = 2 \implies V_2 = 2x$$

$$\begin{aligned} \therefore \quad y_p &= V_1 e^{3x} + V_2\, x e^{3x} \\ &= -x^2 e^{3x} + (2x)\, x e^{3x} \\ &= -x^2 e^{3x} + 2x^2 e^{3x} \\ &= x^2 e^{3x} \end{aligned}$$

$$\begin{aligned} \therefore \quad y &= y_h + y_p \\ &= c_1 e^{3x} + c_2 x e^{3x} + x^2 e^{3x} \end{aligned}$$

3. Laplace Transformation method
4. D. operator method
5. Series solution

Higher Order Differential Equations :

ex. 21 : Solve $(D-3)(D^2-3D+2)y=0$

Soln ch. eq. $(m-3)(m^2-3m+2)=0$

$(m-3)(m-2)(m-1)=0$

$\therefore \quad m_1 = 1 \quad , \quad m_2 = 2 \quad , \quad m_3 = 3$

$\therefore \quad y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

ex. 22 : Solve $(D-2)(D+4)^3 y = 0$

Soln ch. eq. $(m-2)(m+4)^3 = 0$

$\therefore \quad m_1 = 2 \quad \& \quad m_2 = m_3 = m_4 = -4$

$\therefore \quad y = c_1 e^{2x} + c_2 e^{-4x} + c_3 x e^{-4x} + c_4 x^2 e^{-4x}$

$= c_1 e^{2x} + (c_4 x^2 + c_3 x + c_2) e^{-4x}$

ex. 23 : Solve $(D-4)(D^2+4)y = 0$

Soln ch. eq. $(m-4)(m^2+4) = 0$

$\therefore \quad m_1 = 4 \quad \& \quad m_{2,3} = 0 \mp 2i$

$\therefore \quad y = c_1 e^{4x} + e^{0 \cdot x} \{ c_2 \sin 2x + c_3 \cos 2x \}$

University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Chapter ( ): Partial Differentiation
Dr. Akeel Abdullah Mohammed

Partial Differentiation :

$$w = f(x, y)$$

$$\frac{\partial w}{\partial x} = w_x = f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = w_y = f_y = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

ex1: If $w = x^3 + 2x^4y^5 + 5y^3 + \sin(\frac{x}{y})$ then find w_x & w_y .

Sol$_n$

$$w_x = 3x^2 + 2y^5(4x^3) + 0 + \cos(\frac{x}{y}) * \frac{1}{y}$$

$$w_y = 0 + 2x^4(5y^4) + 15y^2 + \cos(\frac{x}{y}) * (-\frac{x}{y^2})$$

ex.2: If $w = f(\frac{x}{y})$ then show that $x w_x + y w_y = 0$

Sol$_n$

$$w_x = \acute{f}(\frac{x}{y}) \cdot \frac{1}{y} = \frac{1}{y} \acute{f}(\frac{x}{y})$$

$$w_y = \acute{f}(\frac{x}{y}) \cdot (-\frac{x}{y^2}) = -\frac{x}{y^2} \acute{f}(\frac{x}{y})$$

$$\therefore \; x w_x + y w_y = x \cdot (\frac{1}{y} \acute{f}(\frac{x}{y})) + y(-\frac{x}{y^2} \acute{f}(\frac{x}{y}))$$

$$= \frac{x}{y} \acute{f}(\frac{x}{y}) - \frac{x}{y} \acute{f}(\frac{x}{y}) = 0$$

ex. 3 : If $w = x^n f\left(\frac{x^2}{x^2+y^2}\right)$ then show that :

$$x\, w_x + y\, w_y = n\, w$$

Sol$\underline{n}$: $w_x = x^n \bar{f}\left(\frac{x^2}{x^2+y^2}\right) * \frac{(x^2+y^2)2x - x^2(2x)}{(x^2+y^2)^2} +$

$$f\left(\frac{x^2}{x^2+y^2}\right) * n\, x^{n-1}$$

$$= \frac{2y^2 x^{n+1}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) + n\, x^{n-1} f\left(\frac{x^2}{x^2+y^2}\right) \quad \text{—(}$$

$$w_y = x^n \bar{f}\left(\frac{x^2}{x^2+y^2}\right) * x^2(-1)(x^2+y^2)^{-2} * 2y$$

$$= -\frac{2y\, x^{n+2}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) \quad \text{———— (2}$$

$$x\, w_x + y\, w_y = x\cdot\left\{\frac{2y^2 x^{n+1}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) + n\, x^{n-1} f\left(\frac{x^2}{x^2+y^2}\right)\right.$$

$$\left. + y\cdot\left\{\frac{-2y\, x^{n+2}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right)\right\}\right\}$$

$$= \frac{2y^2 x^{n+2}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right) + n\, x^n f\left(\frac{x^2}{x^2+y^2}\right)$$

$$- \frac{2y^2 x^{n+2}}{(x^2+y^2)^2} \bar{f}\left(\frac{x^2}{x^2+y^2}\right)$$

$$= n\, x^n f\left(\frac{x^2}{x^2+y^2}\right)$$

$$= n\, w$$

Ex. 4 : If $x y^2 z^3 + x^3 z + y^3 z^2 = 2$ then find $\frac{\partial z}{\partial y}, \frac{\partial y}{\partial x}$ and show that $\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial z} = -1$

Soln :

$$x\left[y^2 \cdot 3z^2 \cdot \frac{\partial z}{\partial y} + z^3 \cdot 2y\right] + x^3 \cdot \frac{\partial z}{\partial y} + \left\{y^3 \cdot 2z \cdot \frac{\partial z}{\partial y} + z^2 \cdot 3y^2\right\} = 0$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{2xyz^3 + 3y^2z^2}{3xz^2y^2 + x^3 + 2zy^3} \quad \text{———} (1)$$

$$z^3\left[x \cdot 2y \cdot \frac{\partial y}{\partial x} + y^2 \cdot 1\right] + z \cdot 3x^2 + z^2 \cdot 3y^2 \cdot \frac{\partial y}{\partial x} = 0$$

$$\therefore \frac{\partial y}{\partial x} = -\frac{y^2z^3 + 3x^2z}{2xyz^3 + 3z^2y^2} \quad \text{———} (2)$$

$$y^2\left\{x \cdot 3z^2 + z^3 \cdot \frac{\partial x}{\partial z}\right\} + \left\{x^3 + z \cdot 3x^2 \cdot \frac{\partial x}{\partial z}\right\} + y^3 \cdot 2z = 0$$

$$\therefore \frac{\partial x}{\partial z} = -\frac{2y^3z + x^3 + 3xy^2z^2}{z^3y^2 + 3x^2z} \quad \text{———} (3)$$

by multiplying Eqns (1), (2) & (3) in each to other, gets:

$$\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial z} = -1$$

رموز المشتقة الجزئية

$$w_{xx} = \frac{\partial w}{\partial x^2} = \frac{\partial}{\partial x}(w_x)$$

Theorem : If $f(x,y)$ is contineous with a contineous partial derivative then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad (\text{or} \quad f_{xy} = f_{yx})$$

ex. 5 : If $w = x^2 + 3x^4y^3$ then ① Find w_{xxyx} ② show that $w_{xy} = w_{yx}$ ③ Prove that $w_{yyyy} = 0$

Sol : ①

$$w = x^2 + 3x^4y^3$$
$$w_x = 2x + 12x^3y^3$$
$$w_{xx} = 2 + 36x^2y^3$$
$$w_{xxy} = 0 + 108x^2y^2$$
$$w_{xxyx} = 216xy^2$$

②

$$\left.\begin{aligned} w_{xy} &= 36x^3y^2 \\ w_y &= 9x^4y^2 \\ w_{yx} &= 36x^3y^2 \end{aligned}\right\} \therefore \quad w_{yx} = w_{xy}$$

③

$$w = x^2 + 3x^4y^3$$
$$w_y = 9x^4y^2$$
$$w_{yy} = 18x^4y$$
$$w_{yyy} = 18x^4$$
$$w_{yyyy} = 0$$

ex. 6 : If $z = f(2u+3v, 3u-2v)$ then show that :

$$\frac{\partial^2 z}{\partial u \partial v} = 6\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2}$$

where $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and $x = 2u+3v$ & $y = 3u-2v$

Soln let $x = 2u+3v$ & $y = 3u-2v$

$\therefore \quad z = f(x, y)$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$z_u = z_x(2) + z_y(3)$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial v}(z_u) = \frac{\partial}{\partial v}(2z_x + 3z_y)$$

$$= \frac{\partial}{\partial x}(2z_x + 3z_y)\cdot\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}(2z_x + 3z_y)\cdot\frac{\partial y}{\partial v}$$

$$= (2z_{xx} + 3z_{yx})\cdot(3) + (2z_{xy} + 3z_{yy})\cdot(-2)$$

$$= 6z_{xx} + 9z_{yx} - 4z_{xy} - 6z_{yy}$$

$$= 6z_{xx} + 5z_{xy} - 6z_{yy}$$

or

$$= 6\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2}$$

Total Differential :

التفاضل الكلي

$$\boxed{d(f(x,y)) = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy}$$

where dx & dy tend to zero

ex. a $d(x^3y^5) = \frac{\partial}{\partial x}(x^3y^5)\,dx + \frac{\partial}{\partial y}(x^3y^5)\,dy$

$= 3x^2y^5\,dx + 5x^3y^4\,dy$

ex. b $d(u^2v^4) = \frac{\partial}{\partial u}(u^2v^4)\,du + \frac{\partial}{\partial v}(u^2v^4)\,dv$

$= 2uv^4\,du + 4u^2v^3\,dv$

ex. 7 : If $z = f(x,y)$ and $g(x,y) = c$ then show that :

$$\frac{dz}{dx} = \frac{g_y f_x - f_y g_x}{g_y}$$

Sol^n $z = f(x,y) \Rightarrow dz = d(f(x,y)) \Rightarrow$

$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Rightarrow dz = f_x\,dx + f_y\,dy$

$\therefore \quad \frac{dz}{dx} = f_x + f_y\left(\frac{dy}{dx}\right)$ ——— (1)

$g(x,y) = c \Rightarrow d(g(x,y)) = d(c)$

$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \Rightarrow g_x\,dx + g_y\,dy = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{g_x}{g_y}$ ——— (2)

by substituting Eq.(2) into Eq.(1), gets :

$$\frac{dz}{dx} = f_x + f_y\left(-\frac{g_x}{g_y}\right) = \frac{g_y f_x - f_y g_x}{g_y}$$

Transformation :

ex. 8 : If $x = e^u \cos v$ & $y = e^u \sin v$, then find $\frac{\partial u}{\partial x}$.

Soln

Note : $\frac{\partial u}{\partial x} \neq \frac{1}{\left(\frac{\partial x}{\partial u}\right)}$

method ① : Inverse Transformation

طريقة التحويل العكسي

$\frac{\partial x}{\partial u} = e^u \cos v$; $\frac{\partial x}{\partial v} = -e^u \sin v$

$\frac{\partial y}{\partial u} = e^u \sin v$; $\frac{\partial y}{\partial v} = e^u \cos v$

hence ;

$x = e^u \cos v \Rightarrow x^2 = e^{2u} \cos^2 v$

$y = e^u \sin v \Rightarrow y^2 = e^{2u} \sin^2 v$

بالجمع

$x^2 + y^2 = e^{2u} (\cos^2 v + \sin^2 v)$
$= e^{2u}$

$\therefore 2u = \ln(x^2 + y^2) \Rightarrow u = \frac{1}{2} \ln(x^2+y^2)$

& $\frac{\sin v}{\cos v} = \frac{y}{x} \Rightarrow \tan v = \frac{y}{x} \Rightarrow v = \tan^{-1} \frac{y}{x}$

$\therefore$ $\boxed{u = \frac{1}{2} \ln(x^2+y^2) \qquad v = \tan^{-1}\left(\frac{y}{x}\right)}$ Inverse Transformation

$\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \Rightarrow \frac{e^u \cos v}{e^{2u}}$ $\therefore$ $\boxed{\frac{\partial u}{\partial x} = \frac{\cos v}{e^u}}$

Note : Polar Transformation

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Inverse Polar Transformation

$$r = \sqrt{x^2 + y^2} \quad , \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

method ② : General Method : It is the best but in which the solution is long " الطريقة العامة "

$x = e^u \cos v \quad \Rightarrow \quad dx = d(e^u \cos v)$

$$dx = \frac{\partial}{\partial u}(e^u \cos v)\,du + \frac{\partial}{\partial v}(e^u \cos v)\,dv$$

$$dx = e^u \cos v\,du - e^u \sin v\,dv \quad \text{———} \quad (1)$$

$y = e^u \sin v \quad \Rightarrow \quad dy = d(e^u \sin v)$

$$dy = \frac{\partial}{\partial u}(e^u \sin v)\,du + \frac{\partial}{\partial v}(e^u \sin v)\,dv$$

$$dy = e^u \sin v\,du + e^u \cos v\,dv \quad \text{———} \quad (2)$$

by using Gramer's Rule between Eqns. (1) & (2)

$$e^u \cos v\,du - e^u \sin v\,dv = dx \quad \text{———} \quad (1)$$
$$e^u \sin v\,du + e^u \cos v\,dv = dy \quad \text{———} \quad (2)$$

$$du = \frac{\begin{vmatrix} dx & -e^u \sin v \\ dy & e^u \cos v \end{vmatrix}}{\begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}}$$

$$dv = \frac{\begin{vmatrix} e^u \cos v & dx \\ e^u \sin v & dy \end{vmatrix}}{\begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}}$$

$$\therefore \quad du = \frac{e^u \cos v\, dx + e^u \sin v\, dy}{e^{2u} \cos^2 v + e^{2u} \sin^2 v} = \frac{e^u \cos v\, dx + e^u \sin v\, dy}{e^{2u}}$$

$$= \frac{\cos v}{e^u}\, dx + \frac{\sin v}{e^u}\, dy$$

$$= \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\cos v}{e^u} \quad \& \quad \frac{\partial u}{\partial y} = \frac{\sin v}{e^u}$$

Home work : If $x = f(u, v)$ & $y = g(u, v)$, then show that

$$\frac{\partial u}{\partial x} = \frac{g_v}{g_v\, f_u - f_v\, g_u}$$

Hint : use general method

Chain Rule :

قاعدة التفاضل المتسلسل

law (1) : If $w = f(x,y)$ and $x = g(t)$, $y = h(t)$

then $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$

law (2) : If $w = f(x,y)$ and $x = g(t,r)$ & $y = h(t,r)$

then $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$

ex. 9 : If $w = u^3 + v^5 + uv$, $u = \sin r$, $v = \cos r$

then find $\frac{dw}{dr}$

Sol$\underline{n}$ $\frac{dw}{dr} = \frac{\partial w}{\partial u} \cdot \frac{du}{dr} + \frac{\partial w}{\partial v} \cdot \frac{dv}{dr}$

$= (3u^2 + v) \cos r + (5v^4 + u)(-\sin r)$

ex. 10 : If $w = f(x-y, y-z, z-x)$ then show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Sol$\underline{n}$ put $u = x-y$, $v = y-z$. $t = z-x$

$\therefore \quad w = f(u, v, t)$

$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial x}$

$= w_u \cdot (1) + w_v \cdot (0) + w_t \cdot (-1)$

$= w_u - w_t$ ———— (1)

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= w_u(-1) + w_v(1) + w_t(0)$$

$$= w_v - w_u \quad \text{———————— (2)}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= w_u(0) + w_v(-1) + w_t(0)$$

$$= w_t - w_v \quad \text{———————— (3)}$$

$$\therefore \quad \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = \text{Eq.(1)} + \text{Eq.(2)} + \text{Eq.(3)} = 0$$

* ~~ * ~~ * ~~ * ~~ *

Gradient Vector :

$$\overrightarrow{\nabla f} = \overrightarrow{\text{grad } f} = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j + \frac{\partial f}{\partial z}\, k$$

$$\vec{\nabla}(\ \ ) = \frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k$$

$\vec{\nabla}$ = Del - operator

Note : $D = \frac{d}{dx}$ = differential operator

ex. 11 : If $f(x,y,z) = x^3 y + z^5 x$ then find $\overrightarrow{\nabla f}$

Soln

$$\frac{\partial f}{\partial x} = 3x^2 y + z^5 \quad ; \quad \frac{\partial f}{\partial y} = x^3 \quad ; \quad \frac{\partial f}{\partial z} = 5z^4 x$$

$$\overrightarrow{\nabla f} = (3x^2 y + z^5)\, i + x^3\, j + 5z^4 x\, k$$

Divergence Vector

$\vec{\nabla} \cdot \vec{F}$ = Divergence of $\vec{F}$

$\vec{F}(x,y,z) = f_1(x,y,z)\,i + f_2(x,y,z)\,j + f_3(x,y,z)\,k$

= Vector Function

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(f_1) + \frac{\partial}{\partial y}(f_2) + \frac{\partial}{\partial z}(f_3)$$

Ex.12: Find $\operatorname{div}(\vec{F})$ for:

① $\vec{F}(x,y,z) = x^3 y\, i + y z^5\, j + x z^2\, K$

② $\vec{F}(x,y,z) = x\, i + y\, j + z\, K$

Soln ① $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x^3 y) + \frac{\partial}{\partial y}(y z^5) + \frac{\partial}{\partial z}(x z^2)$

$= 3x^2 y + z^5 + 2xz$

② $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 1 + 1 + 1$

$= 3$

Curl $\vec{F}$

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= +\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) K$$

Directional Derivative : المشتقة الاتجاهية

$$\boxed{\frac{df}{ds} = D_{\vec{u}} = \vec{\nabla f} \cdot \vec{u}}$$

where $\vec{u}$ is a unit vector

Theorem :

1. $\text{Max.}\left(\frac{df}{ds}\right) = |\vec{\nabla f}|$

2. $\text{Min.}\left(\frac{df}{ds}\right) = -|\vec{\nabla f}|$

and the direction is $\vec{u} = \frac{\vec{\nabla f}}{|\vec{\nabla f}|}$

Proof :

$$\frac{df}{ds} = \vec{\nabla f} \cdot \vec{u}$$

$$= |\vec{\nabla f}|\,\overbrace{|\vec{u}|}^{1}\cos\theta$$

$$= |\vec{\nabla f}|\cos\theta$$

but

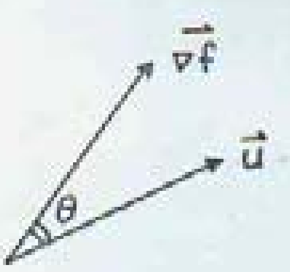
$$-1 \leq \cos\theta \leq 1$$

$$-|\vec{\nabla f}| \leq |\vec{\nabla f}|\cos\theta \leq |\vec{\nabla f}|$$

$$-|\vec{\nabla f}| \leq \frac{df}{ds} \leq |\vec{\nabla f}|$$

$\therefore$ $\text{Max.}\left(\frac{df}{ds}\right) = |\vec{\nabla f}|$ when $\theta = 0$

$\text{Min.}\left(\frac{df}{ds}\right) = -|\vec{\nabla f}|$



if $\theta = 0$ then $\vec{u} \parallel \vec{\nabla f}$

$\therefore$ $\vec{\nabla f} = t\,\vec{u}$

$$|\vec{\nabla f}| = t\,|\vec{u}|$$

$\therefore \quad t = |\vec{\nabla} f|$

$\therefore \quad \vec{\nabla} f = |\vec{\nabla} f| \; \vec{u}$

$\therefore \quad \vec{u} = \dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$

__ex.__ 13 : let $w = f(x, y, z) = x^2 + xy + z^3$ and let $P_1(2, 1, 1)$, Find

1. The maximum value of the directional derivative of f at P_1 (what is the direction)
2. The value of the directional derivative at P_1 towards $P_2(5, 4, 2)$.

Soln

$\dfrac{\partial f}{\partial x} = 2x + y \qquad \qquad \dfrac{\partial f}{\partial x}\Big|_{P_1} = 5$

$\dfrac{\partial f}{\partial y} = x \qquad \qquad \dfrac{\partial f}{\partial y}\Big|_{P_1} = 2$

$\dfrac{\partial f}{\partial z} = 3z^2 \qquad \qquad \dfrac{\partial f}{\partial z}\Big|_{P_1} = 3$

$\therefore \quad \vec{\nabla} f = 5i + 2j + 3K$

$$\text{Max.}\left(\frac{df}{ds}\right) = |\vec{\nabla} f| = \sqrt{5^2 + 2^2 + 3^2} = \sqrt{38}$$

The direction $\quad \vec{u} = \dfrac{\vec{\nabla} f}{|\vec{\nabla} f|} = \dfrac{1}{\sqrt{38}}(5i + 2j + 3K)$

2. $\quad \overrightarrow{P_1P_2} = 3i + 3j + K$

$$\vec{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{3i + 3j + K}{\sqrt{3^2 + 3^2 + 1^2}} = \frac{1}{\sqrt{19}}(3i + 3j + K)$$

$$\frac{df}{ds} = \vec{\nabla} f \cdot \vec{u} = (5i + 2j + 3K) \cdot \frac{1}{\sqrt{19}}(3i + 3j + K)$$

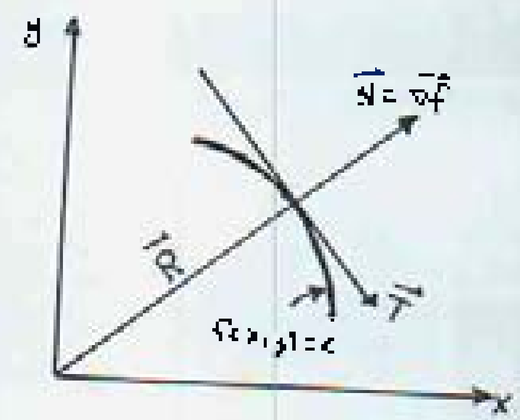
$$= \frac{24}{\sqrt{19}}$$

Equation of Tangent Plane and Normal line to the Surface $f(x,y,z)=c$

Theorem (1) : If $w=f(x,y)$ then the vector

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j \quad \text{is normal to the curve } f(x,y)=c$$

Proof : $\vec{R} = x\,i + y\,j$



$$\vec{T} = \frac{d\vec{R}}{ds} = \frac{dx\,i + dy\,j}{ds}$$

$$\vec{T} = \left(\frac{1}{ds}\right) d\vec{R}$$

$\therefore \quad d\vec{R} \parallel \vec{T} \qquad (\text{i.e } d\vec{R} \perp \vec{N})$

$$f(x,y)=c \quad \Rightarrow \quad d(f(x,y)) = d(c)$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \Rightarrow \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j\right)\cdot(dx\,i + dy\,j) = 0$$

$\therefore \quad \vec{\nabla f} \cdot d\vec{R} = 0 \qquad (\text{i.e. } \vec{\nabla f} \perp \text{curve})\ (\vec{\nabla f} = \vec{N})$

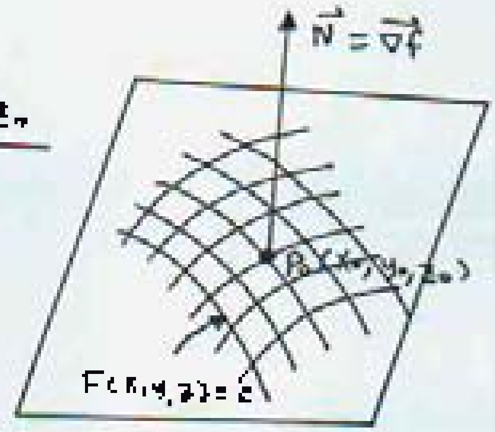
Theorem (2) : If $w=f(x,y,z)$ then the vector

$$\vec{\nabla f} = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j + \frac{\partial f}{\partial z}\, k \quad \text{is normal to the surface}$$

$$f(x,y,z) = c$$

Eq. of Normal line :

$$\frac{x - x_0}{\frac{\partial f}{\partial x}} = \frac{y - y_0}{\frac{\partial f}{\partial y}} = \frac{z - z_0}{\frac{\partial f}{\partial z}}$$



Eq. of Tangent Plane :

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(z - z_0) + \frac{\partial f}{\partial z}(z - z_0) = 0$$

ex. 14 : Find the equation of the tangent plane and the normal line for surface $x^2 + 3xy + z^3 = 5$ at $P_0(1, 1, 1)$.

Soln

$$F(x, y, z) = x^2 + 3xy + z^3$$

$$F_x = 2x + 3y \quad \Rightarrow \quad F_x = 5$$

$$F_y = 3x \quad \Rightarrow \quad F_y = 3$$

$$F_z = 3z^2 \quad \Rightarrow \quad F_z = 3$$

N.L.

$$\frac{x - 1}{5} = \frac{y - 1}{3} = \frac{z - 1}{3}$$

T.P.

$$5(x - 1) + 3(y - 1) + 3(z - 1) = 0$$

$$5x + 3y + 3z - 11 = 0$$

Maximum and Minimum Points for the Surface $z = f(x,y)$

Definition:

$f(a+h, b+k) < f(a,b)$

max. point

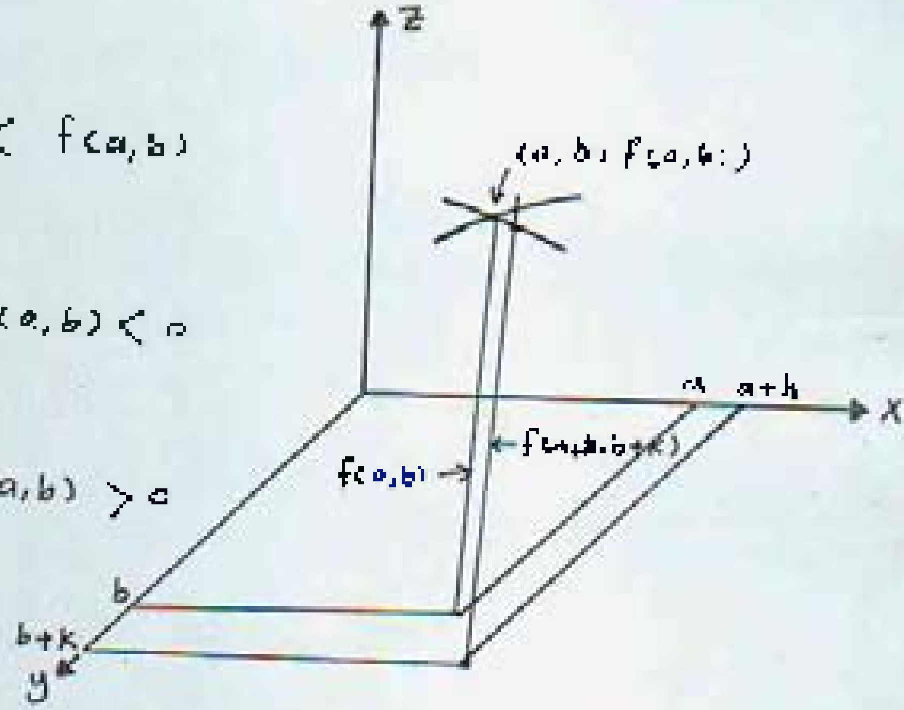
$f(a+h, b+k) - f(a,b) < 0$

min. point

$f(a+h, b+k) - f(a,b) > 0$

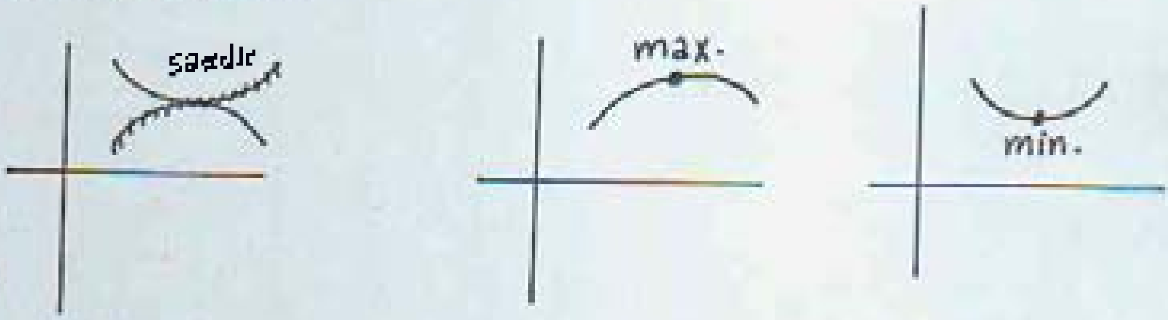
for all values of

h & K



M – Test (for $z = f(x,y,z)$)

1. Find $f_x = 0$, $f_y = 0$ and solve (say at $x=a$, $y=b$)
2. Find Point $(a, b, f(a,b))$ is a critical point
3. Find $M = f_{xx} f_{yy} - (f_{xy})^2$
4. If $M > 0$ and $f_{xx} < 0$ then $(a, b, f(a,b))$ is a max. point
5. If $M > 0$ and $f_{xx} > 0$ then $(a, b, f(a,b))$ is a min. point.
6. If $M < 0$ then $(a, b, f(a,b))$ is a saddle point.
7. If $M = 0$ or $f_{xy} \neq f_{yx}$ then the test fails, use the definition.



ex. 15 : Find the max., min., or a saddle points (if any) for $z = f(x,y) = x^2 + 2y^2 - 2xy - y + x$

Sol.

$f_x = 2x - 2y + 1 = 0 \Rightarrow 2x - 2y = -1$

$f_y = 4y - 2x - 1 = 0 \Rightarrow -2x + 4y = 1$

add: $2y = 0$

$\therefore \; y = 0$

$2x = -1 + 2y \Rightarrow x = \frac{1}{2}(-1 + 2*0) = -\frac{1}{2}$

$\therefore \; z = f(-\frac{1}{2}, 0) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$\therefore \; (-\frac{1}{2}, 0, -\frac{1}{4})$ is a critical point

$$\left.\begin{array}{l} f_{xx} = 2 \\ f_{yy} = 4 \\ f_{xy} = f_{yx} = -2 \end{array}\right\} M = (2)(4) - (-2)^2 = 8 - 4 = 4$$

$\therefore \; M = 4 > 0$

$f_{xx} = 2 > 0$

$\therefore \; (-\frac{1}{2}, 0, -\frac{1}{4})$ is a minimum point

University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Chapter ( ): Double & Triple Integrals
Dr. Akeel Abdullah Mohammed

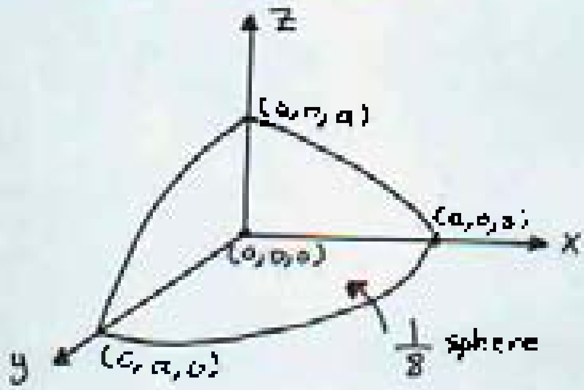
Double Integrals :

The equation of surface is $f(x,y,z)=0$ (or $z=f(x,y)$ which may be 1st order or 2nd order.

The Equations of Some Geometric Figures

1. Sphere

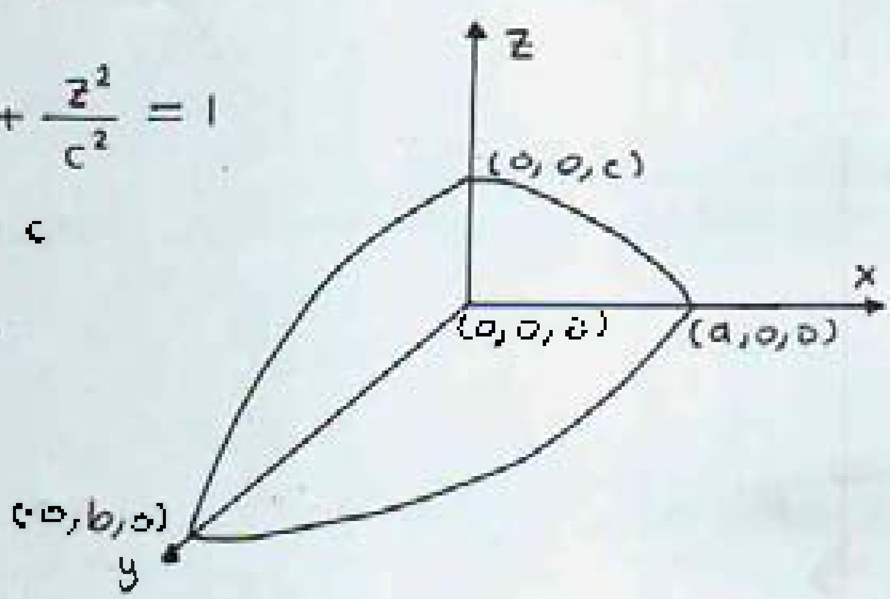
$$x^2+y^2+z^2=a^2$$



2. Ellipisoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$a \neq b \neq c$$

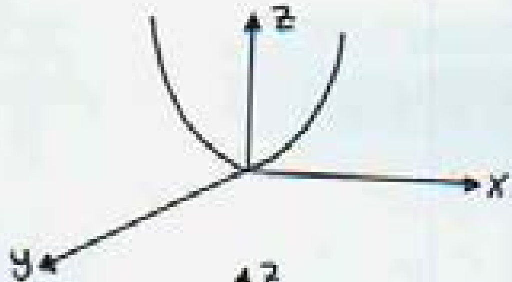


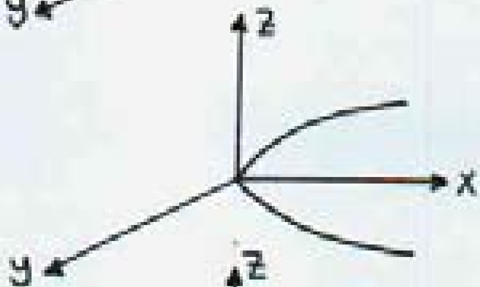
3. Paraboloid

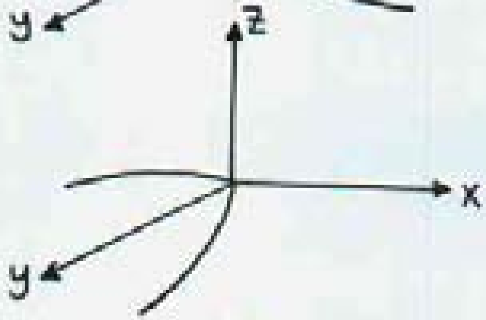
$z = x^2 + y^2 \quad ; \quad z \geqslant 0$

$x = z^2 + y^2 \quad ; \quad x \geqslant 0$

$y = z^2 + x^2 \quad ; \quad y \geqslant 0$





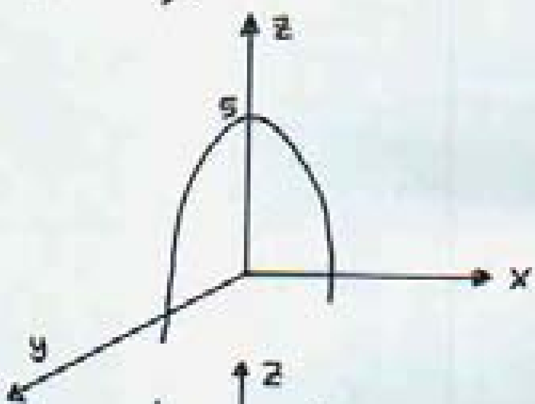


ex. sketch $z = 5 - x^2 - y^2$

$x^2 + y^2 = 5 - z$

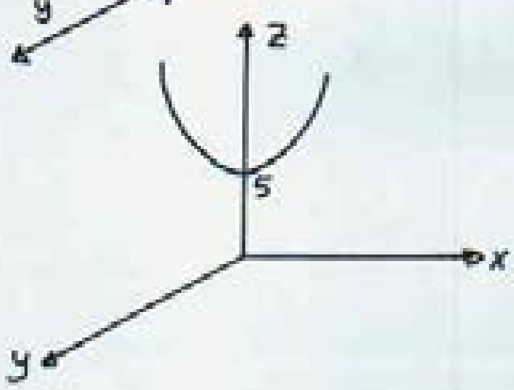
$5 - z \geqslant 0$

$z \leqslant 5$



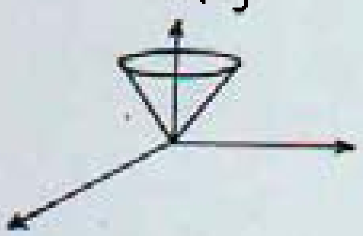
ex. sketch $z = 5 + x^2 + y^2$

$z - 5 \geqslant 0 \quad \therefore \quad z \geqslant 5$

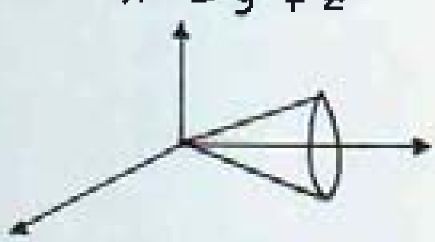


4. Cone

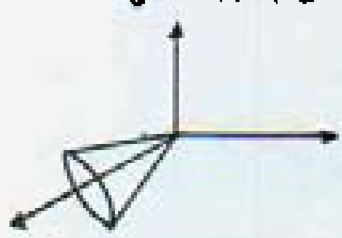
$z^2 = x^2 + y^2$



$x^2 = y^2 + z^2$



$y^2 = x^2 + z^2$



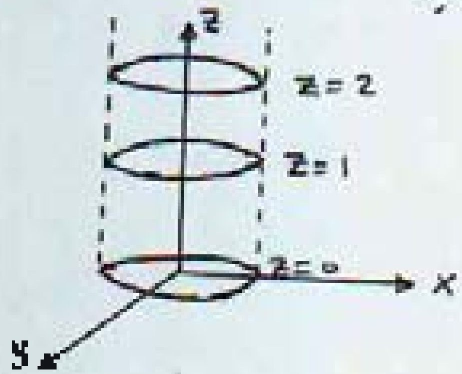
5- Cylinder

الأسطوانة : مجسم جميع مقاطعه المتوازية متساوية

* اي معادلة تحتوي على متغيرين فقط هي أسطوانة

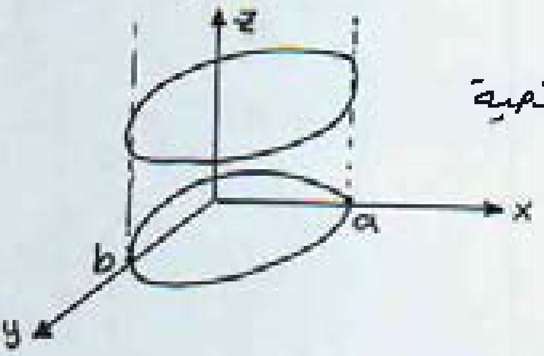
وتكون معادلة ذات درجة ثانية

__ex. 1:__ $x^2 + y^2 = a^2$ for all z



أسطوانة دائرية

__ex. 2:__ sketch $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all z



أسطوانة ناقصية

Triple Integrals :

$$V = \iiint dz\, dx\, dy$$

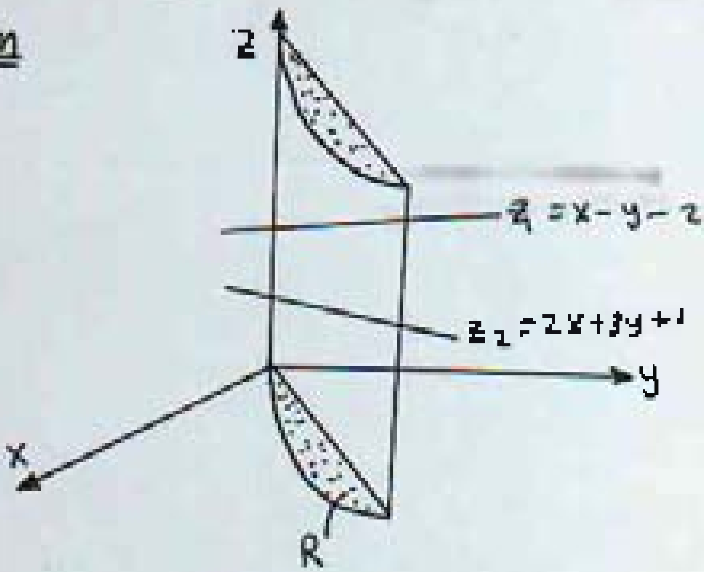
$$\text{Volume} = \iint_R z\, dA = \begin{cases} \iint_R f(x,y)\, dy\, dx \\ \iint_R f(x,y)\, dx\, dy \end{cases}$$

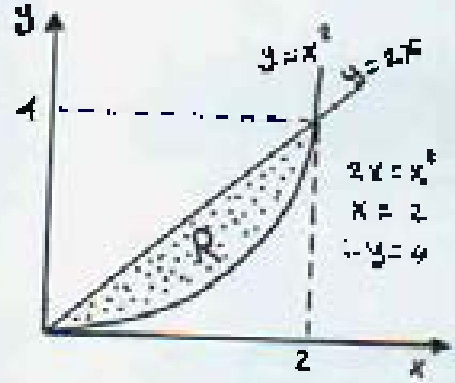
if $z = 1$

$$\therefore \text{area} = A = \iint_R dA = \iint_R dx\, dy = \iint_R dy\, dx$$

Ex. 3: Find the volume of solid bounded by $z_1 = x - y - 2$, $z_2 = 2x + 3y + 1$, for the region $y = x^2$, $y = 2x$ for all z

Soln



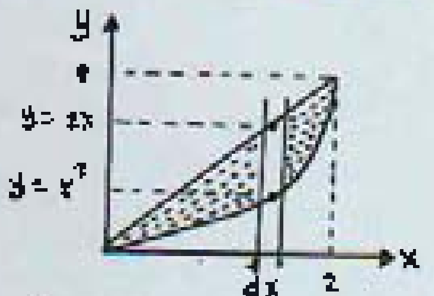


R in x-y plane

$$V = \left| \iint_R (z_2 - z_1)\, dA \right|$$

$$= \iint_R \{(2x+3y+1) - (x-y-2)\}\, dA = \iint_R (x + 4y + 3)\, dA$$

method① تغيرات dy مع ثبات dx

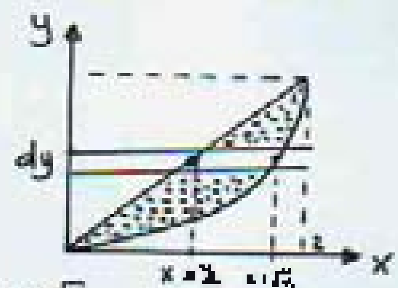


$$V = \int_0^2 \left(\int_{y=x^2}^{y=2x} (x + 4y + 3)\, dy \right) dx$$

$$= \int_0^2 xy + \frac{4y^2}{2} + 3y \Big]_{x^2}^{2x} dx$$

$$= \int_0^2 (x^3 + \frac{4x^4}{2} + 3x^2) - (2x^2 + 8x^2 + 6x)\, dx$$

method② تغيرات (dx) مع ثبات dy



$$V = \int_0^4 \left(\int_{x=\frac{y}{2}}^{x=\sqrt{y}} (x + 4y + 3)\, dx \right) dy$$

$$= \int_0^4 \frac{x^2}{2} + 4yx + 3x \Big]_{\frac{y}{2}}^{\sqrt{y}} dy$$

$$= \int_0^4 \left\{ \frac{y}{2} + 4y^{3/2} + 3y^{1/2} - \left(\frac{y^2}{8} + 2y^2 + \frac{3}{2}y \right) \right\} dy$$

Ex. 4: Find the area of the region bounded by $y=\sqrt{x}$, $x+y=6$, $y=0$ (using double integral).

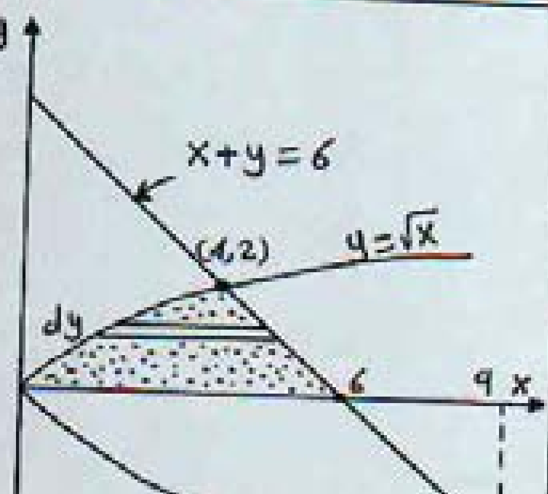
Sol: ايجاد نقطة تقاطع المستقيم مع المنحني

$$x+y=6 \Rightarrow x+\sqrt{x}=6 \Rightarrow \sqrt{x}=6-x \Rightarrow$$
$$x=36-12x+x^2 \Rightarrow x^2-13x+36=0 \Rightarrow$$
$$(x-9)(x-4)=0$$

either $x=9$ is neglected

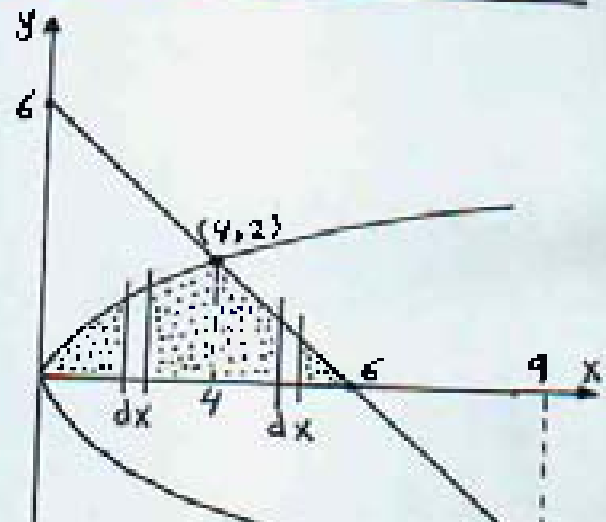
or $x=4$ $\therefore y=2$

method ①



$$A=\int_0^2\int_{y^2}^{6-y} dx\,dy$$

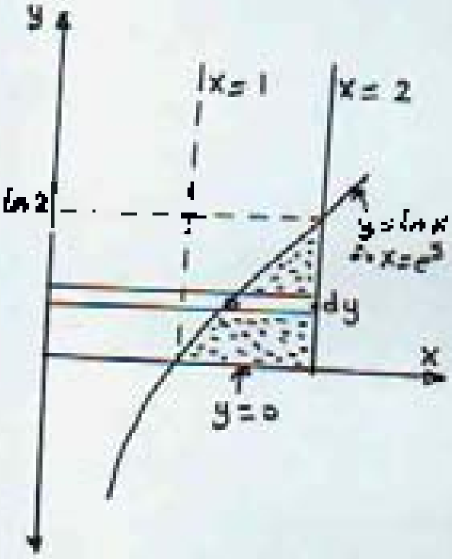
method ②



$$A=\int_0^4\int_0^{\sqrt{x}} dy\,dx+\int_4^6\int_0^{6-x} dy\,dx$$

ex. 5 : Reverse the order of the integral and evaluate .

① $\displaystyle\int^{2}\int_{0}^{\ln x} x \, dy \, dx$



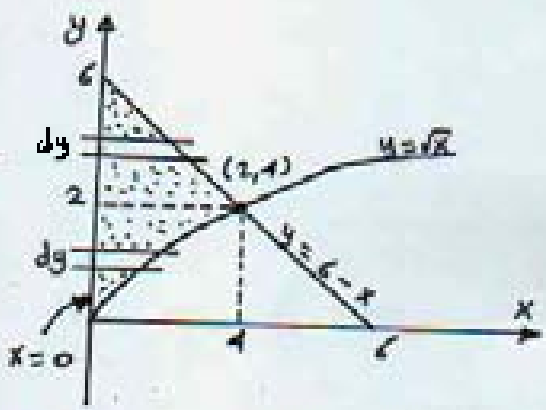
Sol$\underline{n}$ R : $x = 1$ to $x = 2$

$y = 0$ to $y = \ln x$

$$\therefore \int_{1}^{2}\int_{0}^{\ln x} x \, dy \, dx = \int_{0}^{\ln 2}\int_{e^y}^{2} x \, dx \, dy$$

$$= \int_{0}^{\ln 2} \left. \frac{x^2}{2} \right|_{e^y}^{2} dy$$

② $\displaystyle\int_{0}^{4}\int_{\sqrt{x}}^{6-x} y \, dy \, dx$



Sol$\underline{n}$ R : $x = 0$ to $x = 4$

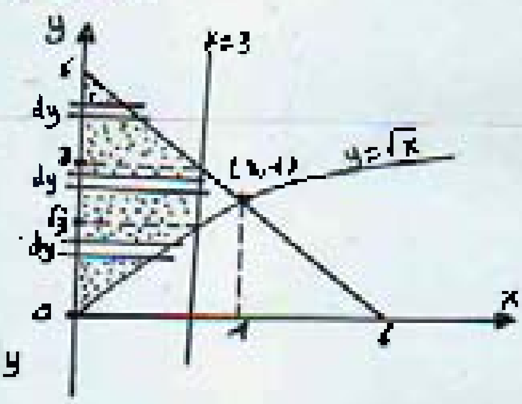
$y = \sqrt{x}$ to $y = 6 - x$

$\sqrt{x} = 6 - x \Rightarrow$ نجد نقاط التقاطع

$\therefore$ $x = 4$ & $y = 2$ (from previos example)

$$\therefore \int_{0}^{4}\int_{\sqrt{x}}^{6-x} y \, dy \, dx = \int_{0}^{2}\int_{0}^{y^2} y \, dx \, dy + \int_{2}^{6}\int_{0}^{6-y} y \, dx \, dy$$

③ $\displaystyle\int_{0}^{3}\int_{\sqrt{x}}^{6-x} y \, dy \, dx$



Sol$\underline{n}$ R : $x = 0$ to $x = 3$

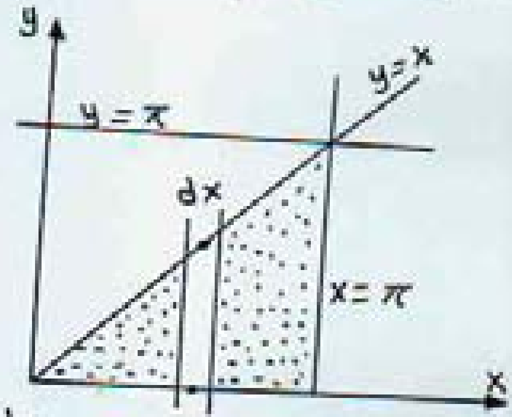
$y = \sqrt{x}$ to $y = 6 - x$

$$\int_{0}^{3}\int_{\sqrt{x}}^{6-x} y \, dy \, dx = \int_{0}^{\sqrt{3}}\int_{0}^{y^2} y \, dx \, dy + \int_{\sqrt{3}}^{3}\int_{0}^{3} y \, dx \, dy$$

$$+ \int^{6}\int^{6-y} y \, dx \, dy$$

ex. 6: Find $\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\, dx\, dy$

Sol$_{\underline{n}}$ the order of integral should be reversed because of the difficult integral



R : $y=0$ to $y=\pi$

$x=y$ to $x=\pi$

$$\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\, dx\, dy = \int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^{\pi} \frac{\sin x}{x}\, y \Big|_0^{x} dx = \int_0^{\pi} \frac{\sin x}{x}\,(x - \;)\, dx$$

$$= \int_0^{\pi} \sin x\, dx = \cos x \Big|_0^{\pi} = -(\cos\pi - \cos 0) = 2$$

<u>ملاحظة</u> : هناك تكاملات صعبة جداً ولكي تكون سهلة نعكس رتبة التكامل ، مثل :

$$\iint e^{x^2} dx\, dy \quad \iint \frac{\cos x}{\sqrt{4-x}} \quad \iint \frac{dx}{1+x^4}\, dy \quad \iint \frac{e^x}{x^3}\, dx\, dy$$

Area in Polar curve :

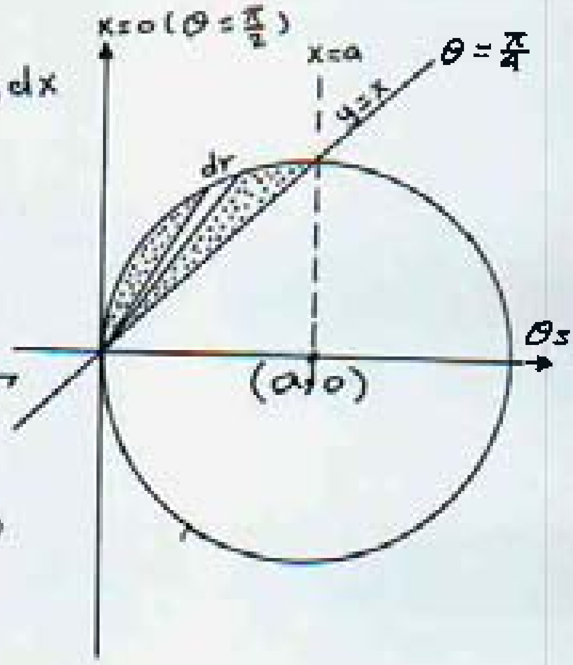
$$V = \iint_R f(x,y)\, dy\, dx = \iint_R f(r,\theta)\, r\, dr\, d\theta = \text{Volume}$$

$$A = \iint_R r\, dr\, d\theta = \frac{1}{2}\int r^2\, d\theta = \text{Area}$$

$x = r\cos\theta$, $y = r\sin\theta$, $\theta = \tan^{-1}\frac{y}{x}$

$r = \sqrt{x^2+y^2}$, $dx\, dy = r\, dr\, d\theta$

ex.7 : Find $\int_0^a \int_x^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx$



Soln R :

$x=0$ to $x=a$

$y=x$ to $y=\sqrt{2ax-x^2}$

hence, change into polar

$y=x \Rightarrow r\sin\theta = r\cos\theta$

$\therefore \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

& $y=\sqrt{2ax-x^2} \Rightarrow y^2 = 2ax - x^2 \Rightarrow x^2+y^2=2ax$

$\therefore r^2 = 2ar\cos\theta$ or $r = 2a\cos\theta$

which is eq. of circle and can be written as

$y^2 = 2ax - x^2 \Rightarrow x^2 - 2ax + a^2 - a^2 + y^2 = 0$

$\therefore (x-a)^2 + y^2 = a^2$ (eq. of circle)

$$I = \int_{\frac{\pi}{4}}^{\pi/2}\int_0^{2a\cos\theta} r^2 \cdot r\, dr\, d\theta = \int_{\frac{\pi}{4}}^{\pi/2} \frac{r^4}{4}\Big|_0^{2a\cos\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\pi/2} 4a^4 (\cos^2\theta)^2 d\theta = 4a^4 \int_{\frac{\pi}{4}}^{\pi/2} \left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$

$$= a^4 \int_{\frac{\pi}{4}}^{\pi/2} \left\{1 + \frac{1}{2}(1+\cos 4\theta) + 2\cos 2\theta\right\} d\theta$$

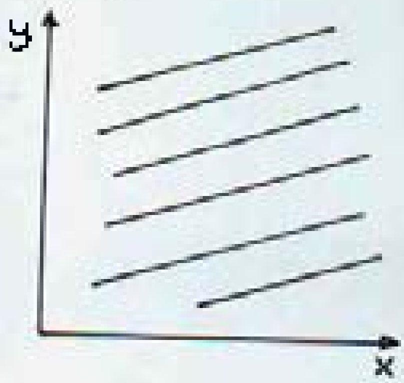
$$= a^4 \left[\frac{3}{2}\theta + \frac{1}{8}\sin 4\theta + \sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= a^4\left\{\frac{3}{2}\frac{\pi}{2} + 0 + 0 - \frac{3}{2}\frac{\pi}{4} - 0 - 1\right\} = \left(\frac{3}{8}\pi - 1\right)a^4$$

ex. 8 : Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dy\, dx$

Soln R: $x = 0$ to $x = \infty$

$y = 0$ to $y = \infty$



$$I = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left(-\frac{1}{2}\right) e^{-r^2} \Big|_0^{\infty} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \frac{1}{e^{r^2}} \Big|_0^{\infty} d\theta = -\frac{1}{2}\int_0^{\pi/2} (0-1)\, d\theta = \frac{\pi}{4}$$

* ~ ~ ~ ~ ~ ~ *

ex. 9 : Find the volume of a solid bounded above by $x^2+y^2+z^2 = 2a^2$ and bounded below by $az = x^2+y^2$

Soln

$\boxed{\text{الشيء الناتج من تقاطع مجسمين هو منحنى (curve)}}$

$(x^2+y^2)+z^2 = 2a^2$

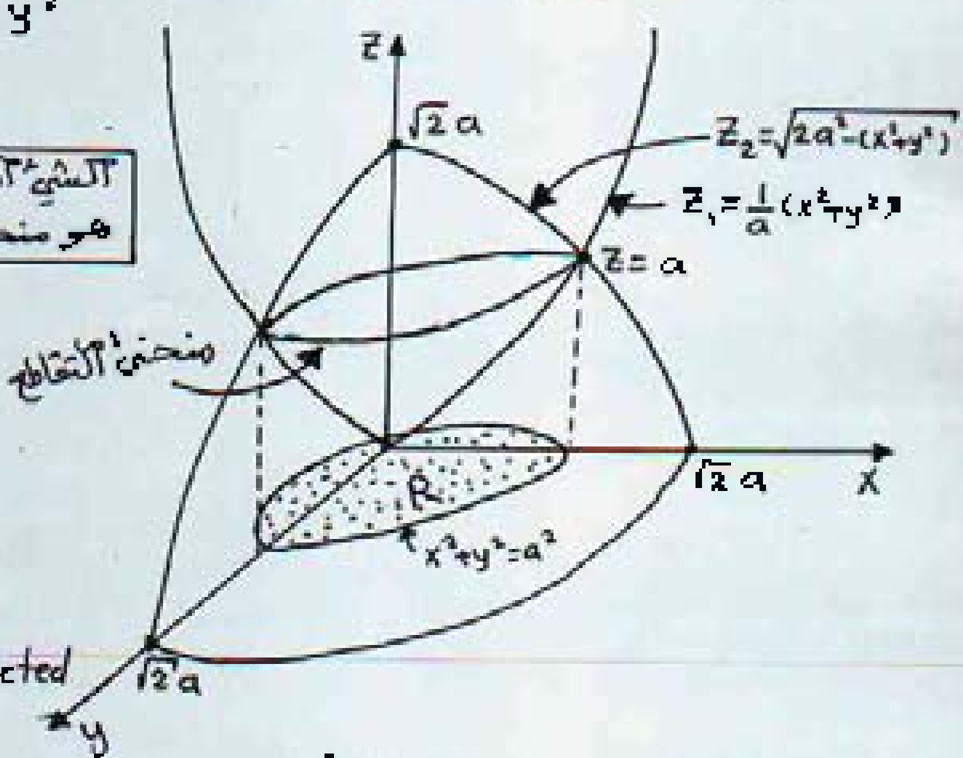
$az + z^2 = 2a^2$

$z^2 + az - 2a^2 = 0$

$(z-a)(z+2a) = 0$

$\boxed{z = a}$ مستوى التقاطع

or $z = -2a$ neglected

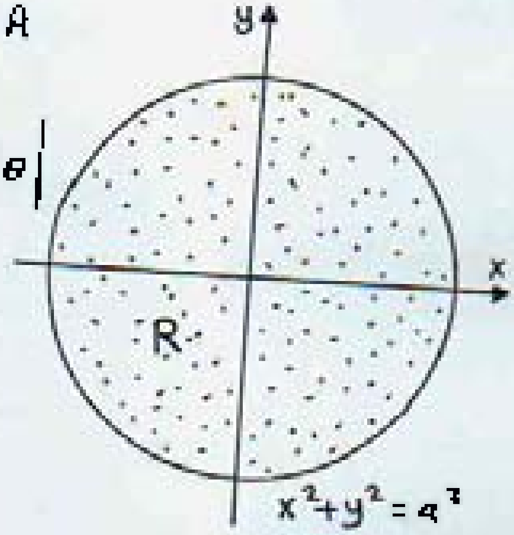


by substituting $z = a$ in one of the two equations gets hence, $V = \iint_R (z_2 - z_1)\, dA$

$$V = \left| \int_0^{2\pi} \int_0^{a} \left\{ \sqrt{2a^2 - r^2} - \frac{1}{a} r^2 \right\} r\, dr\, d\theta \right|$$

$$= \int_0^{2\pi} -\frac{1}{2} \frac{(2a^2 - r^2)^{3/2}}{3/2} - \frac{r^4}{4a} \Bigg|_0^{a} d\theta$$

$= (\qquad)$ unit volume



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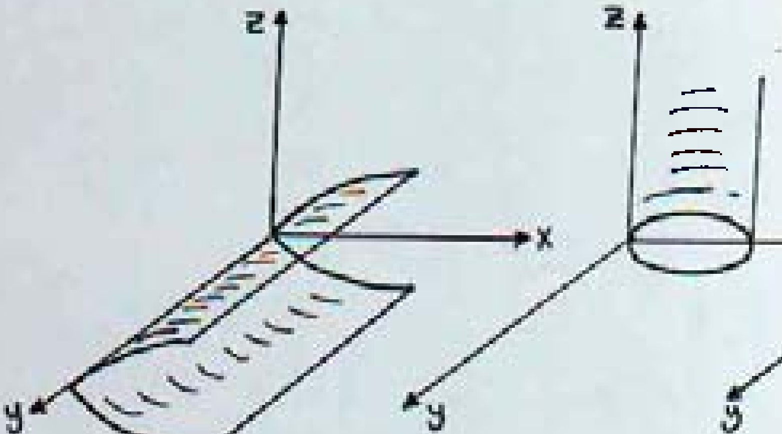
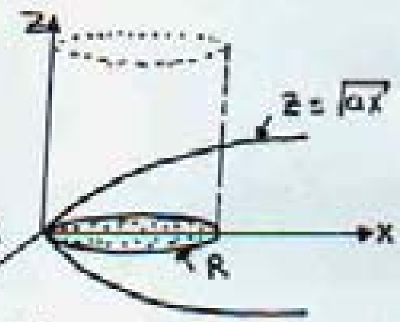
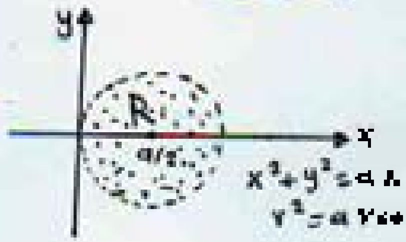
Ex. 10: Find the volume common to the surfaces $x^2+y^2=ax$ & $z^2=ax$

Sol$\underline{n}$ the two surfaces are cylinder because their equations have two variables and they are 2^{nd} degree







$$V = \left(\iint_R z\, dA \right) * 2 = 2 \iint_R \sqrt{a}\sqrt{x}\, dA$$

$$= 2 \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a} \cdot \sqrt{r\cos\theta} \cdot r\, dr\, d\theta \quad * 2$$

$$= 4\sqrt{a} \int_0^{\pi/2} \left(\int_0^{a\cos\theta} r^{3/2} (\cos\theta)^{1/2} dr \right) d\theta$$

$$\therefore V = 4\sqrt{a} \int_0^{\pi/2} (\cos\theta)^{1/2} \left. \frac{r^{5/2}}{5/2} \right|_0^{a\cos\theta} d\theta$$

$$= \frac{8}{5}\sqrt{a} \int_0^{\pi/2} (\cos\theta)^{1/2} \cdot \left\{ a^{5/2} (\cos\theta)^{5/2} \right\} d\theta$$

$$= \frac{8}{5} a^3 \int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{8}{5} a^3 \int_0^{\pi/2} (\cos\theta - \cos\theta \sin^2\theta) d\theta$$

$$= \frac{8}{5} a^3 \sin\theta - \frac{8}{5}\frac{a^3}{3} \sin^3\theta \Big|_0^{\pi/2} = \frac{16}{15} a^3$$

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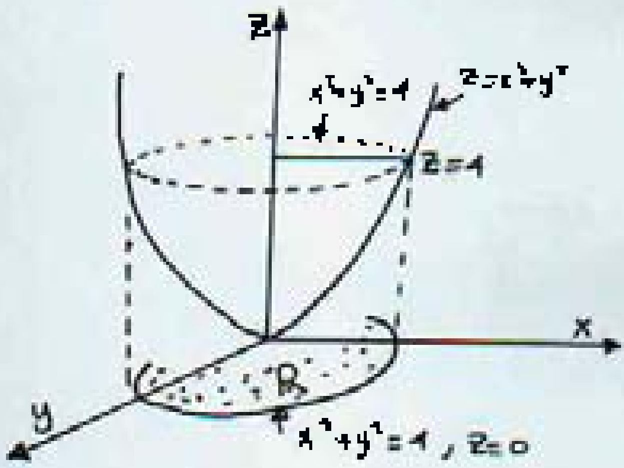
Surface area :

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

ex. 11 : Find the area of the surface $z = x^2 + y^2$ cut by the plane $z = 4$



Soln $z = x^2 + y^2$

$$\therefore \frac{\partial z}{\partial x} = 2x \quad , \quad \frac{\partial z}{\partial y} = 2y$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

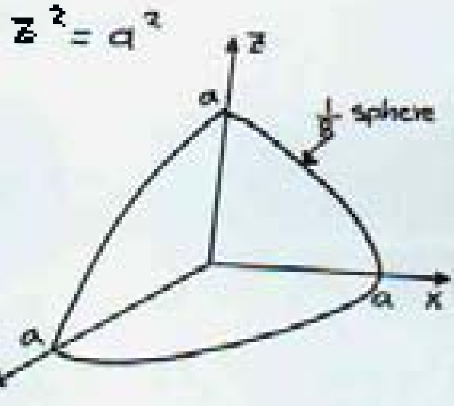
$$\therefore S = \iint_R \sqrt{1 + 4(x^2+y^2)} \, dA$$

$$= \int_0^{2\pi}\int_0^{2} \sqrt{1 + 4r^2} \cdot r \, dr \, d\theta$$

$$= \frac{\pi}{4} \left. \frac{(1+4r^2)^{3/2}}{3/2} \right|_0^2 = \frac{\pi}{6} \left((\sqrt{17})^3 - 1 \right)$$

ex. 12 : For the sphere $x^2+y^2+z^2=a^2$

1. show that $V = \frac{4}{3}\pi a^3$
2. Show that $S = 4\pi a^2$



Soln $\quad V = \iint z\, dA$

$$= \left\{ \int_0^{\pi/2} \int_0^a a^2 - r^2 \cdot r\, dr\, d\theta \right\} * 8$$

$$= -\frac{8}{3} \int_0^{\pi/2} (a^2 - r^2)^{3/2} \Big|_0^a d\theta$$

$$= \frac{8}{3} \int_0^{\pi/2} a^3\, d\theta = \frac{8}{3} a^3 \frac{\pi}{2} = \frac{4}{3}\pi a^3$$

2. $\quad z^2 = a^2 - x^2 - y^2$

$$2z \frac{\partial z}{\partial x} = -2x \qquad \therefore \left(\frac{\partial z}{\partial x}\right)^2 = \frac{x^2}{a^2 - x^2 - y^2}$$

$$2z \frac{\partial z}{\partial y} = -2y \qquad \therefore \left(\frac{\partial z}{\partial y}\right)^2 = \frac{y^2}{a^2 - x^2 - y^2}$$

$$\therefore \quad 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2 + y^2}{a^2 - (x^2 + y^2)}$$

$$= \frac{a^2 - x^2 - y^2 + x^2 + y^2}{a^2 - (x^2 + y^2)} = \frac{a^2}{a^2 - r^2}$$

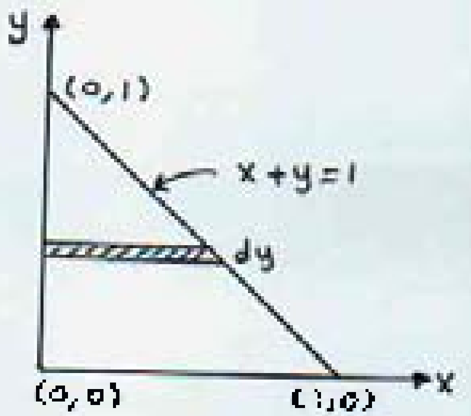
$$\therefore \quad S = \left[\int_0^{\pi/2} \int_0^a \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta \right] * 8$$

$$= \frac{8a}{-2} \cdot \frac{\pi}{2} \cdot \frac{(a^2 - r^2)^{1/2}}{1/2} \Big|_0^a = 4\pi a^2$$

University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Sheet No.(5): Double & Triple Integrals
Dr. Akeel Abdullah Mohammed

Prob. 1 : Evaluate the integral $\iint_R \sin(x+y)\cos(x-y)\,dx\,dy$ where R is the triangle whose vortices are (0,0), (1,0), (0,1).

Soln : The above vortices can be represented by the following figure.



$$\iint_R \sin(x+y)\cos(x-y)\,dx\,dy$$

$$\iint_R (\sin x\cos y + \sin y\cos x) * (\cos x\cos y + \sin x\sin y)\,dx\,dy$$

$$\iint_R \{\sin x\cos x\,(\sin^2 y + \cos^2 y) + \sin y\cos y\,(\sin^2 x + \cos^2 x)\,dx\,dy$$

$$\int_0^1\int_0^{1-y} ((\sin x\cos x + \sin y\cos y)\,dx)\,dy$$

$$\int_0^1 \frac{\sin^2 x}{2} + x\sin y\cos y \Big|_0^{1-y} dy$$

$$\int_0^1 \left(\frac{\sin^2(1-y)}{2} + (1-y)\sin y\cos y\right)dy$$

$$\int_0^1 \frac{1}{2}\cdot\frac{1}{2}(1-\cos 2(1-y)) + \frac{\sin 2y}{2} - \frac{y}{2}\sin 2y\;dy$$

$$= \int_0^1 \frac{1}{4} - \frac{1}{4}(\cos 2 \cos 2y + \sin 2 \sin 2y) + \frac{\sin 2y}{2} - \frac{y}{2}\sin 2y \, d$$

$$= \frac{1}{4}\int_0^1 dy - \frac{\cos 2}{4}\int_0^1 \cos 2y \, dy + \left(\frac{1}{2} - \frac{\sin 2}{4}\right)\int_0^1 \sin 2y \, dy$$

$$- \frac{1}{2}\int_0^1 \underbrace{y}_{u} \; \underbrace{\sin 2y \, dy}_{dv}$$

$$= \left[\frac{y}{4} - \frac{1}{4}\cos 2 \frac{\sin 2y}{2} - \left(\frac{1}{2} - \frac{\sin 2}{4}\right)\frac{\cos 2y}{2} \right]_0^1 - \frac{1}{2}\left[y\left(-\frac{\cos 2y}{2}\right) \right.$$

$$\left. + \frac{1}{2}\int_0^1 \cos 2y \, dy \right]$$

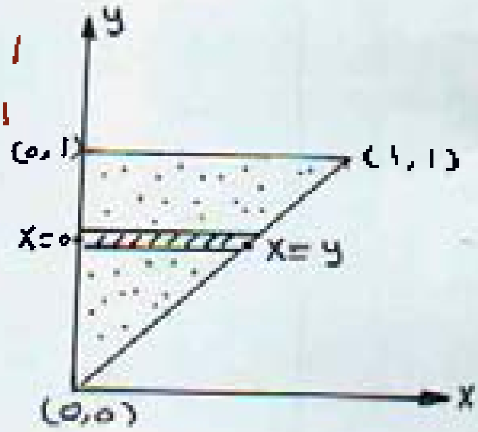
$$= \frac{1}{4} - \frac{1}{8}\cos 2 \sin 2 - \left(\frac{1}{2} - \frac{\sin 2}{4}\right)\left(\frac{\cos 2}{2} - \frac{1}{2}\right) + \frac{\cos 2}{4} + \frac{\sin 2}{8}$$

= ( ) unit . . volume.

Prob. 2 : Evaluate the following integral $\int_0^1 \int_x^1 \frac{1}{y} \sin\frac{x}{y} \cos\frac{x}{y} \, dy\, dx$

sol$^{\underline{n}}$: R : $y = x$ to $y = 1$

$x = 0$ $\quad x = 1$



$$\int_0^1 \int_0^y \frac{1}{y} \sin\frac{x}{y} \cos\frac{x}{y} \, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \left(-\cos^2\frac{x}{y}\right)\Big|_0^y \, dy$$

$$= \frac{1}{2}\int_0^1 (1 - \cos^2 1)\, dy =$$

$$= \frac{1}{2}(1 - \cos^2 1)\, y$$

$$= \frac{1}{2}(1 - \cos^2 1)$$

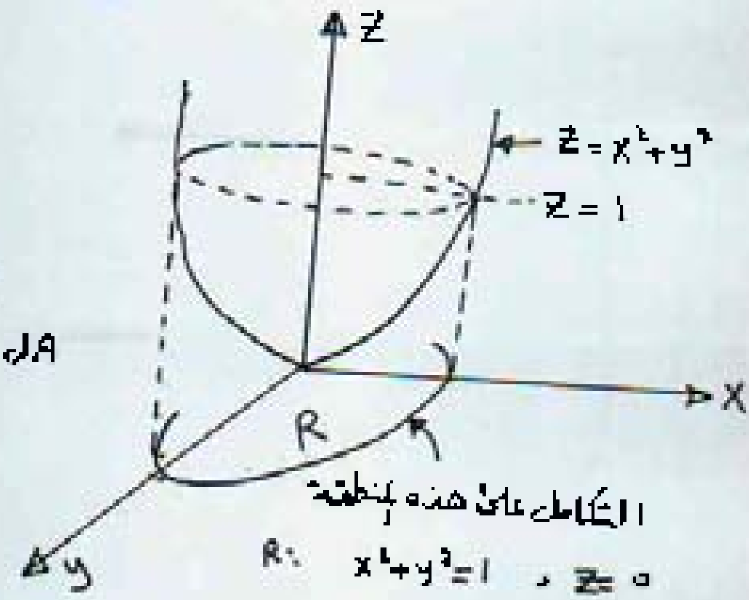
* حوّلنا الشريحة من التوازي مع
y إلى التوازي مع x كي
تكون عملية التكامل سهلة

Prob. 3 Find the surface area of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$

Sol$\underline{n}$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 2y$$



$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint \sqrt{1 + 4(x^2+y^2)}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2} \cdot r\, dr\, d\theta$$

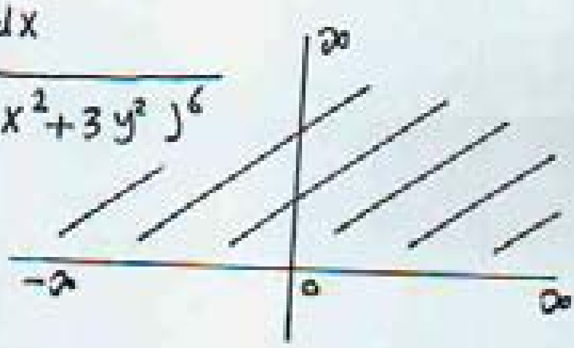
$$= \int_0^{2\pi} \frac{1}{8} \cdot \frac{(1+4r^2)^{3/2}}{3/2} \Big|_0^1 \, d\theta$$

$$= \frac{\pi}{6}\left[(5)^{3/2} - 1\right] = (\qquad) \text{ unit area.}$$

Prob. 4: Find $\displaystyle\int_{-\infty}^{\infty}\int_0^{\infty} \frac{dy\, dx}{(4 + 3x^2 + 3y^2)^6}$

Sol$\underline{n}$: R: $x = -\infty$ to $x = \infty$

$y = 0$ $\quad y = \infty$



$$I = \int_0^{\pi}\int_0^{\infty} \frac{r\, dr\, d\theta}{(4+3r^2)^6} = \int_0^{\pi}\int_0^{\infty} (4+3r^2)^{-6}\, \frac{1}{6} \cdot 6r\, dr\, d\theta$$

$$= -\frac{\pi}{30}\left[\left(\frac{1}{4+3(\infty)^2}\right)^5 - \frac{1}{(4+3(0)^2)^5}\right]$$

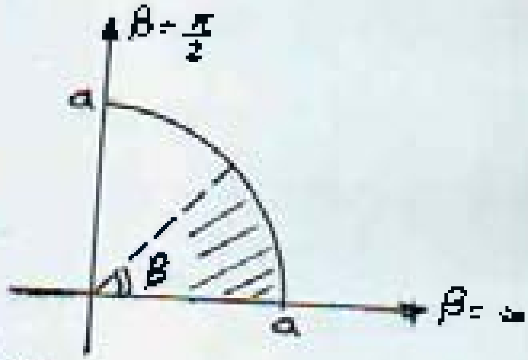
(A)

Prob. 5: show by transforming to polar coordinates that

$$\int_0^{a\sin\beta}\int_0^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy = a^2\beta\left(\ln a-\frac{1}{2}\right)$$

$$0<\beta<\frac{\pi}{2}$$

Soln: R:



$x=0$ to $x=\sqrt{a^2-y^2}$

$\therefore x^2+y^2=a^2$

$r=a$

$y=0$ $\quad$ $y=a\sin\beta$

$$\int_0^{\beta}\int_0^{a} \ln r^2 \; r\,dr\,d\theta = 2\int_0^{\beta}\int_0^{a} \underbrace{\ln r}_{u}\;\underbrace{r\,dr}_{dv}\,d\theta$$

$$= 2\int_0^{\beta}\left\{ \ln(r)\cdot\frac{r^2}{2}\Big|_0^a - \int_0^a \frac{r^2}{2}\cdot\frac{1}{r}\,dr\right\}d\theta$$

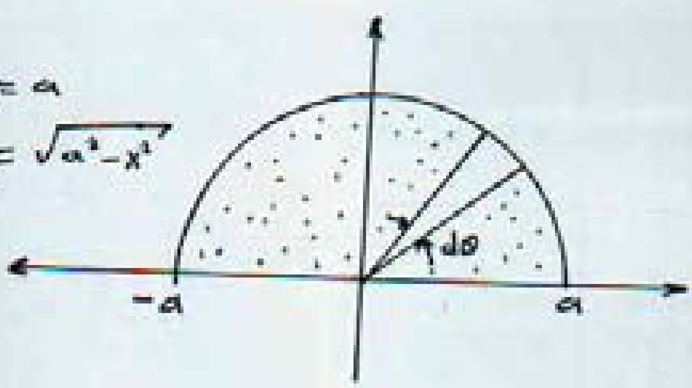
$$= 2\int_0^{\beta}\left(\frac{a^2}{2}\ln a-\frac{a^2}{4}\right)d\theta = a^2\theta\left(\ln a-\frac{1}{2}\right)\Big|_0^{\beta}$$

$$= a^2\beta\left(\ln a-\frac{1}{2}\right)$$

Prob. 6: Evaluate $\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$

Soln: R: $x=-a$ to $x=a$

$y=0$ $\quad$ $y=\sqrt{a^2-x^2}$



$$I=\int_0^{\pi}\int_0^{a} r^3\; r\,dr\,d\theta$$

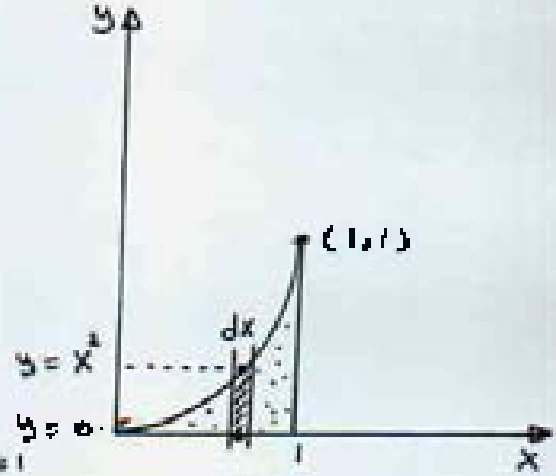
$$=\int_0^{\pi}\frac{r^5}{5}\Big|_0^a\,d\theta$$

$$=\frac{\pi a^5}{5}$$

Prob. 7 : Evaluate $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 e^{5x^3}\, dx\, dy$

Soln : R: $x=\sqrt{y}$ to $x=1$

$y=0 \qquad y=1$



$$I = \int_0^1 \int_0^{x^2} e^{5x^3}\, dy\, dx$$

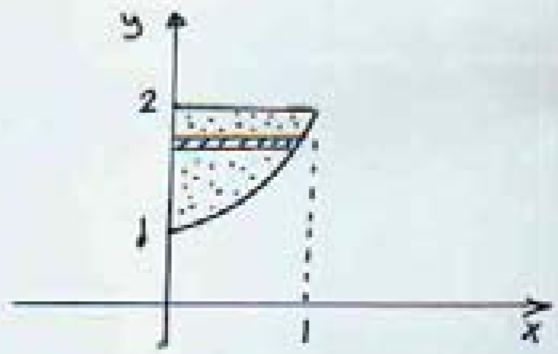
$$= \int_0^1 e^{5x^3} \cdot y \Big|_0^{y=x^2} dx$$

$$= \int_0^1 x^2 e^{5x^3}\, dx = \frac{e^{5x^3}}{15} \Bigg|_{x=0}^{x=1} = \frac{1}{15}(e^5 - 1)$$

Prob. 8 : Evaluate $\displaystyle\int_0^1 \int_{x^2+1}^2 \frac{xye^y}{y-1}\, dy\, dx$

R: $y = x^2+1 \qquad y=2$

$x=0 \qquad x=1$



$$\int_1^2 \int_0^{\sqrt{y-1}} x\, y\, \frac{e^y}{y-1}\, dx\, dy$$

$$= \int_1^2 \frac{x^2}{2}\, \frac{ye^y}{y-1} \Bigg|_0^{x=\sqrt{y-1}} dy$$

$$= \int_1^2 \frac{\cancel{(y-1)}}{2}\, \frac{ye^y}{\cancel{(y-1)}}\, dy = \int_1^2 \underbrace{y}_{u}\, \underbrace{e^y\, dy}_{dv}$$

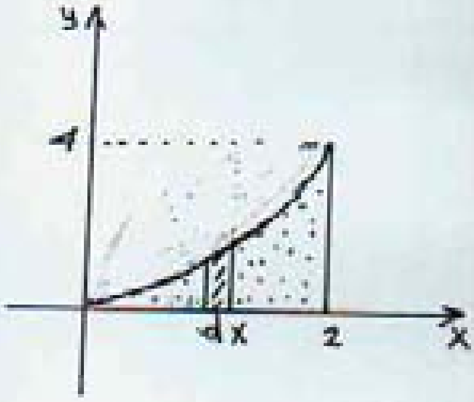
$$= \frac{1}{2}\left\{e^y y - \int_1^2 e^y\, dy\right\} = \frac{1}{2} e^y (y-1)\Big]_1^2$$

$$= \frac{1}{2} e^2$$

Prob. 9 : Evaluate $\int_0^4 \int_{\sqrt{y}}^2 \cos(4x^3+5)\, dx\, dy$

Sol$\underline{n}$: R: $x=\sqrt{y}$ $\quad x=2$

$y=0 \quad y=4$



$$\int_0^2 \int_0^{x^2} \cos(4x^3+5)\, dy\, dx$$

$$= \int_0^2 y \cos(4x^3+5) \Big|_0^{x^2} dx$$

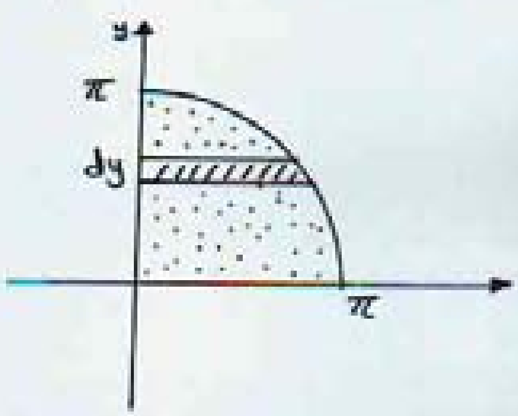
$$= \int_0^2 x^2 \cos(4x^3+5)\, dx = \frac{\sin(4x^3+5)}{12} \Bigg|_{x=0}^{x=2}$$

$$= \frac{1}{12}(\sin 37 - \sin 5) = (\qquad)$$

Prob. 10 : Evaluate $\int_0^{\pi} \int_0^{\sqrt{\pi^2-y^2}} \frac{x^2 y}{\sqrt{x^2+y^2}}\, dx\, dy$

Sol$\underline{n}$: R: $x=0 \quad x=\sqrt{\pi^2-y^2}$

$y=0 \quad y=\pi$



R: $r=0 \quad r=\pi$

$\theta=0 \quad \theta=\frac{\pi}{2}$

$$I = \int_0^{\pi/2} \int_0^{\pi} \frac{(r\cos\theta)^2 (r\sin\theta)}{r} \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi} r^3 \cos^2\theta \sin\theta\, dr\, d\theta$$

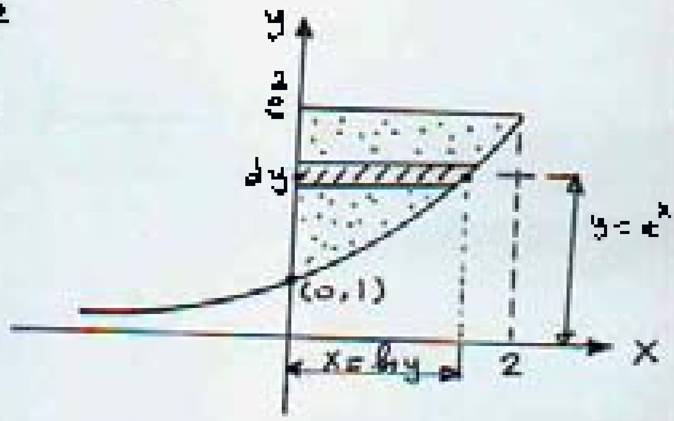
$$= \int_0^{\pi/2} \frac{r^4}{4} \Big|_0^{\pi} \sin\theta \cos^2\theta\, d\theta = \frac{\pi^4}{4}\left(\frac{-\cos^3\theta}{3}\right) \Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^4}{12}$$

Prob. 11 : Evaluate $\displaystyle\int_0^2 \int_{e^x}^{e^2} \frac{y \sin y}{\ln y} \, dy \, dx$

Soln : R: $y = e^x$ $\quad y = e^2$

$x = 0 \quad x = 2$



$$\int_1^{e^2} \int_0^{\ln y} \frac{y \sin y}{\ln y} \, dx \, dy$$

$$= \int_1^{e^2} x \Big|_{x=0}^{x=\ln y} \frac{y \sin y}{\ln y} \, dy$$

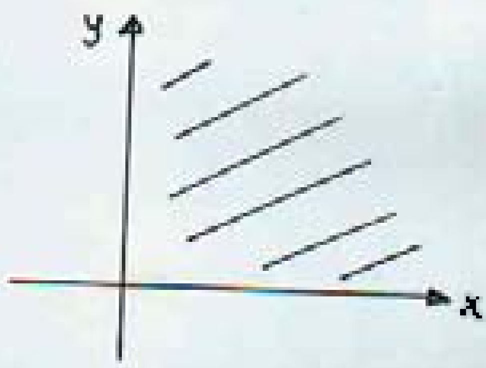
$$= \int_1^{e^2} \ln y \, \frac{y \sin y}{\ln y} \, dy = \int_1^{e^2} \underset{u}{\underline{y}} \; \underset{dv}{\underline{\sin y \, dy}}$$

$$= -y \cos y \Big|_1^{e^2} + \int_1^{e^2} \cos y \, dy = -y \cos y + \sin y \Big|_1^{e^2} = (\quad)$$

Prob. 12 : Evaluate $\displaystyle\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} \, dx \, dy$

soln : R: $x = 0 \quad x = \infty$

$y = 0 \quad y = \infty$



$$I = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} \, r \, dr \, d\theta$$

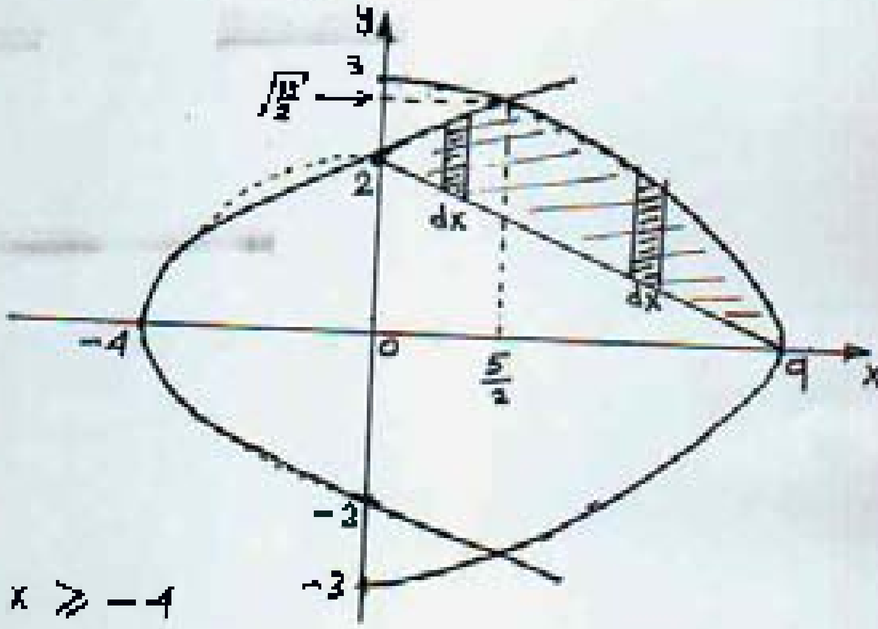
$$= -\frac{1}{2} \int_0^{\pi/2} e^{-r^2} \Big|_0^{\infty} d\theta$$

$$= -\frac{\pi}{4} \left[\frac{1}{\infty} - 1 \right] = \frac{\pi}{4}$$

Prob. 13 : Evaluate $\displaystyle\int_0^2 \int_0^x \frac{dy \, dx}{\sqrt{x^2 + y^2}}$

Prob. 14: Using double integral to find the area of the region that is bounded by $x = 9 - y^2$ & $x = y^2 - 4$ & $\frac{x}{9} + \frac{y}{2} = 1$

Sol$\underline{n}$:



① $x = 9 - y^2$

$9 - x = y^2$

$9 - x \geq 0$

$\therefore x \leq 9$

② $x = y^2 - 4$

$x + 4 = y^2$

$x + 4 \geq 0 \quad \therefore x \geq -4$

To find the intersection points of two curves

$9 - y^2 = y^2 - 4 \implies y = \sqrt{\frac{13}{2}}, \quad x = \frac{5}{2}$

$$\text{area} = \iint_R dy\,dx = \int_0^{5/2}\int_{2-\frac{2}{9}x}^{\sqrt{x+4}} dy\,dx + \int_{\frac{5}{2}}^{9}\int_{2-\frac{2}{9}x}^{\sqrt{9-x}} dy\,dx$$

$$= \int_0^{5/2} (\sqrt{x+4}) - (2 - \tfrac{2}{9}x)\,dx + \int_{5/2}^{9} (\sqrt{9-x}) - (2 - \tfrac{2}{9}x)\,dx$$

$$= \frac{2}{3}(x+4)^{3/2} - 2x + \frac{x^2}{9}\Bigg|_{x=0}^{x=\frac{5}{2}} - \frac{2}{3}(9-x)^{3/2} - 2x + \frac{x^2}{9}\Bigg|_{x=\frac{5}{2}}^{x=9}$$

$$= \left[\frac{2}{3}\left(\frac{13}{2}\right)^{3/2} - 5 + \frac{25}{36} - \frac{16}{3} - 18 + 9 + \frac{2}{3}\left(\frac{13}{2}\right)^{3/2} + 5 - \frac{25}{36}\right]$$

$$= \frac{4}{3}\left(\frac{13}{2}\right)^{3/2} - \frac{16}{3} - 9$$

University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Chapter ( ): Vectors
Dr. Akeel Abdullah Mohammed

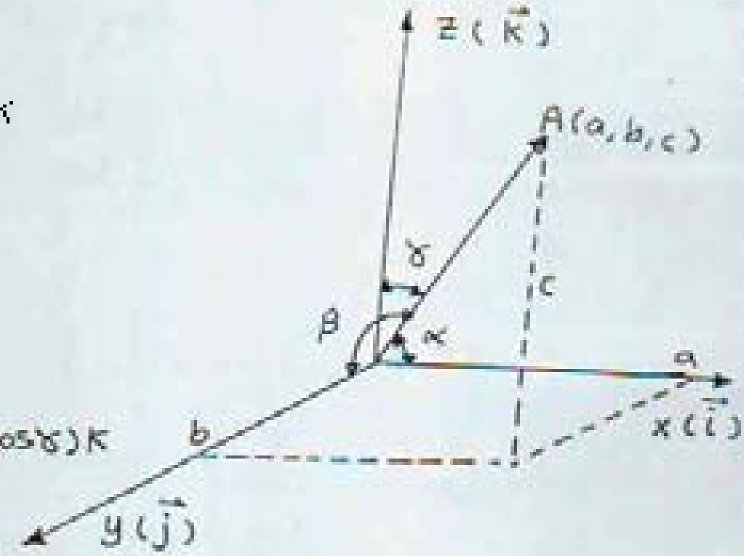
Vectors :

$\vec{A} = \vec{OA} = a\,i + b\,j + c\,k$

$|\vec{A}| = \sqrt{a^2 + b^2 + c^2}$

unit vector $= \vec{u} = \dfrac{\vec{A}}{|\vec{A}|}$

$= (\cos\alpha)\,i + (\cos\beta)\,j + (\cos\gamma)\,k$



where

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$(a, b, c) \Rightarrow$ direction numbers

$(\alpha, \beta, \gamma) \Rightarrow$ direction angles

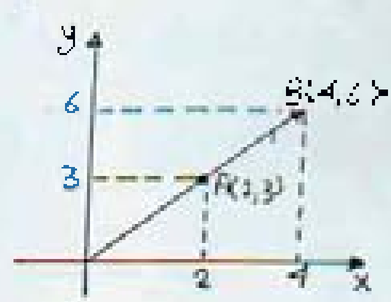
$(\cos\alpha, \cos\beta, \cos\gamma) \Rightarrow$ direction cosines

Paralel Vectors :

let $\vec{A}$ & $\vec{B}$ are two vector quantities , then

if $\vec{A} \parallel \vec{B} \Rightarrow \vec{B} = t\,\vec{A}$

where t is a scalar quantity



ex. 1 : $\vec{A} = 2i + 3j$ & $\vec{B} = 4i + 6j$

$\therefore \vec{B} = 2(2i + 3j)$

$= 2\,\vec{A}$ parallel

Product of two vectors :

let $\vec{A} = a_1 i + a_2 j + a_3 k$

$\vec{B} = b_1 i + b_2 j + b_3 k$

① Dot . Product (scalar product)

وهي طريقة مهمة لمعرفة الزاوية بين متجهين

$$\vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos\theta$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties :

1. $\vec{A} \cdot \vec{A} = |\vec{A}|^2$
2. $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
3. $\vec{A} \perp \vec{B} \Rightarrow \vec{A} \cdot \vec{B} = 0$
4. $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}| \, |\vec{B}|}$
5. $i \cdot i = j \cdot j = k \cdot k = 1$

   $i \cdot j = j \cdot k = k \cdot i = 0$

② cross product (vector product)

طريقة مهمة لمعرفة المتجه العمود على متجهين

$$\vec{A} \times \vec{B} = \vec{n} \, |\vec{A}| \, |\vec{B}| \sin\theta$$

where $\vec{n}$ is a unit vector normal to both $\vec{A}$ & $\vec{B}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = + i (a_2 b_3 - a_3 b_2) - j (a_1 b_3 - a_3 b_1) + k (a_1 b_2 - a_2 b_1)$$

Properties:

1. $\vec{A} \times \vec{A} = 0$
2. $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$
3. $\vec{A} \parallel \vec{B} \Rightarrow \vec{A} \times \vec{B} = 0$

   $(\vec{A} \times \vec{B} \Leftrightarrow \vec{A} \cdot \vec{B} = 0)$

4. $i \times i = j \times j = k \times k = 0$

$i \times j = k$	$j \times i = -k$
$j \times k = i$	$k \times j = -i$
$k \times i = j$	$i \times k = -j$

Triple Product:

1. Vector triple product $= \vec{A} \times (\vec{B} \times \vec{C})$

$$= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

2. scalar triple product $= \vec{A} \cdot (\vec{B} \times \vec{C})$

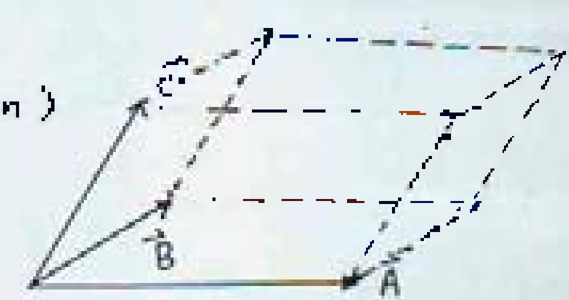
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Volume of a box & pyramid حجم (المكعب والهرم)

volume of the box $= |\vec{A} \cdot (\vec{B} \times \vec{C})|$

volume of pyramid (رباعي السطوح tetrahedron)

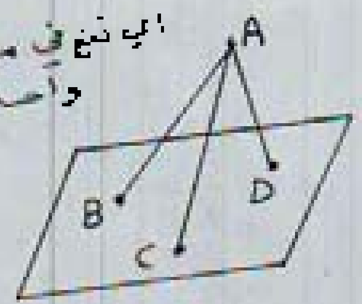
$$= \frac{1}{6} |\vec{A} \cdot (\vec{B} \times \vec{C})|$$



ex-1: let $A(2,1,1)$, $B(3,2,5)$, $C(0,2,2)$, $D(4,5,6)$

show that A, B, C, D are non-coplanar points.

sol: if $\vec{AB}\cdot(\vec{AC}\times\vec{AD}) = 0$ then A, B, C and D are coplanar points. ⇐ اي تقع في مستوى واحد



$\vec{AB} = i + j + 4k$

$\vec{AC} = -2i + j + k$

$\vec{AD} = 2i + 4j + 5k$

* اي ثلاث نقاط يوجد مستوى وحيد يحتويها

$\therefore \quad \vec{AB}\cdot(\vec{AC}\times\vec{AD}) = \begin{vmatrix} 1 & 1 & 4 \\ -2 & 1 & 1 \\ 2 & 4 & 5 \end{vmatrix}$

$= 1(5*1 - 4*1) - 1(5*(-2) - 2*1) + 4(4*(-2) - 2*1)$

$= 1 + 12 - 40 \neq 0$

$\therefore$ A, B, C and D are non-coplanar points. لذلك تشكل حجم مكعب

ex. 2: let $\vec{A} = 2i + 3j + k$, $\vec{B} = i - j - k$ & $\vec{C} = 5i + j + 2k$. Find the vector of length 3 units that is normal to $\vec{C}$ and lies in the plane determined by $\vec{A}$ and $\vec{B}$.

sol: المطلوب ايجاد متجه طوله ثلاث وحدات وعمودي على المتجه $\vec{C}$ ويقع في المستوى $(A B)$ اي الذي يحوي $\vec{B}$, $\vec{A}$ in dr

let $\vec{L}$ is the required vector

and $\vec{N}$ is the vector normal on $\vec{A}$ & $\vec{B}$

and M is the plane determined by $\vec{A}$ & $\vec{B}$

and M is

$\therefore \quad \vec{N} = \vec{A} \times \vec{B}$

$\therefore \quad \vec{N} \perp M$ $\Rightarrow$ { المتجه العمود على متجهين يكون عمودياً على المستوى الذي يحويهما }

$\because \quad \vec{L}$ lies in M

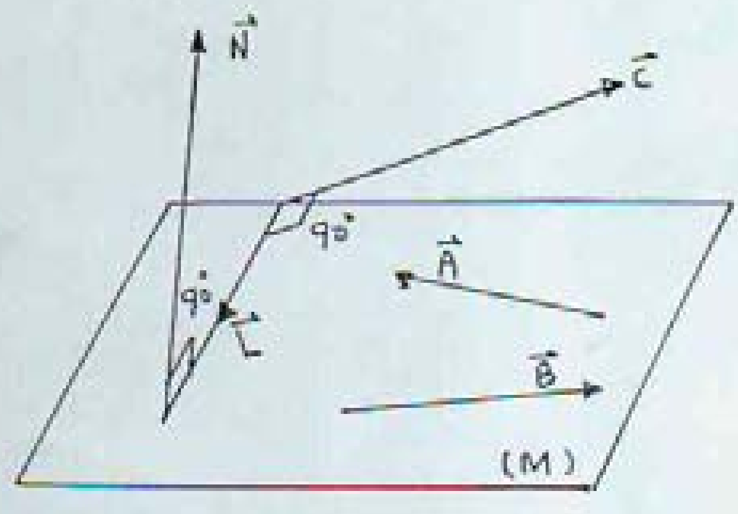
$\therefore \quad \vec{N} \perp \vec{L}$ $\Rightarrow$ { المتجه العمودي على مستوٍ يكون عمودياً على جميع المتجهات المحتواة في ذلك المستوى }

$\because \quad \vec{L} \perp \vec{C}$ $\Rightarrow$ { من منطوق السؤال }

$$\therefore \quad \vec{L} = \vec{C} \times \vec{N} = \vec{C} \times (\vec{A} \times \vec{B})$$
$$= (\vec{C} \cdot \vec{B})\vec{A} - (\vec{C} \cdot \vec{A})\vec{B}$$

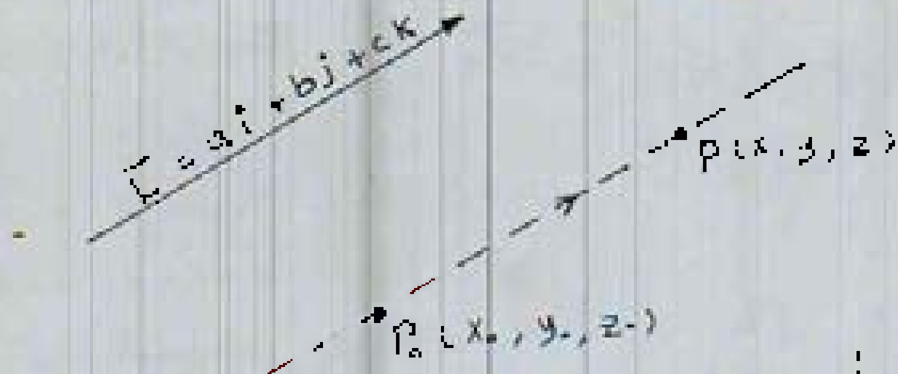
$$\therefore \quad \vec{L} = (5-1-2)(2i+3j+k) - (10+3+2)(i-j-k)$$
$$= 2(2i+3j+k) - 15(i-j-k)$$
$$= 4i + 6j + 2k - 15i + 15j + 15k$$
$$= -11\,i + 21\,j + 17\,k$$

$\therefore$ المتجه المطلوب $= 3 * \dfrac{\vec{L}}{|\vec{L}|} = 3 * \dfrac{-11\,i + 21\,j + 17\,k}{\sqrt{(-11)^2 + (21)^2 + (17)^2}}$



Equation of a Line in a Space :-

المطلوب : ايجاد معادلة المستقيم المار بالنقطة P_0 والموازي للمتجه $\vec{L}$



نفرض أن $p(x, y, z)$ أي نقطة على المستقيم

$$\overrightarrow{P_0P} = (x - x_0)\,i + (y - y_0)\,j + (z - z_0)\,k$$

$$\overrightarrow{P_0P} \parallel \vec{L} \quad \Rightarrow \quad \overrightarrow{P_0P} = t\,\vec{L}$$

$$(x - x_0)\,i + (y - y_0)\,j + (z - z_0)\,k = t\,(ai + bj + ck)$$

$$\therefore \left.\begin{aligned} x - x_0 &= t\,a \\ y - y_0 &= t\,b \\ z - z_0 &= t\,c \end{aligned}\right\}$$ parametric form of the eq. الشكل الوسيطي

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$ standard form الشكل القياسي

$a \neq 0, \quad b \neq 0, \quad c \neq 0$

* ملاحظة : لإيجاد معادلات المستقيم يجب أن يتوفر شرطان :-

① نقطة معلومة ② المتجه الموازي معلوم

ex. 3 : Find the equation of the line that passes through

① $A(2, 1, 4)$, $B(3, 7, 5)$ ② $A(2, 1)$, $B(3, 1)$ ③ $A(2, 1, 4)$, $B(3, 7, 4)$

Soln :

① $\overrightarrow{AB} = (3-2)i + (7-1)j + (6-4)k$
$= i + 6j + 2k$ ⇒ المتجه الموازي

∴ the eq. of Line is :

$$\frac{x-2}{1} = \frac{y-1}{6} = \frac{z-4}{2}$$

② $\overrightarrow{AB} = (3-2)i + (7-1)j$
$= i + 6j$

∴ the eq. of Line is $\frac{x-2}{1} = \frac{y-1}{6}$

③ $\overrightarrow{AB} = (3-2)i + (7-1)j + (4-4)k$
$= i + 6j$
$a = 1$, $b = 6$, $c = 0$

* لا يمكن استعمال الصيغة التناسبية (لأن $c = 0$) لذلك نستعمل الصيغة الوسيطية

$$\left.\begin{aligned} x-2 &= t\cdot 1 \\ y-1 &= t\cdot 6 \end{aligned}\right\} \Rightarrow \frac{x-2}{1} = \frac{y-1}{6} \quad \text{مستقيم}$$

$$z-4 = t\cdot 0 \Rightarrow z = 4 \quad \text{مستوي}$$

~ * ~ * ~ * ~

ex.4 : Find the vector that is parallel to the Line whose equation is $\frac{x-2}{3} = \frac{7-2y}{6} = \frac{3z+4}{6}$ also find at least two points on the Line.

Soln

* نكتب المعادلة بصورتها الصحيحة

$$\frac{x-2}{3} = -2\,\frac{y-\frac{7}{2}}{6} = 3\,\frac{z-(-\frac{4}{3})}{6}$$

$$\frac{x-2}{3} = \frac{y-\frac{7}{2}}{-3} = \frac{z-(-\frac{4}{3})}{2} = t \quad \text{—— (1)}$$

∴ $P(2, \frac{7}{2}, -\frac{4}{3})$ & $\vec{v} = 3i - 3j + 2k$

from Eq(1), gets :

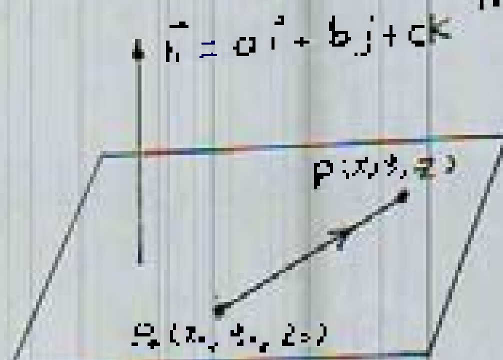
$$x = 3t + 2$$
$$y = \frac{7-11t}{2}$$
$$z = \frac{6t-4}{3}$$

t

$$\left.\begin{array}{ll} t = 0 & \Rightarrow P_0\left(1, \frac{7}{2}, -\frac{4}{3}\right) \\ t = 3 & \Rightarrow P_3\left(11, -\frac{11}{2}, \frac{20}{3}\right) \end{array}\right\} \text{two points}$$

Equation of a Plane : (معادلة المستوى)

المطلوب: إيجاد معادلة المستوى المار بالنقطة P_0 والعمودي على المتجه $\vec{N}$



نفرض أن $P(x, y, z)$ أية نقطة على المستوى

$$\therefore \overrightarrow{P_0P} = (x - x_0)\,i + (y - y_0)\,j + (z - z_0)\,k$$

$$\overrightarrow{P_0P} \perp \vec{N} \Rightarrow \overrightarrow{P_0P} \cdot \vec{N} = 0$$

$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz + \underbrace{(-ax_0 - by_0 - cz_0)}_{d \text{ كمية ثابتة نعرفها}} = 0$$

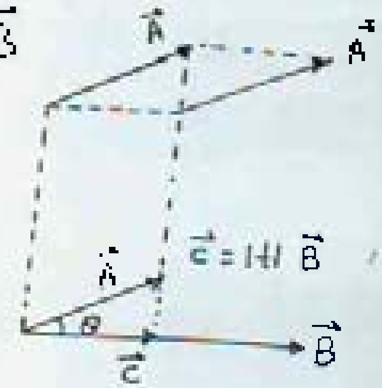
$$\boxed{ax + by + cz + d = 0}$$

الخلاصة: لإيجاد معادلة المستوى يجب أن يتوفر لدينا شرطان

① نقطة معلومة ② المتجه العمودي على المستوى

Projection of two vectors :

$\vec{C}$ = vector projection of $\vec{A}$ onto $\vec{B}$



$$\vec{C} = \overrightarrow{\mathrm{proj}_{\vec{B}}\,\vec{A}}$$

$|\vec{C}|$ = scalar projection of $\vec{A}$ onto $\vec{B}$

$$|\vec{C}| = \mathrm{proj}_{\vec{B}}\,\vec{A}$$

$$|\vec{c}| = |\vec{A}| * |\cos\theta| = |\vec{A}| * \left| \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} \right|$$

$$|\vec{c}| = \frac{|\vec{A}\cdot\vec{B}|}{|\vec{B}|}$$

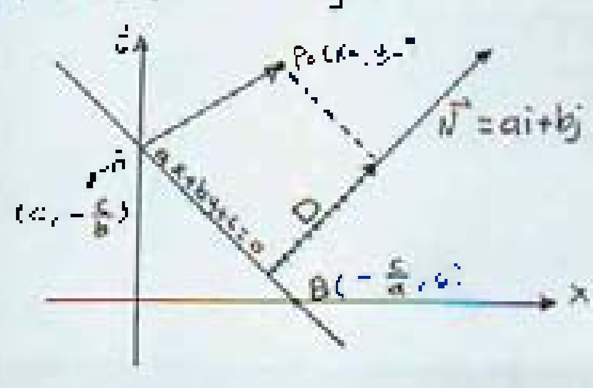
but $\vec{c} = |t|\,\vec{B}$ (parallel vectors)

$$\therefore |\vec{c}| = |t|\,|\vec{B}| \Rightarrow \frac{|\vec{A}\cdot\vec{B}|}{|\vec{B}|} = |t|\,|\vec{B}|$$

$$\therefore |t| = \frac{|\vec{A}\cdot\vec{B}|}{\vec{B}\cdot\vec{B}}$$

$$\therefore \vec{c} = \overrightarrow{\mathrm{proj}_{\vec{B}}\,\vec{A}} = \left(\frac{|\vec{A}\cdot\vec{B}|}{\vec{B}\cdot\vec{B}} \right)\vec{B}$$

ex : show that $D = \dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$ represents the short distance from $P_0(x_0, y_0)$ to the line $ax + by + c = 0$



Sol/

$$\overrightarrow{AP_0} = x_0\,i + (y_0 + \tfrac{c}{b})\,j$$

$$\overrightarrow{AB} = -\frac{c}{a}\,i + \frac{c}{b}\,j$$

$$= \left(-\frac{c}{ab}\right)(b\,i - a\,j)$$

يؤثر على الطول
فقط ولا يؤثر على
اتجاه

∴ $\vec{L} = b\hat{i} - a\hat{j}$ يمثله اتجاه المستقيم

يمثل اتجاه العمود

∴ $\vec{N} = a\hat{i} + b\hat{j}$

the vector $a\hat{i} + b\hat{j}$ is normal to $b\hat{i} - a\hat{j}$

∴ $D = \text{proj}_{\vec{N}} \vec{AP_0} = \left| \frac{\vec{AP_0} \cdot \vec{N}}{|\vec{N}|} \right| \Rightarrow$ وضعنا المطلق لإهمال الإشارة السالبة ان وجدت

$$= \frac{|a x_0 + b(y_0 + \frac{c}{b})|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|a x_0 + b y_0 + c|}{\sqrt{a^2 + b^2}}$$

Note : In three dimensional the extension of the above equation represents the distance between the point $P_0(x_0, y_0, z_0)$ & the plane $ax+by+cz+d=0$ as follows :

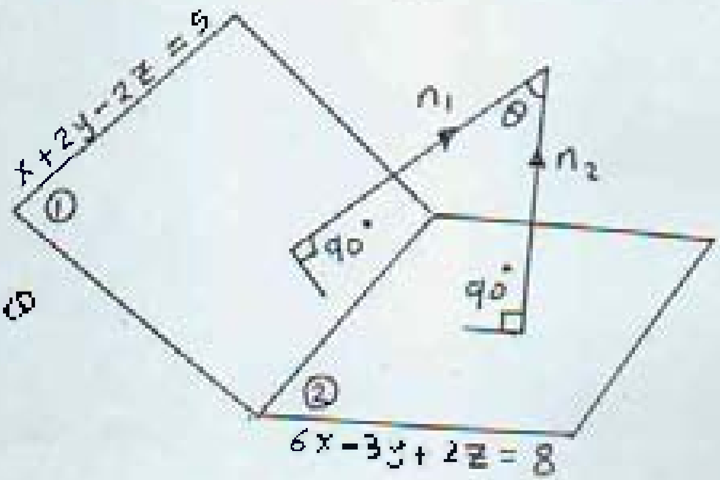
$$D = \frac{|a x_0 + b y_0 + c z_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Sheet No.(4-6): Vector (Solved Problems)
Dr. Akeel Abdullah Mohammed

Prob. 1 : Find the acute angle of intersection of the planes (to the nearest degree)

$x+2y-2z=5$ and $6x-3y+2z=8$

Soln.



$n_1 = i + 2j - 2k$

$n_2 = 6i - 3j + 2k$

$n_1 \cdot n_2 = |n_1||n_2| \cos\theta$

$$\theta = \cos^{-1} \frac{n_1 \cdot n_2}{|n_1|\,|n_2|}$$

$$= \frac{(i+2j-2k)\cdot(6i-3j+2k)}{\sqrt{1^2+2^2+2^2}\ \sqrt{6^2+3^2+2^2}}$$

$$= \cos^{-1} \frac{6-6-4}{(3)(7)} = \cos^{-1} \frac{-4}{21} = 101^\circ$$

$\therefore$ Acute angle $= 180^\circ - 101^\circ = 79^\circ$ Ans.

Prob. 2 : Find the point of intersection of the line and plane shown below

Line : $x = 1+t$, $y = -1+3t$, $z = 2+4t$

Plane : $x - y + 4z = 7$

Soln

point of intersection satisfy the line and plane.

hence substitute line in the equation of plane to get :

$$(1+t) - (-1+3t) + 4(2+4t) = 7$$

$$1 + t + 1 - 3t + 8 + 16t = 7$$

$$14t = -3 \quad \text{to get} \quad t = -\frac{3}{14}$$

$\therefore$ $x = 1 + t \Rightarrow x_o = 1 - \frac{3}{14} = \frac{11}{14}$

$y = -1 + 3t \Rightarrow y_o = -1 + 3\left(-\frac{3}{14}\right) = -1 - \frac{9}{14} = -\frac{23}{14}$

$z = 2 + 4t \Rightarrow z_o = 2 + 4\left(-\frac{3}{14}\right) = \frac{16}{14}$

$\therefore$ point of intersection shall be ;

$$P_o\left(\frac{11}{14}, -\frac{23}{14}, \frac{16}{14}\right)$$

Prob. 3 : Find the coordinates of point of intersection between the line shown below and xy-plane ;

Line ; $x = 3 - t$; $y = 1 + 2t$; $z = 1 + 3t$

Soln :

from the equation of line shown above, intersection with xy-plane gives $z = 0$

$\therefore$ $0 = 1 + 3t \Rightarrow t = -\frac{1}{3}$

$\therefore$ $x_o = 3 - \frac{1}{3} = \frac{8}{3}$

$\therefore$ $y_o = -1 + 2\left(-\frac{1}{3}\right) = -\frac{5}{3}$

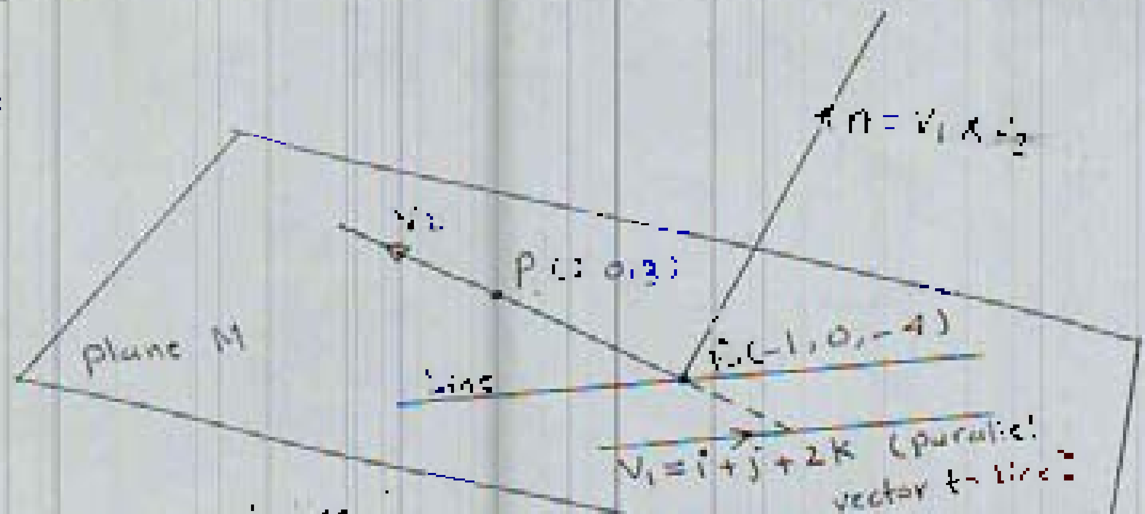
$\therefore$ the point of intersection shall be :

$$P\left(\frac{8}{3}, -\frac{5}{3}, 0\right)$$

Ans

Prob-1 : Find an equation of the plane that contains the point $(2,0,3)$ and the line $x = -1+t$; $y = t$; $z = -4+2t$

Sol^n



from the eq. of line ; Parallel vector shall be ; $\left(\frac{x+1}{1} = \frac{y-0}{1} = \frac{z+4}{2} \right)$

$v_1 = i + j + 2k$

and $P_0(-1,0,-4)$

$v_2 = \vec{P_0P_1} = (3i + 0j + 7k) = 3i + 7k$

$\therefore n = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 3 & 0 & 7 \end{vmatrix} = i(7-0) - j(7-6) + k(0-3)$

$\therefore n = 7i - j - 3k$

$\therefore$ consider the normal vector $n = 7i - 9j + k$ and know point $P_0(-1,0,-4)$ to find the eq of plane where;

$ax + by + cz = d$

$7x - y - 3z = d$

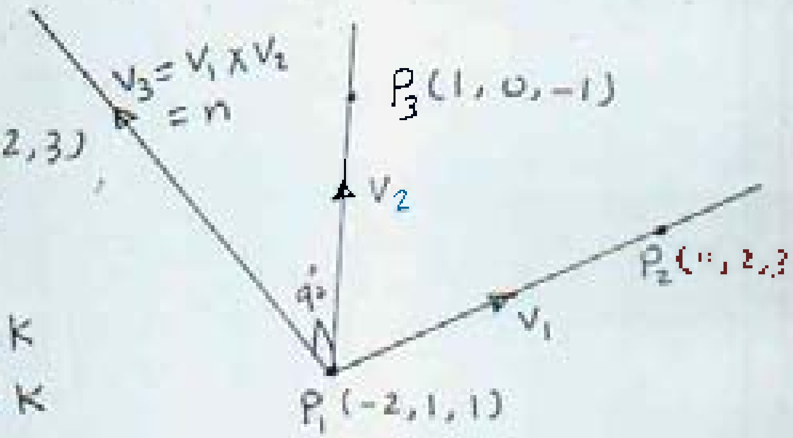
$7(-1) - 0 - 3(-4) = d$

$\therefore d = 5$

$\therefore 7x - 9y + z = 5$ Ans

Prob. 5 : Find the equation of plane passing through the points $(-2,1,1)$, $(0,2,3)$ and $(1,0,-1)$.

Soln :



let $P_1(-2,1,1)$, $P_2(0,2,3)$ and $P_3(1,0,-1)$

$$V_1 = \overrightarrow{P_1P_2} = 2i + j + 2K$$
$$V_2 = \overrightarrow{P_1P_3} = 3i - j - 2K$$

V_3 = normal vector $n = V_1 \times V_2$

$$V_3 = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 3 & -1 & -2 \end{vmatrix} = i(-2+2) - j(-4-6) + K(-2-3)$$

$\therefore$ $V_3 = 10j - 5k$ From which $a = 0$, $b = 10$ and $c = -5$

from $P_1(-2,1,1)$ $x_0 = -2$, $y_0 = 1$, $z_0 = 1$

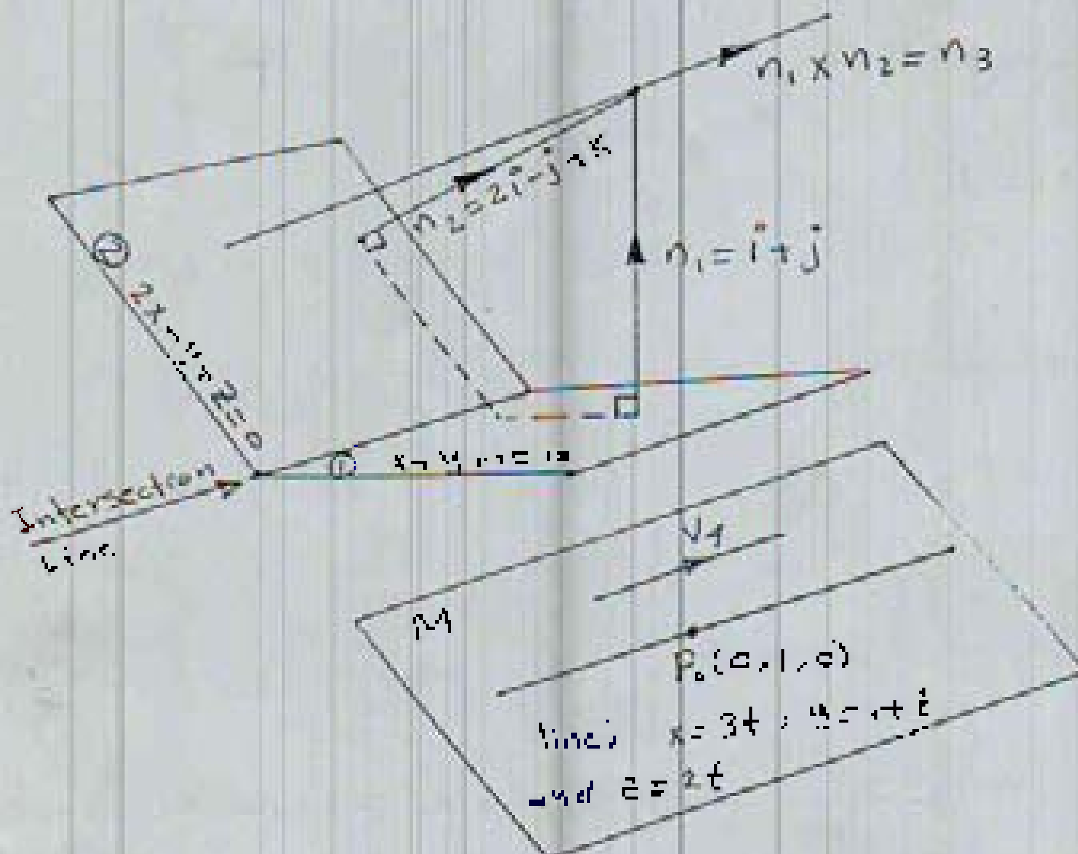
$\therefore$ Equation of plane shall be :

$$ax + by + cz + d = 0$$
$$0x + 10y - 5z + d = 0$$
$$0 + 10(1) - 5(1) + d = 0$$
$$\therefore d = -5$$

$\therefore$ $\boxed{10y - 5z = 5}$ Ans

Prob. 6 : Find an equation of the plane containing the line $x = 3t$, $y = 1 + t$, $z = 2t$ and parallel to the intersection of the plane $2x - y + z = 0$ and $x + y + 1 = 0$.

Sol:



let M is the required plane.

From Equation of Line: $\dfrac{x-0}{3} = \dfrac{y-1}{1} = \dfrac{z-0}{2}$

$V_4 = 3\hat{i} + \hat{j} + 2k$

$P_0\ (0, 1, 0)$

From Equation of Planes:

$n_1 = \hat{i} + \hat{j}$; $n_2 = 2\hat{i} - \hat{j} + k$

find n_3

$$n_3 = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \hat{i}(1-0) - \hat{j}(1-0) + k(-1-2)$$

$\therefore\ n_3 = \hat{i} - \hat{j} - 3k$

then find $n_4 = n_3 \times v_4 =$

or $n_4 = \begin{vmatrix} i & j & k \\ 1 & -1 & -3 \\ 3 & 1 & 2 \end{vmatrix} = i(-2+3) - j(2+9) + k(1+3)$

$\therefore \quad n_4 = i - 11j + 4k$

$\therefore$ There are normal vector $n_4 = i - 11j - 4k$ with $a = 1$, $b = -11$, $c = 4$ and point $P_o(0, 1, 0)$ with $x_o = 0$, $y_o = 1$, and $z_o = 0$ to find equation of the plane as follows:

$$x - 11y + 4z = d$$

$$0 - 11(1) + 4(0) = d \qquad \therefore \; d = -11$$

$\therefore$ The equation of required plane N is:

$$\boxed{x - 11y + 4z = -11}$$

Ans

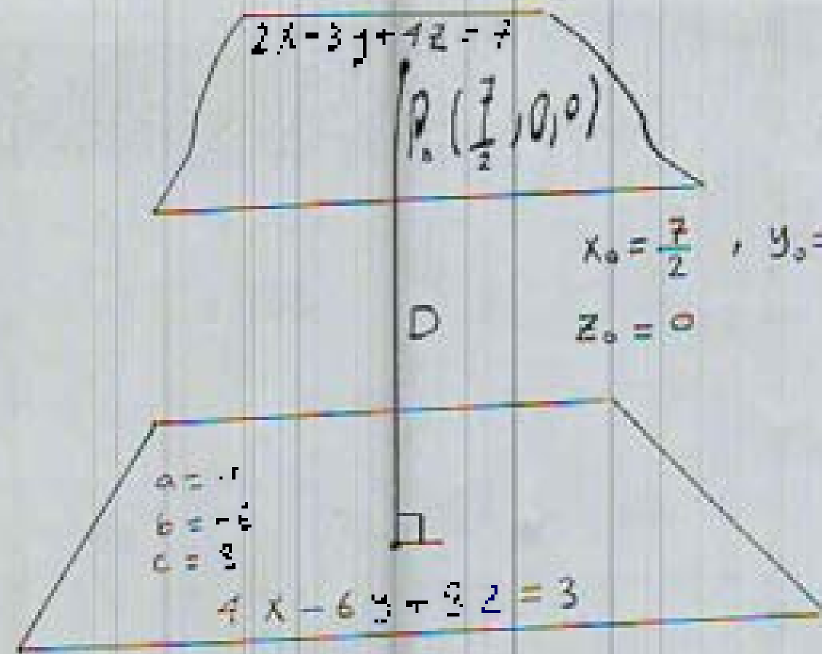
* ~ * ~ * ~ * ~ *

Prob. 7: Find distance between the given parallel planes $2x - 3y + 4z = 7$ and $4x - 6y + 8z = 3$

Sol'n: To find the distance between the planes, we may select an arbitrary point in one of the planes. hence, by selecting $y = z = 0$ in the equation $2x - 3y + 4z = 7$ we obtain $P_o(\frac{7}{2}, 0, 0)$ then;

$$D = \left| \frac{a x_o + b y_o + c z_o + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$= \left| \frac{(4)(\frac{7}{2}) + (-6)(0) + 8(0) - 3}{\sqrt{4^2 + 6^2 + 8^2}} \right| = \frac{11}{\sqrt{116}} = \frac{11}{2\sqrt{29}}$$

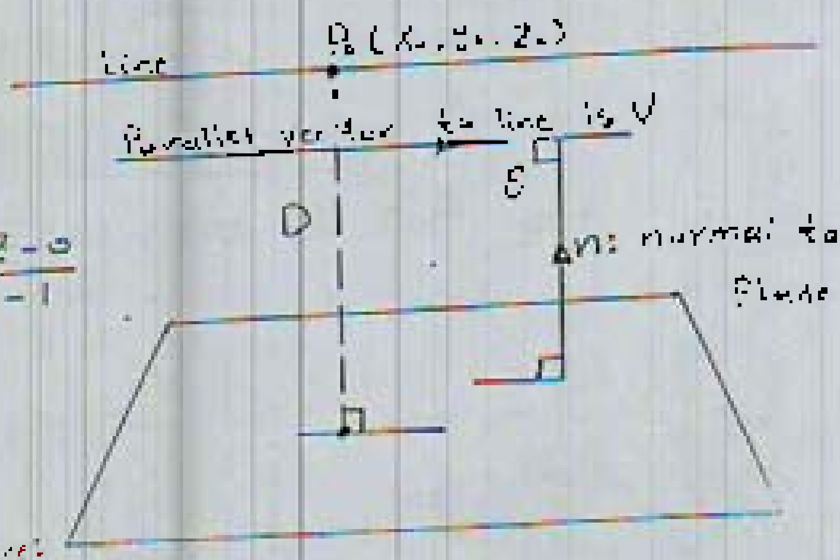


$x_0 = \frac{7}{2}$, $y_0 = 0$

$z_0 = 0$

Prob. 8 : Show that the line $x = -1 + t$; $y = 3 + 2t$, $z = -t$ and the plane $2x - 2y - 2z + 3 = 0$ are parallel then find the distance between them.

Sol :



From Eq. of Line:

$$\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-0}{-1}$$

$$V = i + 2j - k$$

$$P_0 = P_0(-1, 3, 0)$$

From Eq of Plane

$$\vec{n} = 2i - 2j - 2k$$

• لإثبات حالة التوازي بين المستقيم والمستوي يجب ان تكون الزاوية المحصورة بين المتجه العمود n على المستوي والمتجه الموازي للمستقيم V هي 90° .

$$n \cdot v = |n|\,|v| \cos\theta$$

$$\cos\theta = \frac{n \cdot v}{|n|\,|v|} = \frac{(2i-2j-2k)\cdot(i+2j-k)}{\sqrt{2^2+2^2+2^2}\,\sqrt{1^2+2^2+1^2}}$$

$$= \frac{2-4+2}{\sqrt{12}\,\sqrt{6}} = 0$$

$$\therefore \quad \theta = \cos^{-1} 0 = \frac{\pi}{2}$$

$\therefore$ The plane and line are parallel

Hence, to find the distance between the line and the plane, we will find the distance between the point P_0 that lies on the line and the plane ;

$P_0(-1, 3, 0)$ and plane $2x-2y+2z+3=0$

where $x_0 = -1$; $y_0 = 3$ and $z_0 = 0$

and $a = 2$; $b = -2$; and $c = 2$; $d = 3$

and $$D = \left| \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}} \right|$$

$$= \left| \frac{(2)(-1)+(-2)(3)+(2)(0)+3}{\sqrt{2^2+2^2+2^2}} \right|$$

$$= \left| \frac{-2-6+0+3}{\sqrt{12}} \right| = \left| \frac{-5}{2\sqrt{3}} \right|$$

$$= \frac{5}{2\sqrt{3}}$$

Ans

Prob. 9 : Let L_1 and L_2 be the lines;

$$L_1: \quad \begin{aligned} & [illegible] \\ y &= 5 - 4t \\ z &= -1 + 5t \end{aligned} \qquad ; \qquad L_2: \quad \begin{aligned} x &= 2 + 8t \\ y &= 4 - 3t \\ z &= 5 + t \end{aligned}$$

a) Are the lines parallel.
b) Does the lines intersect or they are skew.
c) If they are skew find the distance between them.

Sol$\underline{n}$: From equation of L_1 the parallel vector shall be $\vec{V}_1 = 4i - 4j + 5k$ and from the L_2 the parallel vector $\vec{V}_2 = 8i - 3j + k$
hence L_1 shall be parallel to L_2 if $\vec{V}_1 \parallel \vec{V}_2$ when the angle between them is either (0) or $(180°)$.

hence,
$$\vec{V}_1 \cdot \vec{V}_2 = |\vec{V}_1| |\vec{V}_2| \cos\theta$$

$$\theta = \cos^{-1} \frac{\vec{V}_1 \cdot \vec{V}_2}{|\vec{V}_1||\vec{V}_2|} = \cos^{-1} \frac{(4i - 4j + 5k)\cdot(8i - 3j + k)}{\sqrt{4^2 + 4^2 + 5^2}\sqrt{8^2 + 3^2 + 1^2}}$$

$$= \frac{32 + 12 + 5}{\sqrt{57}\sqrt{74}} = \cos^{-1} \frac{49}{\sqrt{57}\sqrt{74}} = 41°$$

$\therefore$ the two lines are non-parallel
hence, to determine whether they intersect or not.

ملاحظة : ان المستقيمين المتقاطعين يشتركان في نقطة التقاطع التي تحققهما ولذا لنفرض ان نقطة التقاطع هي $P_0(x_0, y_0, z_0)$ والتي تحقق معادلات المستقيمين اعلاه نستخرج ما يلي :-

$$\begin{aligned} x_0 &= 1 + 4t_1 \\ y_0 &= 5 - 4t_1 \\ z_0 &= -1 - 5t_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_0 &= 2 + 8t_2 \\ y_0 &= 4 - 3t_2 \\ z_0 &= 5 + t_2 \end{aligned}$$

This leads to three conditions on t_1 & t_2

$$1 + 4t_1 = 2 + 8t_2 \quad \text{———} (1)$$
$$5 - 4t_1 = 4 - 3t_2 \quad \text{———} (2)$$
$$-1 + 5t_1 = 5 + t_2 \quad \text{———} (3)$$

بحل المعادلتين (1) و (2) آنياً نحصل على t_1 , t_2 :-

$$1 + 4t_1 = 2 + 8t_2 \quad [\text{from Eq. (1)}]$$
$$\therefore \quad 4t_1 = 1 + 8t_2$$

$$\therefore \quad t_1 = 0.25 + t_2 \quad \text{———} (4)$$

by substituting Eq.(4) into Eq.(2), will get:

$$5 - 4(0.25 + t_2) = 4 - 3t_2$$
$$5 - 1 - 4t_2 = 4 - 3t_2 \Longrightarrow t_2 = 0$$
$$\text{and} \quad t_1 = 0.25$$

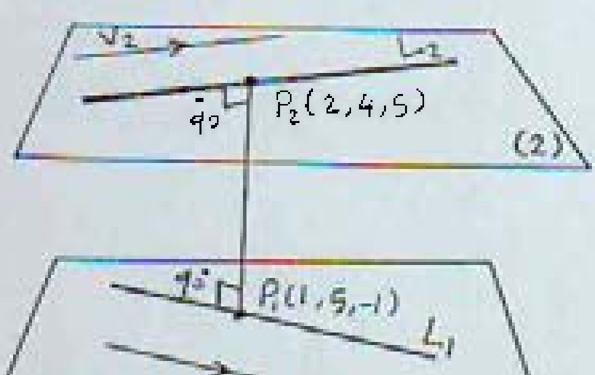
ولكي يكون المستقيمان متقاطعين يجب ان تتحقق المعادلة رقم (3) عند التعويض بقيم t_1 , t_2 فيها من خلال تساوي طرفيها الأيمن والأيسر.

$$\therefore \quad -1 + 5(0.25) \neq 5 + 0$$
$$\text{but} \quad 0.25 \neq 5$$

∴ المستقيمان غير متقاطعين ولا متوازيين

∴ يطلق على المستقيمات متخالفين (skew lines)

والتي سوف نجد البعد بينهما.



$$V_1 = 4\hat{i} - 4\hat{j} - 5\hat{k}$$
$$P_1 = P_1(1, 5, -1)$$

$$V_2 = 8\hat{i} - 3\hat{j} + \hat{k}$$
$$P_2 = P_2(2, 4, 5)$$

المستوي رقم (1) سوف يحوي المستقيم L_1 والنقطة المعلومة P_1

وكذلك المستوي (2) سوف يحوي المستقيم L_2 والنقطة المعلومة P_2 .

سوف يكون n هو حاصل الضرب الاتجاهي للمتجهين الموازيين V_1 و V_2 لكل من المستقيمين L_1 و L_2 على التوالي، والذي يكون عمودياً على كل من المستويين .

$$\therefore \quad n = \vec{V_1} \times \vec{V_2} = \begin{vmatrix} i & j & k \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11i + 36j + 20k$$

hence, the equation of plane containing L_1 & P_1 is:

$$a x_0 + b y + c z + d = 0$$

$$11 x + 36 y + 20 z + d = 0$$

$$11(2) + 36(4) + 20(5) + d = 0$$

$$\therefore \quad d = -266$$

$$\therefore \quad 11 x + 36 y + 20 z - 266 = 0$$

والآن يكون D البعد بين المستوي أعلاه والنقطة P_1 هو البعد بين المستقيمين المتخالفين .

$$D = \left| \frac{a x_0 + b y_0 + c z_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

where $P_1(x_0, y_0, z_0) = P_1(1, 5, -1)$

and $a = 11$, $b = 36$, $c = 20$, $d = -266$

$$\therefore \quad D = \left| \frac{(11)(1) + (36)(5) + (20)(-1) - 266}{\sqrt{11^2 + 36^2 + 20^2}} \right|$$

$$= \frac{95}{\sqrt{1817}} \quad \underline{\underline{Ans}}$$

Prob 10: Show that the lines:

L_1 : $x = -1 + 4t$, L_2 : $x = -13 + 12t$

$y = 3 + t$, $y = 1 + 6t$

$z = 1$, $z = 2 + 3t$

intersect and find the equation of plane they determines.

Sol. -

لكي يتقاطع المستقيمان يجب أن يشتركان في نقطة هي $P_o(x_o, y_o, z_o)$ والتي تحققهما.

$$x_o = -1 + 4t_1 \qquad x_o = -13 + 12t_2$$
$$y_o = 3 + t_1 \qquad y_o = 1 + 6t_2$$
$$z_o = 1 \qquad z_o = 2 + 3t_2$$

or,

$$-1 + 4t_1 = -13 + 12t_2 \quad ----(1)$$
$$3 + t_1 = 1 + 6t_2 \quad ----(2)$$
$$1 = 2 + 3t_2 \quad ----(3)$$

From Eq.(1)

$$4t_1 = -12 + 12t_2$$
$$\therefore t_1 = -3 + 3t_2 \quad ----(4)$$

by substituting Eq.(4) into Eq.(2), will get:

$$3 + (-3 + 3t_2) = 1 + 6t_2$$
$$3 - 3 + 3t_2 = 1 + 6t_2$$
$$\therefore t_2 = -\frac{1}{3} \quad \& \quad t_1 = -4$$

t_1 & t_2 must be satisfy Eq.(3)

$$1 = 2 + 3\left(-\frac{1}{3}\right)$$
$$1 = 1 \quad \text{(i.e. satisfy)}$$

$\therefore$ L_1 & L_2 are intersect in point $P_o(-17, -1, 1)$

where :

$$x_0 = -1 + 4t_1 = -1 + 4(-4) = -17$$

$$y_0 = 3 + t_1 = 3 - 4 = -1$$

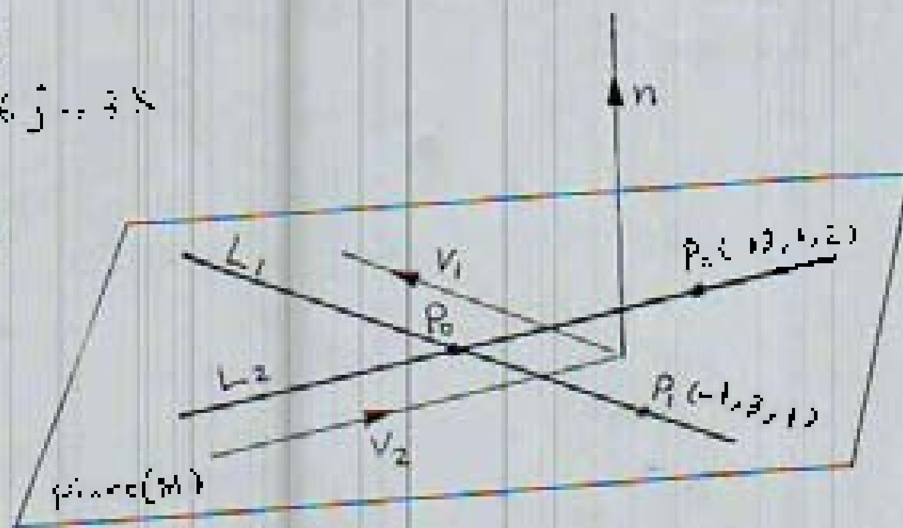
$$z_0 = 1$$

والتي يمكن ان نحصل عليها أيضاً من معادلات المستقيم الثاني.
وذلك بتعويض قيم t_2

$$\vec{V}_1 = 4\hat{i} + \hat{j}$$

$$\vec{V}_2 = 12\hat{i} + 6\hat{j} + 3k$$

$$\vec{n} = \vec{V}_1 \times \vec{V}_2$$



$$\vec{n} = \begin{vmatrix} i & j & k \\ 4 & 1 & 0 \\ 12 & 6 & 3 \end{vmatrix} = i(3-0) - j(12-0) + k(24-12)$$

$$\therefore \quad \vec{n} = 3\hat{i} - 12\hat{j} + 12k$$

The equation of plane (M) that contains L_1 & L_2 (i.e., it contains P_1, P_2 & P_0) is

$$3x - 12y + 12z + d = 0 \qquad \text{---- (5)}$$

by substituting point P_1, (or P_2 or P_0) into eq. (5) will get

$$3(-1) - 12(3) + 12(1) + d = 0$$

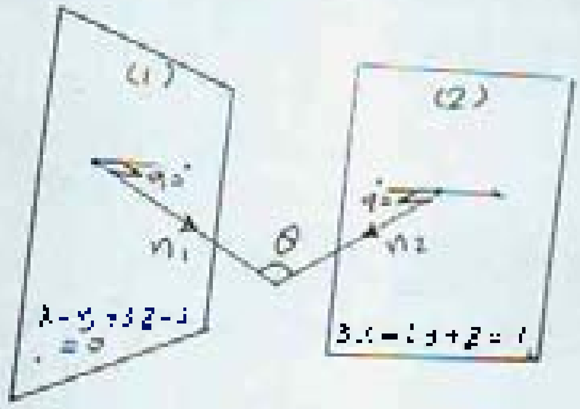
$$\therefore \quad d = 27$$

$$\therefore \quad 3x - 12y + 12z + 27 = 0 \qquad \underline{\text{Ans}}$$

Prob. 11 : Determine weather the following planes are parallel or perpendicular or skew
$x - y + 3z - 2 = 0$ and $3x - 2y + z = 1$

Sol$\underline{n}$: from the equation of planes , normal vectors shall be

$\vec{n}_1 = i - j + 3k$
$\vec{n}_2 = 3i - 2j + k$



$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

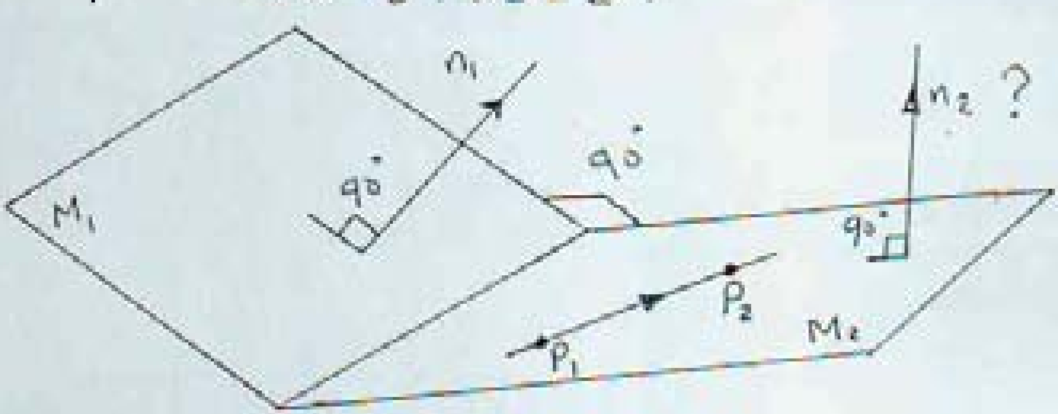
$$\theta = \cos^{-1}\frac{(i - j + 3k)\cdot(3i - 2j + k)}{\sqrt{1^2+1^2+3^2}\sqrt{3^2+2^2+1^2}} = \frac{3+2+3}{\sqrt{11}\sqrt{14}}$$

$= 49.35°$ not $90°$ (not perpendicular)
nor 0 (parallel)

$\therefore$ The two planes are skew .

Prob. 12 : Find the equation of plane through point $P_1(-2, 1, 4)$, $P_2(1, 0, 3)$ and perpendicular to the plane $4x - y + 3z = 2$.

Sol$\underline{n}$:



let M_2 is the required plane that contains P_1 & P_2
M_1 is the given plane

$\therefore \overrightarrow{P_1P_2} = 3i - j - k$

$\overrightarrow{n_2} \perp M_2$

$$\overrightarrow{n_2} = \overrightarrow{n_1} \times \overrightarrow{P_1P_2} = \begin{vmatrix} i & j & k \\ 4 & -1 & 3 \\ 3 & -1 & -1 \end{vmatrix}$$

$$= i(1+3) - j(-4-9) + k(-4+3)$$
$$= 4i + 13j - k$$

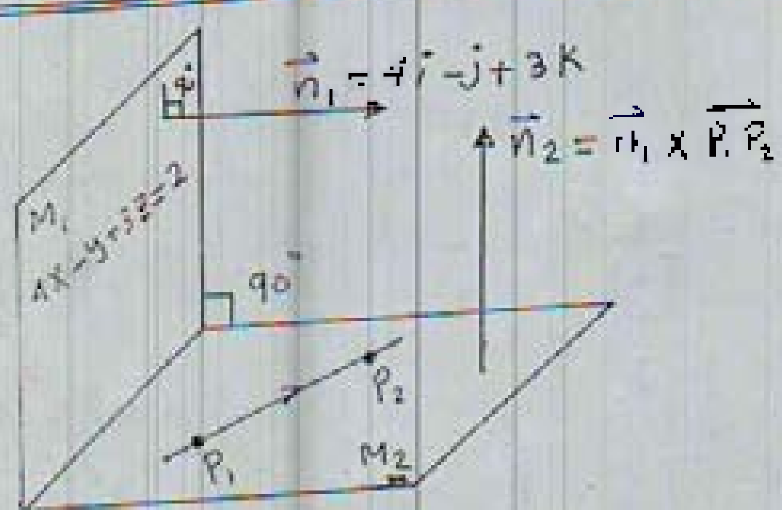
The equation of M_2 is

$4x + 13y - z + d = 0$ (by substituting P_1)
$4(-2) + 13(1) - 4 + d = 0$
$\therefore \quad d = -1$

Ans

$\therefore \quad 4x + 13y - z - 1 = 0$

• ملاحظة : ممكن رسم معالم هذا السؤال بالصورة التالية :-



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University of Technology
Mechanical Engineering Department
Advance Engineering Mathematics
Chapter (3): Series
Dr. Akeel Abdullah Mohammed

Sequence : المتوالية

$$[a_n]_{n=1}^{\infty} = a_1 , a_2 , a_3 , \ldots$$

$$\left[\frac{n+1}{n}\right]_{n=1}^{\infty} = 2 , \frac{3}{2} , \frac{4}{3} , \frac{5}{4} , \frac{6}{5} , \frac{7}{6} \rightarrow 1$$

$$a_n = \frac{n+1}{n} \quad = \text{general term}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

the sequence converges to 1

Convergence of Sequence : تقارب المتوالية

$[a_n]$ converges to L means that

$\lim_{n\to\infty} a_n = L$ where L is a single finite number

otherwise it diverges

Important Rules قوانين للحفظ

1. $\lim_{n\to\infty} x^n = 0 \qquad -1 < x < 1$

2. $\lim_{n\to\infty} x^n = \infty \qquad x > 1 \ , \quad x < -1$

3. $\lim\limits_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ for all x

4. $\lim\limits_{n\to\infty} \left(\frac{a_n^p + \cdots}{b_n^k}\right) = \begin{cases} 0 & p < k \\ \frac{a}{b} & p = k \\ \infty & p > k \end{cases}$

5. $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

6. $\lim\limits_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = \lim\limits_{n\to\infty} \frac{\bar{f}(n)}{\bar{g}(n)}$ (L'Hospital Rule)

تستخدم قاعدة لوبيتال للحالات المبهمة 1^{∞} , ∞^{0} , 0^{0} , $\infty-\infty$,
$0\cdot\infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ على ان تكون مكتوبة
بالصيغة $\frac{0}{0}$ او $\frac{\infty}{\infty}$.

ex. 1: Test for convergence

1. $[2^n] \Rightarrow \lim\limits_{n\to\infty} 2^n = \infty$

$\therefore$ the sequence is diverge

$[(0.2)^n] \Rightarrow \lim\limits_{n\to\infty} (0.2)^n = 0$

$\therefore$ the sequence converges to 0

2. $[(-1)^n] = -1, 1, -1, 1, \ldots = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$

$\therefore [(-1)^n]$ diverges

3. $\left[\frac{2n^2+1}{3n^2+n+2}\right] \Rightarrow \lim\limits_{n\to\infty} \frac{2n^2+1}{3n^2+n+2} = \lim \frac{2+\frac{1}{n^2}}{3+\frac{1}{n}+\frac{2}{n^2}}$

$= \frac{2}{3}$ $\therefore$ the sequence converges to $\frac{2}{3}$

4. $\left[\left(1+\frac{3}{n}\right)^n\right] \Rightarrow \lim\limits_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$

the sequence converges to e^3

5. $\left[\left(1-\frac{2}{n}\right)^{5n}\right] \Rightarrow \lim\limits_{n\to\infty}\left[\left(1-\frac{2}{n}\right)^{5n}\right]$

$= \lim\limits_{n\to\infty}\left[\left(1+\frac{(-2)}{n}\right)^n\right]^5 = (e^{-2})^5 = e^{-10}$

6. $\left[\frac{\ln n}{n}\right] \Rightarrow \lim\limits_{n\to\infty}\left(\frac{\ln n}{n}\right) = \lim\limits_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$

the sequence converges to 0

7. $\left[n^{1/n}\right] \Rightarrow \lim\limits_{n\to\infty}\left[n^{1/n}\right] = \lim\limits_{n\to\infty} e^{\ln(n^{1/n})}$

$= \lim\limits_{n\to\infty} e^{\frac{\ln n}{n}} = \lim\limits_{n\to\infty} e^{1/n} = e^0 = 1$

نحول الصيغة الى بسط ومقام لكي نستطيع استخدام قاعدة لوبيتال

8. $\left[\frac{n+1}{5n^2+2}\right] \Rightarrow \lim\limits_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n^2\left(5+\frac{2}{n^2}\right)} = 0$

the sequence converges to 0

9. $\left[\frac{n^3+1}{n^2+3}\right] \Rightarrow \lim\limits_{n\to\infty}\frac{3n^2}{2n} = \lim\limits_{n\to\infty}\frac{6n}{2} = \infty$ diverges

10. $\left[\left(2+\frac{3}{n}\right)^n\right] \Rightarrow \lim\limits_{n\to\infty}\left(2+\frac{3}{n}\right)^n = \lim\limits_{n\to\infty} 2^n \cdot \lim\limits_{n\to\infty}\left(1+\frac{3/2}{n}\right)^n$

$= \infty \cdot e^{3/2} = \infty$ diverges

11. $\sqrt{n^2+n} - n \Rightarrow \lim\limits_{n\to\infty}\sqrt{n^2+n} - n = \infty$ diverges

12. $\left[\frac{e^n - e^{-n}}{e^n + e^{-n}}\right] \Rightarrow \lim\limits_{n\to\infty}\frac{\left(1-\frac{1}{e^{2n}}\right)}{\left(1+\frac{1}{e^{2n}}\right)} = 1$ converges to 1

Series :

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

∴ The series converges to 1

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots = \infty$$

The series diverges

Definition : **Convergence of a series**

$\sum a_n$ converges to S means that $\lim_{n \to \infty} S_n = S$ where S is a finite single number, otherwise it diverges.

$S \equiv$ total sum

$S_n \equiv$ partial sum $= a_1 + a_2 + \cdots + a_n$

ex. 2 : Use the definition to test whether the series converges or not

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

Soln $\quad a_n = \dfrac{1}{(n+1)(n+2)} = \dfrac{A}{n+1} + \dfrac{B}{n+2} = \dfrac{A(n+2) + B(n+1)}{(n+1)(n+2)}$

$$1 = A(n+2) + B(n+1)$$

if $n=-1 \Rightarrow A = 1$

if $n=-2 \Rightarrow B = -1$

$\therefore \quad a_n = \frac{1}{n+1} - \frac{1}{n+2}$

$a_1 = \frac{1}{2} - \cancel{\frac{1}{3}}$ $\qquad \vdots$

$a_2 = \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$ $\qquad a_{n-2} = \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}$

$a_3 = \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$ $\qquad a_{n-1} = \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}$

$\vdots$ $\qquad a_n = \cancel{\frac{1}{n+1}} - \frac{1}{n+2}$

بالجمع

$$S_n = \frac{1}{2} - \frac{1}{n+2}$$

$\therefore \quad S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2}$

$\therefore$ the series converges to $\frac{1}{2}$

2. $\sum_{n=1}^{\infty} \ln \frac{n+1}{n+3} \Rightarrow a_n = \ln\left(\frac{n+1}{n+3}\right) = \ln(n+1) - \ln(n+3)$

$\therefore \quad a_1 = \ln 2 - \ln 4$ $\qquad a_{n-3} = \ln(n-2) - \ln(n)$

$a_2 = \ln 3 - \ln 5$ $\qquad a_{n-2} = \ln(n-1) - \ln(n+1)$

$a_3 = \ln 4 - \ln 6$ $\qquad a_{n-1} = \ln(n) - \ln(n+2)$

$a_4 = \ln 5 - \ln 7$ $\qquad a_n = \ln(n+1) - \ln(n+3)$

بالجمع

$$S_n = \ln 2 + \ln 3 - \ln(n+2) - \ln(n+3)$$

$\therefore \quad S = \lim_{n\to\infty} S_n = -\infty \quad \therefore$ the series diverges

Geometric Series :

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

عندما نأخذ متغير ونقسمه على الذي قبله فالمقدار يبقى نفسه

مثلاً : x^3 على x^2 $= x$

Theorem :

The series $\sum_{n=0}^{\infty} x^n$ converges to $S = \frac{1}{1-x}$ when $-1 < x < 1$, otherwise it diverges

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \quad , \quad -1 < x < 1$$

ex. 3: Test for convergence $\sum_{n=1}^{\infty} \frac{1}{2^n}$

Soln

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$= \frac{1}{2} \left\{ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \right\} \leftarrow \text{G.S.} \quad x = \frac{1}{2} < 1$$

$\therefore \quad S = \frac{1}{2} \left\{ \frac{1}{1 - \left(\frac{1}{2}\right)} \right\} = \frac{1}{2} * \frac{1}{\frac{1}{2}} = 1$

$\therefore$ the series converges to S

ex. 4: Find the partial sum of $\sum_{n=1}^{\infty} \frac{2}{5^{n-1}}$

Soln

$$S_n = \frac{2}{5^0} + \frac{2}{5^1} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots\cdots + \frac{2}{5^{n-1}} \quad \text{———} (1)$$

Eq. (1) is multiplied by $\frac{1}{5}$

$$\frac{1}{5} S_n = \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \frac{2}{5^4} + \cdots\cdots + \frac{2}{5^n} \quad \text{———} (2)$$

by subtracting Eq. 2 from Eq. (1) (i.e., Eq. 1 − Eq. 2):

$$\frac{4}{5} S_n = 2 + \frac{2}{5^{n-1}} - \frac{2}{5^n}$$

$\therefore \quad S_n = \frac{5}{2}\left\{1 + \frac{4}{5^n}\right\}$

the total sum $\quad S = \lim\limits_{n\to\infty} \frac{5}{2}\left\{1 + \frac{4}{5^n}\right\} = \frac{5}{2}$

<u>ex.5:</u> Test the following series, Find the sum of the convergent series

1. $\sum\limits_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$ $\quad$ 2. $\sum\limits_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^{n-1}$ $\quad$ 3. $\sum\limits_{n=1}^{\infty} \left(-\frac{3}{2}\right)^{n+1}$

<u>Soln</u>

1. $\sum\limits_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1} = 1 + \left(-\frac{3}{4}\right) + \left(-\frac{3}{4}\right)^2 + \cdots$

It is a geometric series $\quad |x| = \frac{3}{4} < 1$

$\therefore$ G.S. converges to $S = \dfrac{1}{1-\frac{3}{4}} = 4$

2. $\sum\limits_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^{n-1} = 1 + \frac{e}{\pi} + \left(\frac{e}{\pi}\right)^2 + \cdots$

It is a geometric series $\quad |x| = \frac{e}{\pi} = 0.865 < 1$

$\therefore$ G.S. converges to $S = \dfrac{1}{1-0.865} = 7.407$

3. $\sum\limits_{n=1}^{\infty} \left(-\frac{3}{2}\right)^{n+1} = \left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^3 + \left(-\frac{3}{2}\right)^4 + \cdots$

$$= \left(-\frac{3}{2}\right)^2 \underbrace{\left\{1 + \left(-\frac{3}{2}\right) + \left(-\frac{3}{2}\right)^2 + \cdots\right\}}_{\text{G.S. } |x| = \frac{3}{2} > 1}$$

$\therefore$ G.S diverges

Homework : Test the following series, Find the sum of the convergent series.

$$\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2} \qquad \sum_{n=2}^{\infty} \frac{1}{n^2-1} \qquad \sum_{n=3}^{\infty} \frac{5}{n-2}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \qquad \sum_{n=1}^{\infty} \frac{4^{n+2}}{7^{n-1}} \qquad \sum_{n=1}^{\infty} -\frac{7}{6^{n-1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n+3} - \frac{1}{n+4} \qquad \sum_{n=1}^{\infty} \frac{2^{n-1}}{4} \qquad \sum_{n=1}^{\infty} -\frac{1}{5^n}$$

Convergence Tests : اختبارات التقارب

1. The general term test اختبار الحد العام

If $\lim_{n \to \infty} a_n \neq 0$ then $\sum a_n$ diverges

If $\lim_{n \to \infty} a_n = 0$ then the test is failure

ex. 6 : Test for convergence $\sum_{n=1}^{\infty} \frac{2n+1}{3n+4}$

Soln $\quad a_n = \frac{2n+1}{3n+4}$

$\therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(\frac{2n+1}{3n+4} \right) = \frac{2}{3} \neq 0$

$\therefore$ the series diverges.

2. The Integral Test "الاختبار التكاملي"

If $a_n = f(n)$ then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ are togather converge or diverge.

أي يتقاربان او يتباعدان سوية

<u>ex. 7</u>: Test for convergence $\sum_{n=1}^{\infty} \frac{1}{n^2}$; $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

<u>Soln</u> let $f(x) = \frac{1}{x^2}$

$$\therefore \int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_1^{n} f(x)\,dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x^2}\,dx$$

$$= \lim_{n\to\infty} \frac{x^{-1}}{-1}\Big|_1^n = \lim_{n\to\infty} -\left(\frac{1}{n} - 1\right) = 1$$

$\therefore \int_1^{\infty} f(x)\,dx$ converges

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

2. $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$, let $f(x) = \frac{1}{\sqrt{x}}$

$$\therefore \lim_{n\to\infty} \int_3^{n} f(x)\,dx = \lim_{n\to\infty} \int_3^{n} \frac{1}{\sqrt{x}}\,dx = \lim_{n\to\infty} 2\sqrt{x}\Big|_3^n$$

$$= \lim_{n\to\infty} 2(\sqrt{n} - \sqrt{3}) = \infty$$

$\therefore \int_3^{\infty} f(x)\,dx$ diverges

3. The Comparison Test : اختبار المقارنة

م. إذا كان $a_n < b_n$ وكانت $\sum b_n$ متقاربة فأن $\sum a_n$ متقاربة وإلا يفشل الاختبار.

ب. إذا كان $a_n > b_n$ وكانت $\sum b_n$ متباعدة فأن $\sum a_n$ متباعدة وإلا يفشل الاختبار.

Theorem (P-Series)

تحفظ

$$\boxed{\sum \frac{1}{n^p} \begin{cases} \text{conveges} & \text{for } p > 1 \\ \text{diverges} & \text{for } p \leq 1 \end{cases}}$$

كيف نجد b_n ؟

هناك نمطان من الاسئلة نتعرف عليهما من خلال المثالين أدناه

ex. 8 : Test for convergence $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

Soln :

$\because \quad \sin^2 n \leq 1 \qquad \div n^3$

$\therefore \quad \frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}$

$\qquad a_n \leq b_n$

but $\sum \frac{1}{n^3}$ (converges, p-series theorem, $p = 3 > 1$)

$\therefore \quad \sum \frac{\sin^2 n}{n^3}$ converges ex. 4: $\sum_{n=1}^{\infty} \frac{n^3}{e^{n^4} + 17}$

Soln by using comparison test

$$17 > 0 \Rightarrow 17 + e^{n^4} > e^{n^4} \Rightarrow$$

$$\frac{1}{17 + e^{n^4}} < \frac{1}{e^{n^4}} \Rightarrow \frac{n^3}{17 + e^{n^4}} < \frac{n^3}{e^{n^4}}$$

$$\Rightarrow a_n < b_n$$

$\sum b_n = \sum \frac{n^3}{e^{n^4}}$ by using integral test

$$\int_1^{\infty} f(x)\,dx = \lim_{m \to \infty} \int_1^m \frac{x^3}{e^{x^4}}\,dx$$

$$= \lim_{m \to \infty} \int_1^m \frac{-4}{-4} x^3 e^{-x^4}\,dx$$

$$= -\frac{1}{4} \lim_{m \to \infty} e^{-x^4} \Big|_1^m$$

$$= \frac{1}{4} \lim_{m \to \infty} \left(\frac{1}{e^1} - \frac{1}{e^{m^4}} \right)$$

$$= \frac{1}{4e^1}$$

$\therefore \sum b_n = \sum \frac{n^3}{e^{n^4}}$ converges

$\therefore \sum a_n = \sum \frac{n^3}{17 + e^{n^4}}$ converges

by comparison test

4. The Limit Comparison Test : اختبار غاية للمقارنة

نستخدم هذه الطريقة عندما تفشل الطريقة الثالثة حيث يجب ان يكون :

$$\lim_{n \to \infty} \frac{a_n}{b_n} \neq 0$$

عندئذ فأن $\sum a_n$ & $\sum b_n$ يتقاربان او يتباعدان سوية و الا تفشل هذه الطريقة .

ex.10: Test for convergence $\sum_{n=1}^{\infty} \frac{n^3}{n^{1/2}+3}$

Soln

$$3 > 0$$

$$3 + n^{1/2} > n^{1/2}$$

$$\frac{1}{3+n^{1/2}} < \frac{1}{n^{1/2}} \qquad * \; n^3$$

$$\frac{n^3}{3+n^{1/2}} < \frac{n^3}{n^{1/2}}$$

$$a_n < b_n$$

$$\sum b_n = \sum \frac{n^3}{n^{1/2}} = \sum \frac{1}{n^{-2.5}}$$ (diverges , p-series theorem, $p=-2.5<1$)

هنا لا يمكن استخدام الـ (comparison test) لأن هناك نقص للقاعدة ؟ لذا نستعيض عنها بـ (limit comparison test)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^{1/2}+3} \cdot \frac{n^{1/2}}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{n^{1/2}+3}{n^{1/2}}} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n^{1/2}}}$$

$$= 1 \neq 0$$

$\therefore$ the choosing is right and $\sum a_n$ diverges ex. 11: Test for convergence $\sum_{n=1}^{\infty} \frac{2n^3+n-2}{5n^4+n^2+3}$

Soln: choose $\sum b_n = \sum \frac{n^3}{n^4} = \sum \frac{1}{n}$

(diverges, p-series theorem, $p=1$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^3+n-2}{5n^4+n^2+3}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2n^4+n^2-2n}{5n^4+n^2+3} = \frac{2}{5} \neq 0$$

$\therefore$ the choosing is right

$\therefore$ $\sum a_n$ diverges

H.W. Test for convergence $\sum \frac{5n^4+n^2+3}{2n^3+n-2}$

5. The Ratio Test الاختبار النسبي

a. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum a_n$ converges

b. If $\lim \frac{a_{n+1}}{a_n} > 1$ then $\sum a_n$ diverges

c. If $\lim \frac{a_{n+1}}{a_n} = 1$ the test is failure

Note: $n! = n(n-1)(n-2)$

$0! = 1$

$(n+1)! = (n+1)n(n-1)(n-2)$

$= (n+1)n!$

* نستخدم الطريقة الخاصة بنا إذا احتوت المتسلسلة على فكتوريل

ex.12: Test for convergence $\sum \frac{n!}{n^n}$

Soln $a_n = \frac{n!}{n^n}$; $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} * \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,n!}{(n+1)^n\cdot(n+1)} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{(n+1)^n}{n^n}} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)^n\right\}^{-1} = \{e^1\}^{-1} = \frac{1}{2.7} < 1$$

$\therefore$ $\sum \frac{n!}{n^n}$ is convergence

6. <u>The Root Test</u> الاختبار الجذري

a. If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\sum a_n$ converges

b. If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ then $\sum a_n$ diverges

c. If $\lim_{n\to\infty} \sqrt{a_n} = 1$ then the test is failure

ex.13: Test for convergence $\sum_{n=1}^{\infty} \left(\frac{3n+1}{2n+3}\right)^n$

Soln $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+1}{2n+3}\right)^n} = \lim_{n\to\infty} \frac{3n+1}{2n+3}$

$$= \frac{3}{2} > 1$$

$\therefore$ the series diverges

Alternating Series : المتسلسلات المتناوبة

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

Definition 1 : The series $\sum (-1)^n a_n$ converges absolutely if $\sum |(-1)^n a_n|$ converges.

Definition 2 : The series $\sum (-1)^n a_n$ converges conditionally if $\sum |(-1)^n a_n|$ diverges & $\lim_{n\to\infty} a_n = 0$

Definition 3 : The series $\sum (-1)^n a_n$ diverges if $\sum |(-1)^n a_n|$ diverges & $\lim_{n\to\infty} a_n \neq 0$

ex. 14 : Test for convergence $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k+2}{3k-1}\right)^k$

Sol$\underline{n}$ $|(-1)^{k+1} a_k| = \sum \left(\frac{k+2}{3k-1}\right)^k$ using root test

$\lim_{k\to\infty} \frac{k+2}{3k-1} = \frac{1}{3} < 1$ it converges

$\therefore$ $\sum |(-1)^{k+1} a_k|$ converges

$\therefore$ The alternating series converges absolutely

ex. 15 : Test for convergence $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+3)}$

Sol$\underline{n}$ $|(-1)^{k+1} a_k| = \frac{k+2}{k^2+3k}$ using limit comparison test choose $b_k = \frac{k}{k^2} = \frac{1}{k}$ (p-series, diverges, p = 1)

$$\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| = \lim_{k\to\infty} \left(\frac{k+2}{k^2+3k}\right) * k$$

$$= \lim_{k\to\infty} \frac{k^2+2k}{k^2+3k} = 1 \neq 0$$

∴ The choosing is right & the series $|a_k|$ diverges

$$\therefore \lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+2}{k(k+3)} = 0$$

$$\therefore \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+3)}$$ converges conditionally

Power Series (Series of Function) متسلسلة دوال

$$\sum a_n(x) = a_1(x) + a_2(x) + a_3(x) + \cdots$$

ex. $\sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ هذه الدالة متقاربة لقيم معينة من x ومتباعدة لقيم أخرى

Interval of Convergence :

هي قيم x التي عندها تكون المتسلسلة متقاربة ولمعرفتها عادة نستعمل الاختبار النسبي . أحياناً قد نستخدم الاختبار الجذري أو الاختبار بالمقارنة

$$\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| < 1}$$

or

$$\boxed{\lim_{n\to\infty} \sqrt[n]{\left|[a_n(x)]^n\right|} < 1}$$

<u>ex. 16</u>: Test for convergence $\sum \frac{\cos^n x}{n!}$

<u>Soln</u> : by using comparison test

$$|\cos^n x| \leq 1 \quad \Rightarrow \left|\frac{\cos^n x}{n!}\right| \leq \frac{1}{n!}$$

$$a_n(x) < b_n \qquad \text{where } \sum b_n = \sum \frac{1}{n!}$$

$$b_{n+1} = \frac{1}{(n+1)!} = \frac{1}{(n+1)n!}$$

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{1}{(n+1)n!} \cdot \frac{n!}{1} = 0 < 1$$

$\therefore$ $\sum b_n$ converges

$\therefore$ $\sum a_n(x) = \sum \frac{\cos^n x}{n!}$ converges for all values of x

<u>ex. 17</u>: Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n n^3}$

<u>Soln</u>

$$a_n(x) = \frac{(x-2)^n}{5^n n^3}$$

$$a_{n+1}(x) = \frac{(x-2)^{n+1}}{5^{n+1}(n+1)^3}$$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{5^{n+1}(n+1)^3} \cdot \frac{5^n n^3}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)}{5} \cdot \frac{n^3}{(n+1)^3}\right| = \lim_{n\to\infty}\left|\frac{(x-2)}{5(1+\frac{1}{n})^3}\right|$$

$$= \left|\frac{x-2}{5}\right|$$

The series converges when $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}(x)}{a_n(x)}\right| < 1$

$\left|\dfrac{x-2}{5}\right| < 1 \Rightarrow -1 < \dfrac{x-2}{5} < 1$

$-5 < x-2 < 5 \Rightarrow -3 < x < 7$

$\therefore$ the series converges when $-3 < x < 7$ and diverges when $x > 7$ & $x < -3$

hence,

at $x = 7 \Rightarrow \dfrac{(7-2)^n}{5^n n^3} = \sum \dfrac{1}{n^3}$ (P-series, $p = 3 > 1$, it converges

at $x = -3 \Rightarrow \dfrac{(-3-2)^n}{5^n n^3} = \sum \dfrac{(-5)^n}{5^n n^3} = \sum \dfrac{(-1)^n}{n^3}$

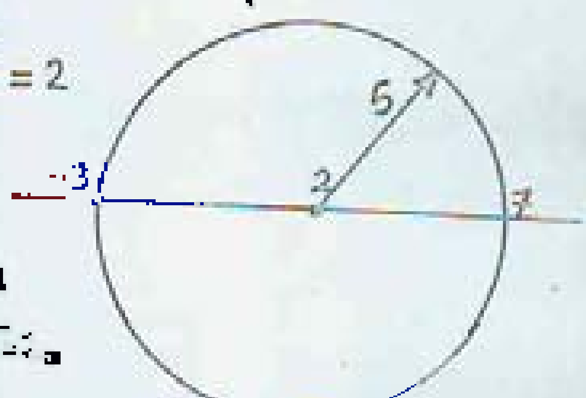
$\left|\sum \dfrac{(-1)^n}{n^3}\right| = \sum \dfrac{1}{n^3}$ (p-series, it converges)

$\therefore \sum \dfrac{(-1)^n}{n^3}$ converges absolutely

$\therefore$ the series converges when $-3 \leq x \leq 7$

center of convergence $= \dfrac{-3+7}{2} = 2$

radius $= 7-2 = 5$



إذا كان الحد الموجب (تعويض 7 في المتسلسلة)

متقارب، نأخذ الحد السالب (تعويض 3- في المتسلسلة)

ex.18: Find the interval of convergence $\sum \frac{(2x-5)^n}{n^2}$

Soln by using root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(2x-5)^n}{n^2}\right|} < 1$$

$$\lim_{n\to\infty} \left|\frac{2x-5}{n^{2/n}}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{2x-5}{e^{\ln n^{2/n}}}\right| < 1$$

$$\lim_{n\to\infty} \frac{|2x-5|}{e^{2\frac{\ln n}{n}}} < 1 \quad \text{using Lopital Rule}$$

$$\lim_{n\to\infty} \frac{|2x-5|}{e^{\frac{2}{n}/1}} < 1 \Rightarrow \frac{|2x-5|}{e^0} < 1$$

$$-1 < 2x-5 < 1 \Rightarrow 4 < 2x < 6$$

$\therefore$ the power series converges for $2 < x < 3$
and diverges for $2 > x$ & $x > 3$

if $x = 3 \Rightarrow \sum \frac{(3*2-5)^n}{n^2} = \sum \frac{1}{n^2}$ converges

if $x = 2 \Rightarrow \sum \frac{(2*2-5)^n}{n^2} = \sum \frac{(-1)^n}{n^2}$ converges absolutely

ex.19: Test for convergence $\sum_{n=1}^{\infty} (-2)^n (n+1)(x-1)^n$

Soln $|(-1)^n a_n| = 2^n (n+1)(x-1)^n$

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$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n(x)|} < 1 \quad \text{for convergence}$$

$$= \lim_{n\to\infty} \left[2^n (n+1)(x-1)^n \right]^{1/n} < 1$$

$$= \lim_{n\to\infty} 2(x-1)(n+1)^{1/n} < 1$$

$$= \lim_{n\to\infty} 2(x-1)\, e^{\ln(n+1)^{1/n}} < 1$$

$$= \lim_{n\to\infty} 2(x-1)\, e^{\frac{\ln(n+1)}{n}} < 1 \quad \text{using L'opital Rule}$$

$$= \lim_{n\to\infty} 2(x-1)\, e^{\frac{1}{n+1}} < 1$$

$x-1 < \frac{1}{2}$ ∴ the series converges when $x < \frac{3}{2}$
and diverges when $x > \frac{3}{2}$

hence, if $x = \frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} (-2)^n (n+1)(\frac{3}{2}-1)^n$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n 2^n (n+1)(\tfrac{1}{2})^n = \sum_{n=1}^{\infty} (-1)^n (n+1)$$

$$\Rightarrow \sum_{n=1}^{\infty} |(-1)^n (n+1)| = \sum_{n=1}^{\infty} (n+1)$$

$\lim_{n\to\infty} (n+1) = \infty \neq 0$ ∴ it diverges

∴ $\sum (-1)^n (n+1)$ diverges also

∴ the alternating power series diverges at $x \geq \frac{3}{2}$

Tayler Series!

if f is defined at x=a and it is differential of order n at x=a then the series

$$f(a)+\frac{f'(a)}{1!}(x-a)^1+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

converges to f(x)

$$f(x)=f(a)+f'(a)\frac{(x-a)}{1!}+f''(a)\frac{(x-a)^2}{2!}+f'''(a)\frac{(x-a)^3}{3!}+\cdots$$

Tayler Series

$$f(\ )=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+\cdots$$

Maclourin Series (special case)

ex. 20: Find Maclourin series (a=0) for sin x

Soln

$f(x)=\sin x$	$f(0)=\sin 0=0$
$f'(x)=\cos x$	$f'(0)=\cos 0=1$
$f''(x)=-\sin x$	$f''(0)=-\sin 0=0$
$f'''(x)=-\cos x$	$f'''(0)=-1$

$$\therefore\ f(x)=f(0)+f'(0)\,x+f''(0)\frac{x^2}{2!}+\cdots$$

$$=0+1\,x+0\,\frac{x^2}{2!}+\frac{(-1)}{3}x^3+\cdots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

قوانين مهمة جداً (للحفظ)

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

3. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

4. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

5. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

6. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

7. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$

8. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$

9. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$

10. $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$

ex. 21: Find the Taylor series expansion of $\cos x$ about the point $a = 2\pi$

Sol: the values of $\cos x$ and its derivatives at $a = 2\pi$ are the same as their values at $a = 0$, therefore;

$$f^{2k}(2\pi) = f^{2k}(0) = (-1)^k$$

$$f^{(2k+1)} \ldots$$

$f(x) = \cos x$ $\quad f'(x) = -\sin x$ $\quad f'(0) = 0$ $\quad f'(2\pi) = 0$

$f(0) = 1$ $\quad f''(x) = -\cos x$ $\quad f''(0) = -1$ $\quad f''(2\pi) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$ $\quad f'''(2\pi) = 0$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$= 1 + 0 - \frac{1}{2!}(x-2\pi)^2 + 0 + \frac{1}{4!}(x-2\pi)^4 - \cdots$$

$$= 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \cdots$$

$$= \sum_{K=0}^{\infty} (-1)^K \frac{(x-2\pi)^{2K}}{(2K)!}$$

ex.22: Find the series that converges to $\tan^{-1} x$ by the use of the series that converges to $\frac{1}{1-x}$

Sol: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

put $x = -t^2$

$$\therefore \frac{1}{1+t^2} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \cdots$$

$$= 1 - t^2 + t^4 - t^6 + t^8$$ أخذ التكامل للطرفين

$$\int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - \cdots)\, dt$$

$$\tan^{-1} t \Big|_0^x = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots \Big|_0^x$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$$

EX. 23: Find the series that converges to $\frac{x^4}{(1+x)^2}$ by differenatiating the series that converges to $\frac{1}{1+x}$

Sol^n

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$

بأخذ المشتقة للطرفين

$$(-1)(1+x)^{-2} = 0 - 1 + 2x - 3x^2 + 4x^3$$

نضرب الطرفين بـ $(-x^4)$

$$\frac{x^4}{(1+x)^2} = x^4 - 2x^5 + 3x^6 - 4x^7 + \dots$$

EX. 24: Find the series that converges to $e^x \cos x$ by multiplication

Sol^n

$$e^x \cdot \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right)$$

$$= 1\, x^0 + 1\, x^1 + \left(-\tfrac{1}{2} + \tfrac{1}{2}\right) x^2 + \left(-\tfrac{1}{2} + \tfrac{1}{6}\right) x^3 + \left(\tfrac{1}{24} + \tfrac{1}{24} - \tfrac{1}{4}\right) x^4 + \dots$$

$$= 1 + x - \frac{1}{3} x^3 - \frac{1}{6} x^4 + \dots$$

EX. 25: Expand using Maclaurine series ($\frac{x}{1-x^2}$, $\ln(1+x^4)$)

Sol^n $f(x) = \frac{x}{1-x^2}$ & $g(x) = \ln(1+x^4)$

① $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$ $\Rightarrow$ put $t = x^2$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$$ بالضرب بـ x

② $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ بالضرب ب dt ثم التكامل

$$\int \frac{dt}{1+t} = \int (1 - t + t^2 - t^3 + \cdots)\, dt$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \cdots$$

put $t = x^4$

$$\ln(1+x^4) = x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \frac{x^{16}}{4} + \cdots = g(x)$$

$$\therefore \frac{x}{1-x^2} + \ln(1+x^4) = x + x^3 + x^4 + x^5 + x^7 - \frac{x^8}{2} + x^9 + x^{11} + \frac{x^{12}}{3} + x^{13} + x^{15} - \frac{x^{16}}{4} + \cdots$$

ex. 26: Find $\int_{0.1}^{0.2} \frac{1-e^x}{x^3}\, dx$ using Maclaurine series

Soln $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$1 - e^x = -\left\{ x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right\}$$

$$\frac{1-e^x}{x^3} = -\left\{ \frac{1}{x^2} + \frac{1}{2!}\frac{1}{x} + \frac{1}{3!} + \frac{x}{4!} + \cdots \right\}$$

$$\therefore \int_{0.1}^{0.2} -\left\{ \frac{1}{x^2} + \frac{1}{2!}\frac{1}{x} + \frac{1}{3!} + \frac{x}{4!} + \cdots \right\} dx$$

$$= \frac{1}{x} - \frac{1}{2}\ln x - \frac{1}{6}x - \frac{x^2}{48} + \cdots \Big|_{0.1}^{0.2}$$

$$\simeq (\quad)$$

ex. 27 : Evaluate $\lim_{x \to 0} \{(\sin x - \tan x)/x^3\}$ using Maclaurine series

Soln

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots$$

$$= x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$\therefore \lim_{x \to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x \to 0} \left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$= -\frac{1}{2}$$

ex. 28 : Evaluate $\lim_{x \to 1} \{(\ln x/(x-1))\}$

Soln let $f(x) = \ln(x)$ & $g(x) = x - 1$

Note :

$$\boxed{\begin{array}{llll} \text{If} & \lim_{x \to a} \frac{f(x)}{g(x)} & \text{use} & \text{T-series} \\ \text{but} & \lim_{x \to 0} \frac{f(x)}{g(x)} & \text{use} & \text{M-series} \end{array}}$$

$f(x) = \ln(x)$ $\qquad f(1) = 0$

$\bar{f}(x) = 1/x$ $\qquad \bar{f}(1) = 1$

$\bar{\bar{f}}(x) = -1/x^2$ $\qquad \bar{\bar{f}}(1) = -1$

$$f(x) = f(a) + \frac{\bar{f}(a)}{1!}(x-a) + \frac{\bar{\bar{f}}(a)}{2!}(x-a)^2 + \cdots$$

$$\ln(x) = 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \cdots$$

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \cdots$$

$$\therefore \quad \frac{\ln(x)}{x-1} = 1 - \frac{1}{2}(x-1) + \cdots$$

$$\therefore \quad \lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \left\{1 - \frac{1}{2}(x-1) + \cdots\right\}$$

$$= 1$$

ex. 29 : Express $\int \sin x^2 \, dx$ as a power series

Soln from the series for $\sin x$ we obtain

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots \qquad -\infty < x < \infty$$

$$\therefore \int \sin x^2 \, dx = \frac{x^3}{3} - \frac{x^7}{7 \times 3!} + \frac{x^{11}}{11 \times 5!} - \cdots + C$$

ex. 30 : Estimate $\int_0^1 \sin(x^2)\, dx$ with an error of less than 0.001

Soln $\int_0^1 \sin(x^2)\, dx = \frac{1}{3} - \frac{1}{7\times 3!} + \frac{1}{11\times 5!} - \frac{1}{15\times 7!} + \cdots$

but $\frac{1}{11\times 5!} \approx 0.00076 < 0.001$ ∴ نكتفي بأول حدين

$$\therefore \int_0^1 \sin(x^2)\, dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.31$$

ex. 31 : Estimate $\int_0^{0.5} \sqrt{1+x^4}\, dx$ with an error $< 10^{-4}$

Sol<u>n</u>

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \cdots$$

$$\int_0^{0.5} \sqrt{1+x^4}\, dx = x + \frac{1}{2*5}x^5 - \frac{1}{8*9}x^9 + \cdots \Big|_0^{0.5}$$

$$= 1 + 0.0031 - 0.00003 + \cdots$$

$$\simeq 1.0031$$

ex. 32 : Find the interval of the convergence for the series represent $\tan^{-1} x$?

Sol<u>n</u> from Maclaurine series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)}$$

$$|a_n(x)| = \frac{x^{2n-1}}{2n-1} \quad ; \quad |a_{n+1}(x)| = \frac{x^{2(n+1)-1}}{2(n+1)-1}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| < 1 \quad \text{for convergence}$$

$$= \lim_{n\to\infty} \frac{x^{2(n+1)-1}}{2(n+1)-1} \cdot \frac{2n-1}{x^{2n-1}} < 1$$

$$= \lim_{n\to\infty} \frac{x^2}{1+\frac{2}{2n-1}} < 1 \quad \Rightarrow \quad x^2 < 1$$

$\therefore$ $x < 1$ [illegible]

Ex. 35 : Evaluate $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

Soln $\lim_{x \to 0} \frac{x - \sin x}{x \sin x}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

$$\therefore \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}}$$

$$= \lim_{x \to 0} \frac{\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}{x^2 - \frac{x^4}{3!}}$$

$$= \lim_{x \to 0} \frac{\frac{x}{3!} - \frac{x^3}{5!}}{1 - \frac{x^2}{3!}} = 0$$

Homework : Use series to evalute the limits in

1. $\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3}$

2. $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3 \cos x}$

3. $\lim_{x \to 0} \frac{\ln(1 + x^2)}{1 - \cos x}$

4. $\lim_{x \to 1} \frac{\ln(x^2)}{x - 1}$

5. $\lim \quad \sin x$