Chapter One/Introduction to Compressible Flow

1.1. Introduction

In general flow can be subdivided into:

i. Ideal and real flow.

For ideal (inviscid) flow viscous effect is ignored. The momentum equations are Euler's equations that derived in 1755 by Euler.

For real (viscose) viscous effect is considered. The momentum equations are Navier-Stokes equations.

ii. Steady and unsteady flow.

For steady flow, flow properties are time independent and mass exits from the system equals the mass enters the system.

For unsteady, flow properties are time dependent and mass exit s from the system may or may not equals the mass enters the system and the difference causes system mass change.

iii. Compressible and incompressible flow

For compressible flow, density becomes an additional variable; furthermore, significant variations in fluid temperature may occur as a result of density or pressure changes. There are four possible unknowns, and four equations are required for the solution of a problem in compressible gas dynamics: equations for the conservation of mass, momentum, and energy, and a thermodynamic relations and equation of state for the substance involved. The study of compressible flow necessarily involves an interaction between thermodynamics and fluid mechanics.

For incompressible flow can be assumed with density is not a variable. For this type of flow, two equations are generally sufficient to solve the problems encountered: the continuity equation or conservation of mass and a form of the Bernoulli equation, derivable from either momentum or energy considerations. Variables are generally pressure and velocity.

iv. One, two and three-Dimensional Flow

One-dimensional flow, by definition, did not consider velocity components in the y or z directions, as in Figure (1.1a). In true one-dimensional flow, area changes are not allowed. For inviscid flow the velocity profile is shown in section (a) and (c). However, the more gradual the area change with x, the more exact becomes the one-dimensional approximation.

For viscose flow the velocity profiles is shown in Figure (1.1b). Actually, due to viscosity, the flow velocity at the fixed wall must be zero as in sections (a) and (c).

Consider the flow in a varying area channel. The velocity profile in a real fluid is shown in Figure (1.1b) section (b).

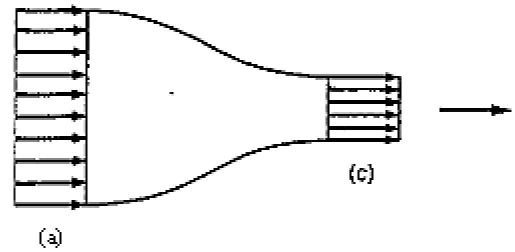

Figure 1.1a: One dimension flow

A complete solution of a problem in a fluid mechanics requires a three-dimensional analysis. However, even for incompressible flow a complete solution in three dimensions is possible only numerically with the aid of computer programs. Fortunately, a great many compressible flow problems can be solved with the use of a one-dimensional analysis. One-dimensional flow implies that the flow variables are functions of only one space coordinate.

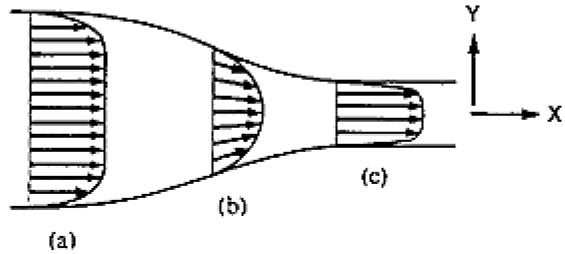

Figure 1.1b: Real flow in varying area duct

1.2. Control volume approach

Figure (1.2) shows an arbitrary mass at time t and the same mass at time $t+\Delta t$, which composes the same mass particles at all times. If Δt is small, there will be an overlap of the two regions as shown, with the common region identified as region 2. At time t the given mass particles occupy regions 1 and 2. At time $t+\Delta t$ the same mass particles occupy regions 2 and 3. Regions 1 & 2, which originally confines of the mass, are called the ***control volume***.

Introducing of concept of ***material derivative*** of any ***extensive property*** (a property which is mass dependent such as mass, enthalpy, internal energy … etc) transforms to a control volume approach gives a valuable general relation called ***Reynolds's Transport Theorem*** that can be used to find property change for many particular situations. Let

$X\ (pronounce\ chi) \equiv$ the total amount of any extensive property in a given mass.

$x \equiv$ the amount of X per unit mass. Thus

$$X = \int x\, dm = \iiint_{c.v.} x\rho d\Upsilon$$

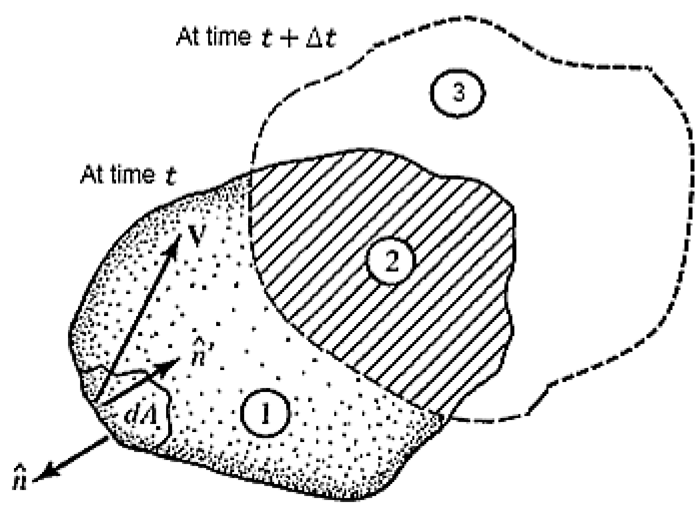

Figure 1.2: Flow into control volume.

We construct our material derivative from the mathematical definition

$$\frac{DX}{dt} = \lim_{\Delta t \to 0} \left[\frac{(final\ value\ of\ X)_{t+\Delta t} - (initial\ value\ of\ X)_t}{\Delta t} \right]$$

$$\frac{DX}{dt} = \lim_{\Delta t \to 0} \left[\frac{(X_2 + X_3)_{t+\Delta t} - (X_1 + X_2)_t}{\Delta t} \right] \qquad (1.1)$$

Now for the term

$$\lim_{\Delta t \to 0} \frac{(X_3)_{t+\Delta t}}{\Delta t}$$

The numerator represents the amount of X in region 3 at time $(t + \Delta t)$, and by definition *region 3 is formed by the fluid moving out of the control volume*. Then;

$$\lim_{\Delta t \to 0} \frac{(X_3)_{t+\Delta t}}{\Delta t} = \iint_{cs,out} x\, \rho\, (\boldsymbol{V}.\hat{n})\, dA \approx total\ amount\ of\ X\ in\ region\ 3 \qquad (1.2)$$

This integral is called a *flux* or *rate* of X flow *out* of the control volume.

Now let us consider the term

$$\lim_{\Delta t \to 0} \frac{(X_1)_t}{\Delta t}$$

Region 1 has been formed by the original mass particles moving into the control volume (during time Δt). Thus

$$\lim_{\Delta t \to 0} \frac{(X_1)_t}{\Delta t} = \iint_{cs,in} x\, \rho\, (\boldsymbol{V}.\breve{n})\, dA \approx total\ amount\ of\ X\ in\ region\ 1 \qquad (1.3)$$

This integral is called a *flux* or *rate* of X flow *into* the control volume.

Now look at the first and last terms of equation (1.1) which is:

$$\lim_{\Delta t \to 0} \left[\frac{(X_2)_{t+\Delta t} - (X_2)_t}{\Delta t} \right] = \frac{\partial\, X_{c.v.}}{\partial t} = \frac{\partial}{\partial t} \iiint_{cv} x\, \rho\, d\Upsilon \qquad (1.4)$$

Note that the partial derivative notation is used since the region of integration is fixed and time is the only independent parameter allowed to vary. Also note that as Δt approaches zero, region 2 approaches the original control volume. Then eq. (1.1) becomes

$$\frac{DX}{dt} = \lim_{\Delta t \to 0} \left[\frac{(X_2 + X_3)_{t+\Delta t} - (X_1 + X_2)_t}{\Delta t} \right]$$

$$= \frac{\partial}{\partial t} \iiint_{cv} x\, \rho\, d\Upsilon + \iint_{cs,out} x\, \rho\, (\boldsymbol{V}.\hat{n})\, dA - \iint_{cs,in} x\, \rho\, (\boldsymbol{V}.\breve{n})\, dA \qquad (1.5)$$

As $\hat{n} = -\breve{n}$ then the last two terms become

$$\iint_{cs,out} x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA - \iint_{cs,in} x\,\rho\,(\boldsymbol{V}.\check{n})\,dA = \iint_{cs} x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA$$

which is the net rate of X passes the control volume surface. The final transformation becomes:

$$\left(\frac{DX}{Dt}\right) = \frac{\partial}{\partial t}\iiint_{cv} x\,\rho\,d\Upsilon + \iint_{cs} x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (1.6)$$

This relation, known as ***Reynolds's Transport Theorem***, which can be interpreted in words as: The rate of change of X property for a fixed mass system of fluid particles as it is moving is equal to the rate of change of X inside the control volume *plus* the *net* efflux of X from the control volume (flow out minus flow in across control volume boundary).

Where

$\frac{D}{Dt}$: Material or total or substantial derivative

$\frac{\partial}{\partial t}$: Partial derivative with respect to time

cv : control volume that containing the mass.

cs : control surface that surrounding the control volume.

X : Mass-dependent (extensive) property; scalar or vector quantity.

x : is the amount of the property per unit mass. For mass it equals one.

ρ : Fluid density (kg/m^3).

$d\Upsilon$: Infinitesimal (very small) control volume.

dA : Infinitesimal control surface.

$\boldsymbol{V}$: Velocity vector.

$\hat{n}$: Outward unit vector which is perpendicular to dA.

$\check{n}$: Inward unit vector which is perpendicular to dA.

Examples of the application of this powerful transformation equation are conservation of mass, energy and momentum equations which are presented in the next chapter.

References:

1. James John & Thie Keith, Gas dynamics, 3td edition, Pearson prentice hall, Upper Saddle, New Jersey, 2006.
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3. منذر اسماعيل الدروبي، مبادئ ديناميك الغازات، بغداد، وزارة التعليم العالي و البحث العلمي، 1980.

Chapter Two/Basic Equation of Compressible Flow

2.1. Conservation of mass:

$$\left(\frac{DX}{Dt}\right) = \frac{\partial}{\partial t}\iiint\limits_{cv} \chi\,\rho\, d\Upsilon + \iint\limits_{cs} \chi\,\rho\,(\boldsymbol{V}.\hat{n})\, dA$$

Let $X \equiv mass$ so $\chi = 1$. For fixed amount of mass that moves through the control volume:

$$\left(\frac{DMass}{Dt}\right) = 0 \qquad (2.1)$$

And for steady flow:

$$\frac{\partial}{\partial t}\iiint\limits_{cv} \rho\, d\Upsilon = 0 \qquad (2.2)$$

So the second term must equals to zero.

$$\iint\limits_{cs} \rho\,(\boldsymbol{V}.\hat{n})\, dA = 0 \qquad (2.3)$$

Let us now evaluate the remaining integral for the case of one-dimensional flow. Figure (2.1) shows fluid crossing a portion of the control surface. Recall that for one-dimensional flow any fluid property will be constant over an entire cross section. Thus both the density and the velocity can be brought out from under the integral sign. If the surface is always chosen perpendicular to V, the integral is very simple to evaluate:

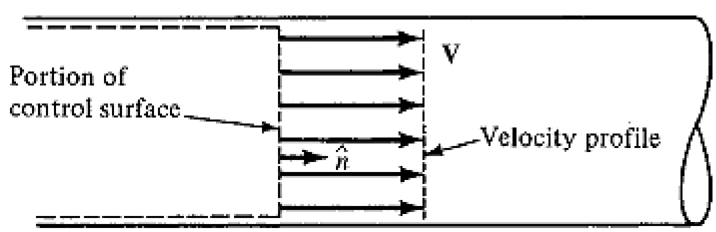


Figure 2.1: One-dimensional velocity profile.

$$\int \rho\,(\boldsymbol{V}.\hat{n})\, dA = \rho V \int dA = \rho\, V\,(A_e - A_i) \qquad (2.4)$$

But integral in eq. 2.3 must be evaluated over the entire control surface, which yields:

$$\iint_{cs} \rho\,(\boldsymbol{V}.\hat{n})\,dA = \sum \rho\, V\, A \tag{2.5}$$

This summation is taken over all sections where fluid crosses the control surface. It is positive where fluid leaves the control volume (since $\boldsymbol{V}.\hat{n}$ is positive here) and negative where fluid enters the control volume.

For steady, one-dimensional flow, the continuity equation for a control volume becomes:

$$\sum \rho\, V\, A = 0 \tag{2.6}$$

If there is only one section where fluid enters and one section where fluid leaves the control volume, this becomes:

$$(\rho\, V\, A)_{out} = (\rho\, V\, A)_{in} \tag{2.7}$$

$$\dot{m} = \rho\, V\, A = const \tag{2.8}$$

V is the component of velocity perpendicular to the area A. If the density ρ is in kg/m^3, the area A is in m^2 and velocity V is in m/s, then $\dot{m}$ is in kg/s.

Note that *as a result of steady flow* the mass flow rate into a control volume is equal to the mass flow rate out of the control volume. But if the mass flow rates into and out of a control volume is the same it doesn't ensure that the flow is steady.

For steady one-dimensional flow, differentiating eq. 2.8 gives:

$$d(\rho\, V\, A) = 0 = V\, A\, d(\rho) + \rho\, V d(\, A) + \rho\, A\, d(V) \tag{2.9}$$

Dividing by $\rho\, V\, A$

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \tag{2.10}$$

This expression can also be obtained by first taking the natural logarithm of equation (2.8) and then differentiating the result. This is called *logarithmic differentiation*.

This differential form of the continuity equation is useful in interpreting the changes that must occur as fluid flows through a duct, channel, or stream-tube. It indicates that if mass is to be conserved, the changes in density, velocity, and cross sectional area must compensate for one another. For example, if the area is constant $(dA = 0)$, any increase in velocity must be accompanied by a corresponding decrease in density. We shall also use this form of the continuity equation in several future derivations.

2.2. Conservation of energy.

From first law of thermodynamics

$$Q = W + \Delta E \qquad (2.11)$$

Where ΔE is the change in total energy of the system i.e. it is the change in internal, kinetic and potential energies, $\Delta(U + K.E. + P.E.)$. Eq. 2.11 can be written on a rate basis to yield an expression that is valid at any instant of time:

$$\frac{\delta Q}{dt} = \frac{\delta W}{dt} + \frac{dE}{dt} \qquad (2.12)$$

$\delta Q/dt$ and $\delta W/dt$ represent instantaneous rates of heat and work transfer between the system and the surrounding. They are rates of energy transfer across the boundaries of the system. These terms are *not* material derivatives since heat and work are not properties of a system. On the other hand, energy is a property of the system and dE/dt is a material derivative, then:

$$\left(\frac{DE}{Dt}\right) = \frac{\partial}{\partial t}\iiint_{cv} e\,\rho\,d\Upsilon + \iint_{cs} e\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (2.13)$$

For one-dimensional, steady flow the last integral is simple to evaluate, as $e, \rho, and\ V$ are constant over any given cross section. Assuming that the velocity V is perpendicular to the surface A, we have

$$\iint_{cs} e\,\rho\,(\boldsymbol{V}.\hat{n})\,dA = \sum(\rho\,V\,A)\,e \sum \dot{m}\,e \qquad (2.14)$$

$$\frac{\partial}{\partial t}\iiint_{cv} e\,\rho\,d\Upsilon = 0 \qquad (2.15)$$

We must be careful to include all forms of work, whether done by pressure forces or shear forces. Figure (2.2) shows a simple control volume. Note that the control surface is chosen carefully so that there is no fluid motion at the boundary, except:

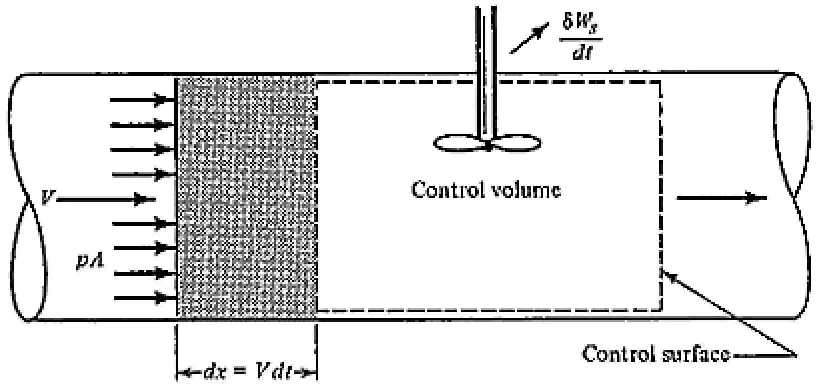


Figure 2.2: Identification of work quantities.

(a) Fluid enters and leaves the system.

(b) A mechanical device crosses the boundaries of the system.

For fluid enters and leaves the system, the pressure forces do work to push fluid into or out of the control volume. The shaded area at the inlet represents the fluid that enters the control volume during time dt. The work done here is:

$$\delta \acute{W} = F.dx = p\,A\,dx = p\,A\,V\,dt \qquad (2.16)$$

The rate of doing work, which called *flow work*, is

$$\frac{\delta \acute{W}}{dt} = pAV = \dot{m}pv \qquad (2.17)$$

The rate at which work is transmitted out of the system by the mechanical device is $\delta W_s/dt$ and

$$\frac{\delta W}{dt} = \frac{\delta W_s}{dt} + \frac{\delta \acute{W}}{dt} = \frac{\delta W_s}{dt} + \dot{m}pv \qquad (2.18)$$

Thus for steady one-dimensional flow the energy equation for a control volume becomes

$$\frac{\delta Q}{dt} = \frac{\delta W_s}{dt} + \sum \dot{m}(e + pv) \qquad (2.19)$$

The summation is taken over all sections where fluid crosses the control surface and is positive where fluid leaves the control volume and negative where fluid enters the control volume.

If there is only one section where fluid leaves and one section where fluid enters the control volume, we have, (from continuity), for steady flow:

$$\dot{m}_{in} = \dot{m}_{out} = \dot{m}$$

Let us take:

$$\frac{\delta Q}{dt} = \frac{\partial}{\partial t}\iiint_{cv} q\,\rho\, d\Upsilon + \iint_{cs} q\,\rho\,(\boldsymbol{V}.\hat{n})\, dA = \dot{m}q \qquad (2.20)$$

$$\frac{\delta W_s}{dt} = \frac{\partial}{\partial t}\iiint_{cv} w_s\,\rho\, d\Upsilon + \iint_{cs} w_s\,\rho\,(\boldsymbol{V}.\hat{n})\, dA = \dot{m}w_s \qquad (2.21)$$

Substitute in eqs (2.20) and (2.21) into eq (2.19) gives:

$$q = w_s + \sum(e + pv) \qquad (2.22)$$

$$q = w_s + \left(u + \frac{V^2}{2} + gz + pv\right)_{out} - \left(u + \frac{V^2}{2} + gz + pv\right)_{in} \qquad (2.23)$$

$$q = w_s + \left(h + \frac{V^2}{2} + gz\right)_2 - \left(h + \frac{V^2}{2} + gz\right)_1 \qquad (2.24)$$

This is the form of the energy equation that may be used to solve many problems. It is often referred as steady flow energy equation (SFEE).

For **unsteady flow**, since change of kinetic and potential energies within the system is negligible, then (Unsteady F.E. E) becomes:

$$\left\{Q + \left[\dot{m}\left(h + \frac{V^2}{2} + gz\right)\right]_{in}\right\} - \left\{W_s + \left[\dot{m}\left(h + \frac{V^2}{2} + gz\right)\right]_{out}\right\} = (\dot{m}u)_2 - (\dot{m}u)_1 \qquad (2.25)$$

$$\dot{m}_{out} - \dot{m}_{in} = \dot{m}_2 - \dot{m}_1 \qquad (2.26)$$

where u_2 and m_2 are internal energy and mass of the working fluid inside the system after change while u_1 and m_1 are internal energy and mass of the working fluid inside the system before change.

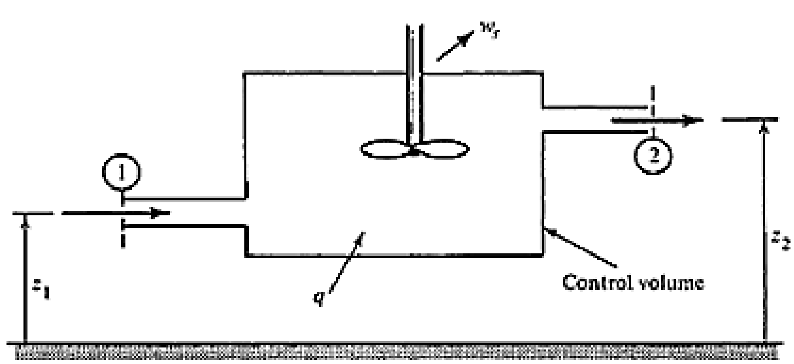


Figure 2.3: Finite control volume for energy analysis.

2.3. Conservation of momentum.

If we observe the motion of a given quantity of mass, Newton's second law tells us that the linear momentum will be changed in direct proportion to the applied forces. This is expressed by the following equation:

$$\sum \mathbf{F} = \frac{D(momentum)}{Dt} = \frac{\partial}{\partial t}\iiint_{cv} \mathbf{V}\,\rho\, d\Upsilon + \iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\, dA \qquad (2.27)$$

Here $\boldsymbol{V}$ besides it is a velocity vector it also represents the momentum per unit mass. This equation is usually called the *momentum* or *momentum flux equation*. $\sum \mathbf{F}$ represents the summation of all forces *on the fluid within the control volume* which maybe forces due to pressure, viscosity, gravity, surface tension … etc..

For steady flow the time rate of change of linear momentum stored inside the control volume is

$$\frac{\partial}{\partial t}\iiint_{cv} \boldsymbol{V}\,\rho\, d\Upsilon = 0 \qquad (2.28)$$

And momentum equation simplify to:

$$\sum \mathbf{F} = \iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\, dA \qquad (2.29)$$

The *x*-component of this equation would appear as

$$\sum F_x = \iint_{cs} V_x\,\rho\, V_x\, dA \qquad (2.30)$$

If there is only one section where fluid enters and one section where fluid leaves the control volume, we know (from continuity) that:

$\dot{m} = \dot{m}_{out} = \dot{m}_{in}$

And the momentum equation for a finite control volume becomes:

$$\sum F_x = \sum \dot{m}\,(V_{out} - V_{in}) \qquad (2.31)$$

The summation is taken over all sections where fluid crosses the control surface and is positive where fluid leaves the control volume and negative where fluid enters the control volume.

2.4. 1st law of thermodynamics.

First law of thermodynamics takes the following form

$$\sum Q = \sum W \tag{2.32}$$

Or

$$Q = W + \Delta E \tag{2.33}$$

First law of thermodynamics is a conservation of energy and we dealt with in 2.2.

2.5. 2nd law of thermodynamics.

Two concepts that are important to a study of compressible fluid flow are derivable from the second law of thermodynamics: the *reversible process* and the *property entropy*. For a thermodynamic system, *a reversible process is one after which the system can be restored to its initial state and leave no change in either system or surroundings*. As a consequence of this definition, it can be shown that a reversible process is quasi-static; changes occur infinitely slowly, with no energy being dissipated

Since thermodynamics, is a study of equilibrium states, definite thermodynamic equations for changes taking place during processes can be derived only for reversible processes; irreversible processes can only be described thermodynamically with the use of inequalities. Irreversible processes involve, for example, the following: friction, heat transfer through a finite temperature difference, sudden expansion, and magnetization with hysteresis, electrical resistance heating, and mixing of different gases.

In general, any natural process is irreversible, so the assumption of reversibility, while it may simplify the thermodynamic equations, necessarily yields an approximation. For many, cases, the assumption of reversibility leads to very accurate results; yet it is well to keep in mind that the reversible process is always an idealization.

The thermodynamic property derivable from the second law is entropy, which is-defined for a system undergoing a reversible process by $dS = (\delta Q/T)_{rev}$.

Entropy changes were defined in the usual manner in terms of reversible processes:

$$\Delta S = \int \frac{\delta Q_{Rev}}{T} \tag{2.34}$$

$$dS = dS_{external} + dS_{internal} \tag{2.35}$$

The term dS_e represents that portion of entropy change caused by the actual heat transfer between the system and its (external) surroundings. It can be evaluated readily from:

$$dS_e = \frac{\delta Q_{Rev}}{T} \tag{2.38}$$

One should note that dS_e can be either positive or negative, depending on the direction of heat transfer. If heat is removed from a system, δQ is negative and thus dSe will be negative. It is obvious that $dS_e = 0$ for an adiabatic process.

The term dS_i represents that portion of entropy change caused by irreversible effects. Moreover, dS_i effects are internal in nature, such as temperature and pressure gradients within the system as well as friction along the internal boundaries of the system. Note that this term depends on the process path and from observations we know that *all irreversibilities generate entropy* (i.e., cause the entropy of the system to increase). Thus we could say that

$$dS_i \geq 0 \tag{2.36}$$

Obviously, $dS_i = 0$ only for a reversible process. An isentropic process is one of constant entropy. This is also represented by $dS = 0$.

$$dS = 0 = dS_e + dS_i \tag{2.37}$$

A reversible-adiabatic process is isentropic, but an isentropic process does not have to be reversible and adiabatic we only know that $dS = 0$.

2.6. Equation of State.

An equation of state for a pure substance is a relation between pressure, .density, and temperature for that substance. Depending on the phase of the substance and on the range of conditions to which it is subjected, one of a number of different equations of state is applicable. However, for liquids or solids, these equations become so cumbersome and have such a limited range of application that it is generally more convenient to use tables of thermodynamic properties. For gases, an equation exists that does have a reasonably wide range of application, the *perfect gas law;* in its usual form, it is expressed as

$$p = \rho R T \qquad (2.38)$$

For the derivation of the perfect gas law from kinetic theory, the volume of the gas molecules and the forces between the molecules are neglected. These assumptions are satisfied by a real gas only at very low pressures. However, even at reasonably high pressures, a real gas approximates a perfect gas as long as the gas temperature is great enough

2.7. Thermodynamics Relations.

Also the following relations are very useful equations. Starting with the thermodynamic property relation:

$$\delta q = du + \delta w \qquad (2.39)$$

$$T ds = du + p dv = c_v dT + RT \frac{dv}{v} \qquad (2.40)$$

$$T ds = dh - v dp = c_p dT - RT \frac{dp}{p} \qquad (2.41)$$

For perfect gas with constant specific heats

$$\Delta s = c_v \int \frac{dT}{T} + R \int \frac{dv}{v} = c_v \ln T + R \ln v \qquad (2.42)$$

$$\Delta s = c_p \int \frac{dT}{T} - R \int \frac{dp}{p} = c_p \ln T - R \ln p \qquad (2.43)$$

$$R = c_p - c_v \qquad and \qquad \gamma = c_p / c_v$$

Example 2.1 Ten kilograms per second of air enters a tank $100\ m^3$ in volume while $2\ kg/s$ is discharged from the tank (Figure 2.4). If the temperature of the air inside the tank remains constant at $300\ K$, and the air can be treated as a perfect gas, find the rate of pressure rise inside the tank.

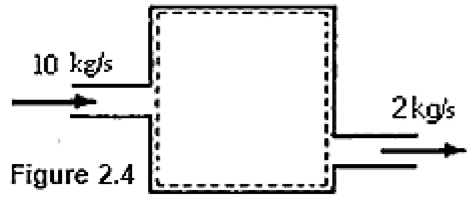


Figure 2.4

Solution:

Select a control volume as shown in the sketch. For this case the net rate of efflux of mass from the control volume is

$$\iint_{cs} \rho\,(\boldsymbol{V}.\hat{n})\,dA = -\,8\ kg/s$$

The volume is constant and also density is assumed constant inside the tank as temperature is constant, but it is time dependent.

$$0 = \frac{\partial \rho}{\partial t}\iiint_{cv} d\Upsilon + \iint_{cs} \rho\,(\boldsymbol{V}.\hat{n})\,dA$$

$$\iiint_{cv} d\Upsilon = \Upsilon = 100\ m^3$$

$$0 = 100\frac{\partial \rho}{\partial t} - 8$$

From equation of state for a perfect gas

$$p = \rho R T$$

$$\frac{dp}{dt} = RT\frac{d\rho}{dt}$$

$$\frac{dp}{dt} = 287 * 300 * \frac{8}{100} = 6.888\ \ kPa/s$$

Example 2.2 Two kilograms per second of liquid hydrogen and eight kg/s of liquid oxygen are injected into a rocket combustion chamber in steady flow (Figure 2.5). The gaseous products of combustion are expelled at high velocity through the exhaust nozzle. Assuming uniform flow in the rocket nozzle exhaust plane, determine the exit velocity. The nozzle exit diameter is $30\ cm$. and the density of the gases at the exit plane is $0.18\ kg/m^3$

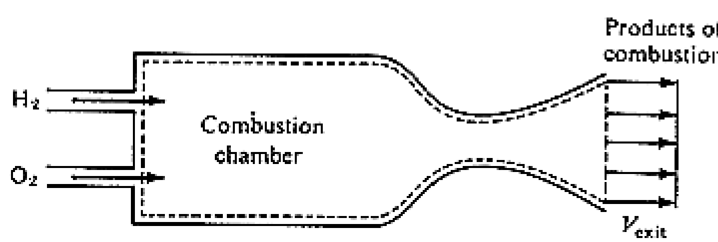


Figure 2.5

Solution

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4}(0.30)^2 = 0.07069\ m^2$$

Select a control volume as shown in the sketch. For this case of steady flow, Eq. (1.12) is applicable

$$\iint_{cs} \rho\ (\boldsymbol{V}.\hat{n})\ dA = 0 = \sum \rho\ V\ A$$

The rate of influx into the control volume is

$2 + 8 = 10.0\ kg/s.$

The rate of efflux is

$$(\rho\ V\ A)_{exit} = (\rho\ V\ A)_{in} = 10.0\ kg/s$$

$$V = \frac{10}{(0.18)(0.07069)} = 785.9\ m/s$$

Example 2.3 An air stream at a velocity of $100\ m/s$ and density of $1.2\ kg/m3$ strikes a stationary plate and is deflected by 90°. Determine the force on the plate. Assume standard atmospheric pressure surrounding the jet and an initial jet diameter of $2\ cm$.

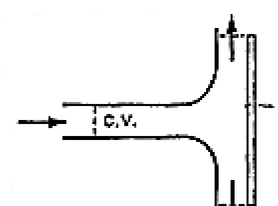


Figure 2.6a

solution

Select a control volume as shown in Figure (2.6a). Writing the x component of eq. (2.30) for steady flow to determine fluid force on the plate

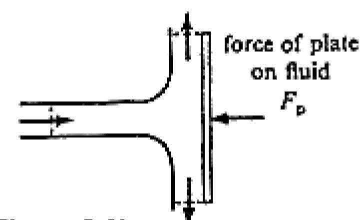


Figure 2.6b

$$\sum F_x = \iint_{cs} V_x\ \rho\ (\boldsymbol{V}.\hat{n})\ dA$$

$$F_{x,fluid} = 100 * \left[1.2(100)\frac{\pi}{4}(0.02)^2\right] = 3.770\ N$$

This force is opposite by F_{plate}

Example 2.4 A rocket motor is fired in place on a test stand. The rocket exhausts $10\ kg/s$ at an exit velocity of $800\ m/s$. Assume uniform steady conditions at the exit plane with an exit plane static pressure of $50\ kPa$. For an ambient pressure of $101\ kPa$, determine the rocket motor thrust transmitted to the test stand as shown in Figure (2.7).

Solution

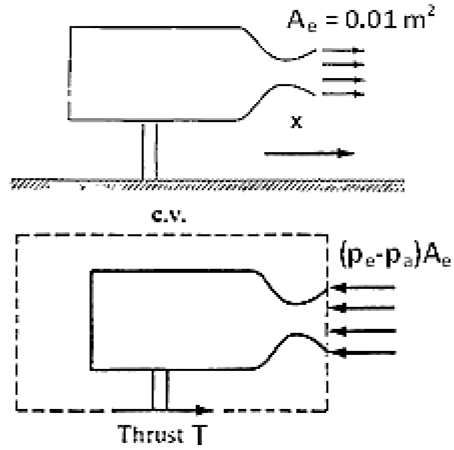


Figure 2.7

$$\sum F_x = \iint_{cs} V_x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA$$

$$\sum F_x = F_{thrust} + F_{pressure}$$

$$\iint_{cs} V_x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA = V_x\,\rho\,V_x\,A = \dot{m}_x V_x$$

$$F_{thrust} - (p_e - p_a)A_e = \dot{m}_x V_x$$

$$F_{thrust} = (50 - 101) \times 10^3 * 0.01 + 10 * 800$$

$$= -510 + 8000 = 7490\ N$$

Example 2.5 A rigid, well-insulated vessel is initially evacuated. A valve is opened in a pipeline connected to the vessel, which allows air at $3\ MPa$ and $300\ K$ to flow into the vessel. The valve is closed when the pressure in the vessel reaches $3\ MPa$. Determine the final equilibrium temperature of the air in the vessel over the temperature range of interest.

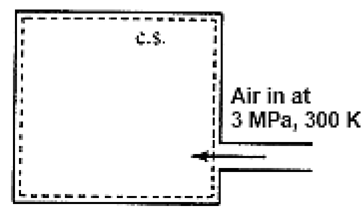


Figure 2.8

Solution

Select a control volume as shown in Figure (1.9). With no heat transfer, no work, and negligible ΔkE and ΔpE, the energy equation is

$$\left[Q + \left[\dot{m}\left(h + \frac{V^2}{2} + gz\right)\right]_{in}\right] - \left[W_s + \left[\dot{m}\left(h + \frac{V^2}{2} + gz\right)\right]_{out}\right] = (\dot{m}u)_2 - (\dot{m}u)_1$$

$$\dot{m}_{out} - \dot{m}_{in} = \dot{m}_2 - \dot{m}_1$$

$$\dot{m}_{in} = \dot{m}_2 = m$$

$$\dot{m}_{out} = \dot{m}_1 = 0$$

So eq. (1.32) is simplify to

$$(\dot{m}h)_{in} = \quad (\dot{m}u)_2$$

and

$$c_p T_{in} = c_v T_2$$

$$T_{final} = T_2 = \frac{c_p}{c_v} T_{in} = \frac{1.005}{0.718} * 300 = 421.1\ \text{K}$$

Example 2.6 Steam enters an ejector (Figure 2.9) at the rate of $0.0454\ kg/sec$ with an enthalpy of $3023.8\ kJ/kg$ and negligible velocity. Water enters at the rate of $0.454\ kg/sec$ with an enthalpy of $93\ kJ/kg$ and negligible velocity. The mixture leaves the ejector with an enthalpy of $349\ kJ/kg$ and a velocity of $27.432\ m/s$. All potentials may be neglected. Determine the magnitude and direction of the heat transfer.

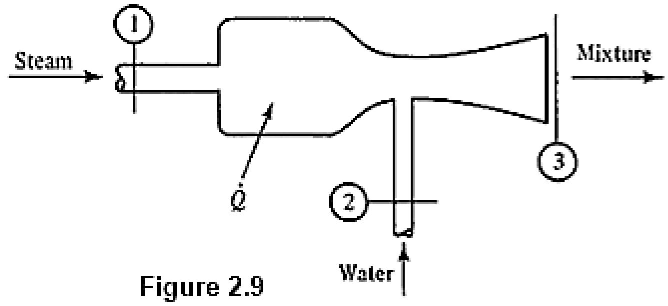


Figure 2.9

$\dot{m}_1 = 0.0454\ kg/\sec, \quad \dot{m}_2 = 0.454\ kg/\sec,$

$h_1 = 3023.8\ kJ/kg, \qquad h_2 = 93\ kJ/kg, \qquad h_3 = 349\ kJ/kg$

$V_1 \approx 0.0\ m/s, \qquad V_2 \approx 0.0\ m/s, \qquad V_3 = 27.432\ m/s$

$\dot{m}_3 = \dot{m}_1 + \dot{m}_2 = 0.0454 + 0.454 = 0.4994\ kg/sec$

$$\dot{Q} + \dot{m}_1\left(h_1 + \frac{V_1^2}{2} + gz_1\right) + \dot{m}_2\left(h_2 + \frac{V_2^2}{2} + gz_2\right) = \dot{W}_s + \dot{m}_3\left(h_3 + \frac{V_3^2}{2} + gz_3\right)$$

$$\dot{Q} + \dot{m}_1 h_1 + \dot{m}_2 h_2 = \dot{W}_s + \dot{m}_3\left(h_2 + \frac{V_3^2}{2}\right)$$

$$\dot{Q} + 0.0454 * 3023.8 + 0.454 * 93 = 0.4994\left(349 + \frac{27.432^2 * 10^{-3}}{2}\right)$$

$$\dot{Q} + 137.281 + 42.222 = 550.1$$

$$\dot{Q} = -5.0245\ kW$$

Example 2.7 A horizontal duct of constant area contains CO2 flowing isothermally (Figure 2.10). At a section where the pressure is $14\ bar$ absolute, the average velocity is know to be $50\ m/s$. Farther downstream the pressure has dropped to $7\ bar$ abs. Find the heat transfer.

Solution

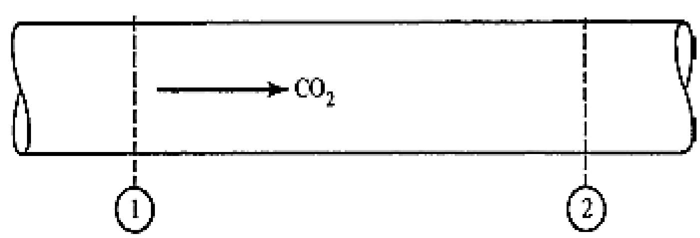


Figure 2.10

$p_1 = 14 \times 10^5\ N/m^2$

$p_7 = 7 \times 10^5\ N/m^2$

$V_1 = 50\ \ m/s$

$V_2 = ?\ \ m/s$

From state equation between 1 and 2, as T is constant:

$P_1\, v_1 = p_2 v_2$

$$\frac{\rho_1}{\rho_2} = \frac{p_1}{p_2} = \frac{14}{7} = 2$$

From continuity equation

$$\dot{m} = \rho_1\, V_1\, A_1 = \rho_2\, V_2\, A_2$$

$$V_2 = V_1 * \frac{\rho_1}{\rho_2} = 50 * 2 = 100\ m/s$$

$$q = w_s + \left(u_2 + \frac{p_2}{\rho_2} + \frac{V_2^2}{2} + gz_2\right) - \left(u_1 + \frac{p_1}{\rho_1} + \frac{V_1^2}{2} + gz_1\right)$$

$$q = \left(\frac{V_2^2 - V_1^2}{2}\right) = \frac{(100^2 - 50^2)}{2} = 3750\ J/kg$$

Example 2.8 Hydrogen is expanded isentropically in a nozzle from an initial pressure of $500\ kPa$, with negligible velocity, to a final pressure of $100\ kPa$. The initial gas temperature is $500\ K$. Assume steady flow with the hydrogen behaving as a perfect gas with constant specific heats, where $c_v = 14.5\ kJ/kg.K$ and $R = 4.124\ kJ/kg.K$. Determine the final gas velocity and the mass flow through the nozzle for an exit area of $500\ m^2$.

Solution

$$\gamma = \frac{c_p}{c_v} = \frac{c_p}{c_p - R} = \frac{14.5}{14.5 - 4.124} = 1.397$$

From isentropic relation

$$T_2 = T_1 \frac{p_2}{p_1}^{\gamma - 1/\gamma} = 500\left(\frac{100}{500}\right)^{1.397-1/1.397} = 316.5\ K$$

From energy equation

$$q = w_s + \left(h + \frac{V^2}{2} + gz\right)_{out} - \left(h + \frac{V^2}{2} + gz\right)_{in}$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$V_2 = \sqrt{2(h_1 - h_2)} = \sqrt{2cp(T_1 - T_2)} = \sqrt{2 * 14.5 * 10^3(500 - 316.5)} = 2306.84\ \ m/s$$

From equation of state

$$\rho_2 = \frac{p_2}{RT_2} = \frac{100}{4.124 * 316.5} = 0.0766\, kg/m^3$$

From continuity equation

$$\dot{m} = \rho_2 V_2 A_2 = 0.0766 * 2306.84 * (500 * 10^4) = 8.837\ \ kg/s$$

Example 2.9 There is a steady one-dimensional flow of air through a $30.48\,cm$ diameter horizontal duct (Figure 1.12). At a section where the velocity is $140.208\,m/s$, the pressure is $344.379\,kN/m^2$ and the temperature is $305.5\,K$. At a downstream section the velocity is $268.224\,m/s$ and the pressure is $164.7847\,kN/m^2$. Determine the total wall shearing force between these sections.

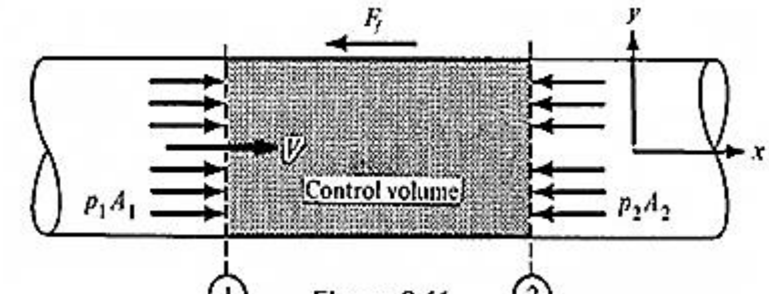


Figure 2.11

Solution

From eq.

$$\sum \mathbf{F} = \sum \dot{m}\,(\mathbf{V}_{out} - \mathbf{V}_{in})$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{344.379}{0.287 * 305.5}$$

$$= 3.928\ kg/m^3$$

$$\dot{m} = \rho_1 V_1 A = 3.928 * 140.208 * \pi * 0.3048^2/4 = 40.182\ kg/s$$

$$\sum \mathrm{F} = (pA)_1 - (pA)_2 - F_f$$

$$F_f = (pA)_1 - (pA)_2 + m(V_{exit} - V_{in})$$

$$F_f = (344.379 - 164.7847) * 10^3 * \frac{\pi}{4} 0.3048^2 + 40.182\,(268.224 - 140.379)$$

$$= 13104.256 - 5137.067 = 7967.2\,N$$

Chapter Three/Wave Propagation

3.1. Introduction

The method by which a flow adjusts to the presence of a body can be shown visually by a plot of the flow streamlines about the body. Figures (3.1) and (3.2) show the streamline patterns obtained for uniform, steady, incompressible flow over an airfoil and over a circular cylinder, respectively.

Note that the fluid particles are able to sense the presence of the body before actually reaching it. At points 1 and 2, for example, the fluid particles have been displaced vertically, yet 1 and 2 are points in the flow field well ahead of the body. This result, true in the general case of anybody inserted in an incompressible flow, suggests that a signaling mechanism exists whereby a fluid particle can be forewarned of a disturbance in the flow ahead of it. The velocity of signal waves sent from the body, relative to the moving fluid, apparently is greater than the absolute fluid velocity, since the flow is able to start to adjust to the presence of a body before reaching it.

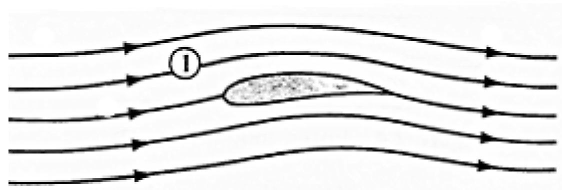


Figure 3.1

Thus, when a body is inserted into incompressible flow, a smooth, continuous streamlines result, which indicate gradual changes in fluid properties as the flow passes over the body. If the fluid particles were to move faster than the signal waves, the fluid would not be able to sense the body before actually reaching it. and very abrupt changes in velocity vectors and other properties would ensue.

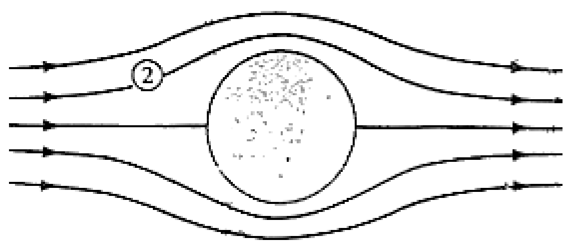


Figure 3.2 Stream patterns for steady incompressible flow

In this chapter, the mechanism by which the signal waves are propagated through incompressible and compressible flows will be studied. An expression for the velocity of propagation of the waves will be derived.

3.2. Wave formulation

To examine the means by which disturbances pass through an elastic medium. A disturbance at a given point creates a region of compressed molecules that is passed along to its neighboring molecules and in so doing creates a *traveling wave*. Waves come in various *strengths*, which are measured by the amplitude of the disturbance. The speed at which this disturbance is propagated through the medium is called the *wave speed*. This speed not only depends on the type of medium and its thermodynamic state but is also a function of the strength of the wave. The *stronger* the wave is, the faster it moves.

If we are dealing with waves of *large amplitude*, which involve relatively large changes in pressure and density, we call these *shock waves*. These will be studied later. If, on the other hand, we observe waves of *very small amplitude*, their speed is characteristic only by the medium and its state. These waves are of vital importance since sound waves fall into this category. Furthermore, the presence of an object in a medium can only be felt by the object's sending out or reflecting infinitesimal waves which propagate at the *sonic velocity*.

Consider a long constant-area tube filled with fluid and having a piston at one end, as shown in Figure (3.3). The fluid is initially at rest. At a certain instant the piston is given an incremental velocity dV to the left. The fluid particles immediately next to the piston are compressed a very small amount as they acquire the velocity of the piston. As the piston (and these compressed particles) continue to move, the next group of fluid particles is compressed and the *wave front* is observed to propagate through the fluid at *sonic velocity* of magnitude a. All particles between the wave front and the piston are moving with velocity dV to the left and have been compressed from ρ to $\rho + d\rho$ and have increased their pressure from p to $p + dp$.

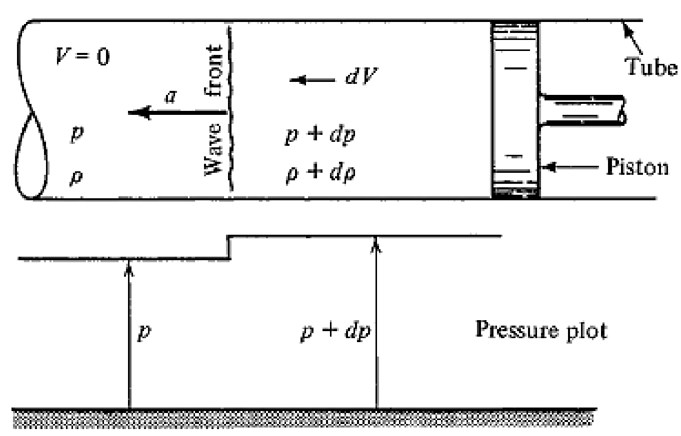


Figure 3.3 Initiation of infinitesimal pressure pulse.

The flow is unsteady and the analysis is difficult. This difficulty can easily be solved by superimposing on the entire flow field a constant velocity to the right of magnitude a.

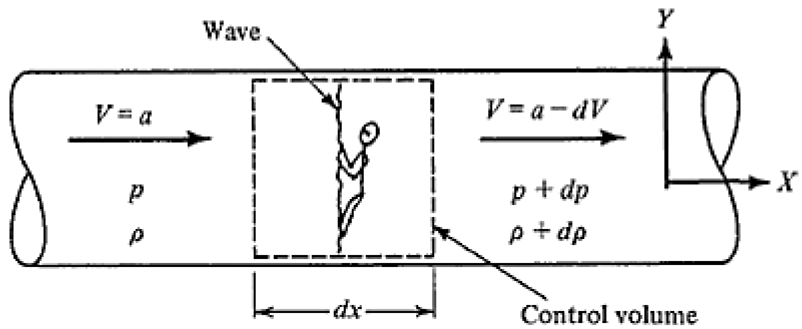


Figure 3.4 Steady-flow picture corresponding to Figure 3.3.

3.3. Sonic Velocity

Figure (3.4) shows the problem. Since the wave front is extremely thin, we can use a control volume of infinitesimal thickness. For steady one-dimensional flow, we have from continuity equation

$\dot{m} = \rho A V = const$

But $A = const$; thus

$$\rho V = const \tag{3.1}$$

Application of this to our problem yields

$\rho a = (\rho + d\rho)(a - dV)$

$\rho a = \rho a - \rho\, dV + a\, d\rho - d\rho\, dV$

Neglecting the higher-order term and solving for dV, we have

$$dV = \frac{a\, d\rho}{\rho} \tag{3.2}$$

Since the control volume has infinitesimal thickness, we can neglect any shear stresses along the walls. We shall write the *x*-component of the momentum equation, taking forces and velocity as positive if to the right. For steady one-dimensional flow we may write from momentum equation

$\sum \mathrm{F}_x = \sum \dot{m}\, (\mathrm{V}_{out} - \mathrm{V}_{in})$

$pA - (p + dp)A = \rho A a[(a - dV) - a]$

$A dp = \rho A a\, dV$

Canceling the area and solving for dV, we have

$$dV = \frac{dp}{\rho a} \tag{3.3}$$

Equations (3.2) and (3.3) may now be combined, the result is:

$$a^2 = \frac{dp}{d\rho} \qquad (3.4a)$$

However, the derivative $dp/d\rho$ is not unique. It depends entirely on the process. For example

$$\left(\frac{\partial p}{\partial \rho}\right)_T \neq \left(\frac{\partial p}{\partial \rho}\right)_s$$

Thus it should really be written as a *partial* derivative with the appropriate subscript.

Since we are analyzing an infinitesimal disturbance, we can assume negligible losses and heat transfer as the wave passes through the fluid. Thus the process is both reversible and adiabatic, which means that it is isentropic. Equation (4.4) should properly be written as:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_{ise} \qquad (3.4b)$$

For substances other than gases, sonic velocity can be expressed in an alternative form by introducing the *bulk* or *volume modulus of elasticity Ev*.

$$E_v = -v\left(\frac{\partial p}{\partial v}\right)_{ise} \equiv \rho\left(\frac{\partial p}{\partial \rho}\right)_{ise} \qquad (3.5)$$

$$a^2 = \frac{E_v}{\rho} \qquad (3.6)$$

Equations (3.4) and (3.6) are equivalent general relations for sonic velocity through *any* medium. The bulk modulus is normally used in connection with liquids and solids. Table 4.1 gives some typical values of this modulus, the exact value depending on the temperature and pressure of the medium. For solids it also depends on the type of loading. The reciprocal of the bulk modulus is called the *compressibility*.

Equation (3.4) is normally used for gases and this can be greatly simplified for the case of a gas that

Table 4.1 Bulk Modulus Values for Common Media

Medium	Bulk Modulus (psi)
Oil	185,000–270,000
Water	300,000–400,000
Mercury	approx. 4,000,000
Steel	approx. 30,000,000

obeys the perfect gas law. For an isentropic process:

$$pv^{\gamma} = c \quad or \quad p = c\,\rho^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_{ise} = c\,\gamma\,\rho^{\gamma-1} = \gamma\,\rho^{\gamma-1}\frac{p}{\rho^{\gamma}} = \gamma RT$$

$$a = \sqrt{\gamma RT} \qquad (3.7)$$

For perfect gases, sonic velocity is a function of the γ , R and T only.

$$Mach\ number, M = \frac{V}{a} \qquad (3.8)$$

It is important to realize that both V and a are computed *locally* for the same point. For other point within the flow we must seek further information to compute on the sonic velocity, which has probably changed.

Subsonic flow, $M <$, the velocity is less than the local speed of sound.

Supersonic flow, $M > 1$, the velocity is greater than the local speed of sound.

We shall soon see that the Mach number is the most important parameter in the analysis of compressible lows.

3.4: Wave Propagation

Let us examine a point disturbance that is at rest in a fluid. *Infinitesimal* pressure pulses are continually being emitted and thus they travel through the medium at *sonic* velocity in the form of spherical wave fronts. To simplify matters we shall keep track of only those pulses that are emitted every second. At the end of 3 seconds the picture will appear as shown in Figure (3.5). Note that the wave fronts are concentric.

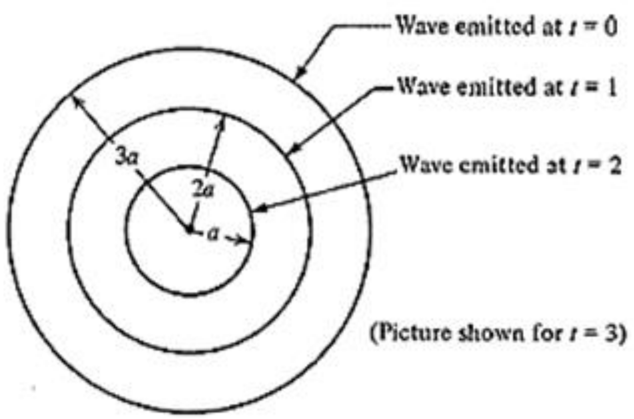


Figure 3.5 Wave fronts from a stationary disturbance.

Now consider a similar problem in which the disturbance is moving at a speed less than sonic velocity, say $a/2$. Figure (3.6) shows such a situation at the end of 3 seconds. Note that the wave fronts are no longer concentric.

Furthermore, the wave that was emitted at $t = 0$ is always in front of the disturbance itself. *Therefore, any person, object, or fluid particle located upstream will feel the wave fronts pass by and know that the disturbance is coming*.

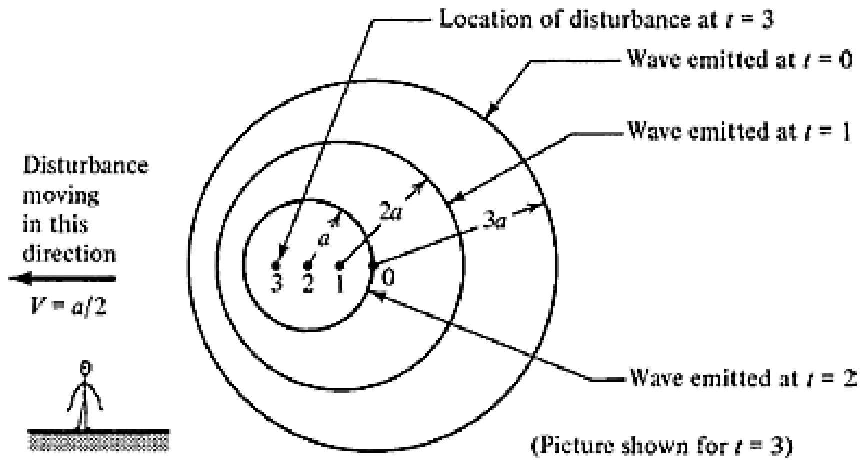
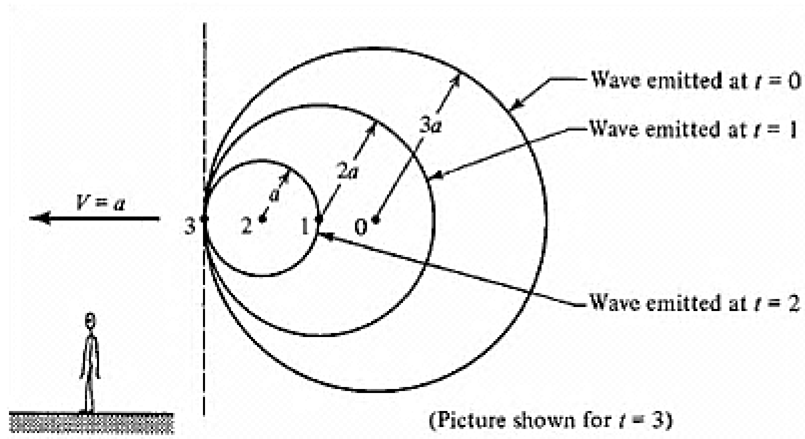


Figure 3.6 Wave fronts from subsonic disturbance.

Next, let the disturbance move at exactly sonic velocity. Figure (3.7) shows this case and you will note that all wave fronts coalesce on the left side and move along with the disturbance. After a long period of time this wave front would approximate a plane indicated by the dashed line. In this case, no region upstream is forewarned of the disturbance as the disturbance arrives at the same time as the wave front.



The only other case to consider is that of a disturbance moving at velocities greater than the speed of sound. Figure (3.8) shows a point disturbance moving at Mach number = 2 (twice sonic velocity). The wave

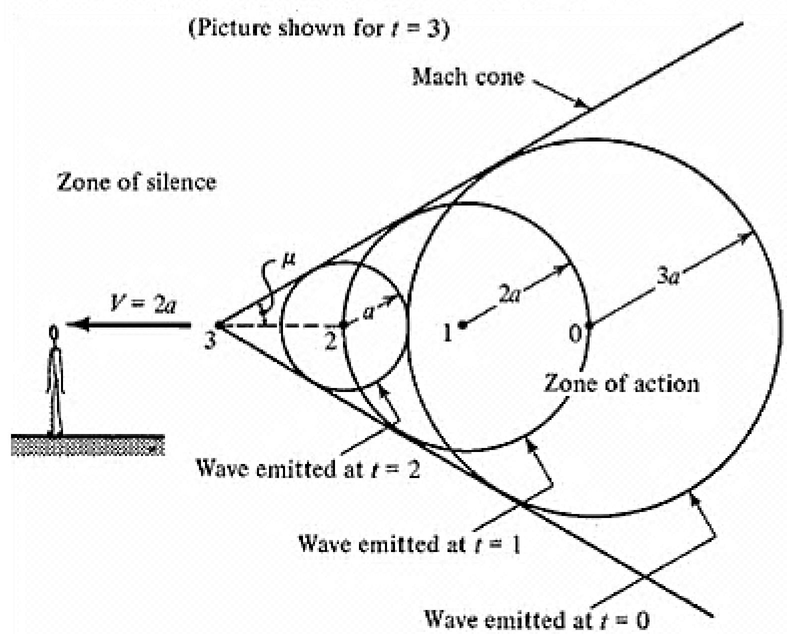


Figure 3.8 Wave fronts from supersonic disturbance.

fronts have coalesced to form a cone with the disturbance at the apex. This is called a *Mach cone*. The region inside the cone is called the *zone of action* since it feels the presence of the waves. The outer region is called the *zone of silence*, as *this entire region is unaware of the disturbance*. The surface of the Mach cone is sometimes referred to as a *Mach wave*; the half-angle at the apex is called the *Mach angle* and is given the symbol μ. It should be easy to see that:

$$\sin\mu = \frac{a}{V} = \frac{1}{M} \tag{3.9}$$

For subsonic flow, no such zone of silence exists. If the disturbance caused by a projectile, the entire fluid is able to sense the projectile moving through it, since the signal waves move faster than the projectile. No concentration of pressure disturbances can occur for subsonic flow; Mach lines cannot be defined.

Let us now compare steady, uniform, subsonic and supersonic flow over a finite wedge-shaped body. If the fluid velocity is less than the velocity of sound, flow ahead of the body is able to sense its presence. As a result, gradual changes in flow properties take place; with smooth, continuous streamlines (see Figure 3.9).

If the fluid velocity is greater than the velocity of sound, the approach flow, being in the zone of silence, is unable to sense the presence of the body. The body now presents a finite disturbance to the flow. The wave pattern obtained is a result of the addition of individual Mach waves emitted from each point on the wedge. This nonlinear addition yields a compression shock wave across which occur finite changes in velocity, pressure, and other flow properties. A typical flow pattern obtained for supersonic flow over the wedge is shown in Figure (3.10).

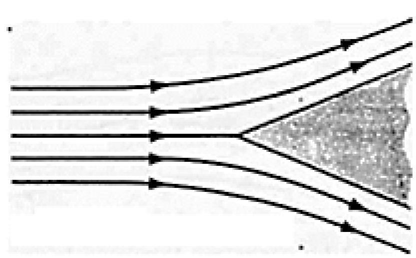


Figure 3.9 Subsonic wedge Flow

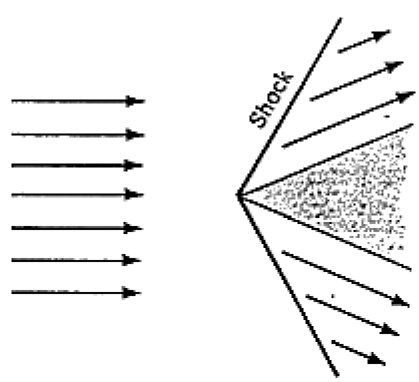


Figure 3.10 Supersonic wedge flow

Chapter Four/Isentropic flow of a perfect gas in varying area duct

To study the compressible, isentropic flow through varying area channels such as nozzles, diffusers and turbine blade passages, the following assumptions are considered:

1. One dimensional, steady flow of a perfect gas.
2. Friction is zero.
3. No heat and work exchange.
4. Variation in properties is brought about by area change.
5. Changes in potential energy and gravitational forces are negligible.

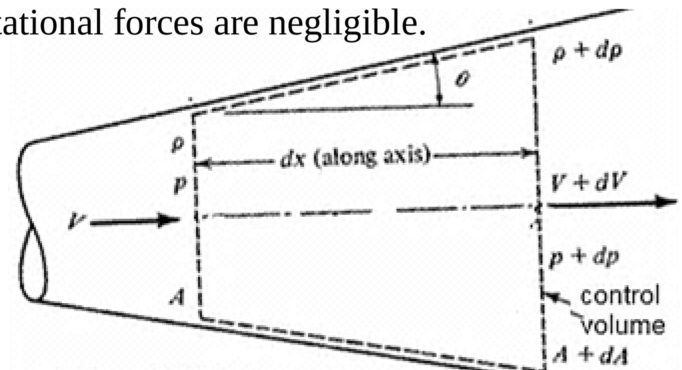


Figure 4.1 Control volume for varying area flow

4.1 Equations of motion.

- **Continuity equation:**

$$\iint_{cs} \rho\,(\boldsymbol{V}.\hat{n})\,dA = \sum \rho\,V\,A = 0 \qquad (4.1)$$

$$\dot{m} = \rho\,V\,A = const \qquad (4.2)$$

$$(\rho + d\rho)(\,V + dV)(\,A + dA) = \rho\,V\,A \qquad (4.3)$$

Simplifying and ignoring high order

$$\rho VA + \rho VdA + \rho AdV + VAd\rho = \rho\,V\,A \qquad (4.4)$$

Divided by $\rho\,V\,A$

$$\boldsymbol{\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0} \qquad (4.5)$$

- **Momentum equation:**

$$\sum \mathbf{F} = \iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (4.6)$$

$$\iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\,dA = \rho VA[(V + dV) - V] \qquad (4.7)$$

If there is no electromagnetic force and friction force is negligible, the only acting force is the pressure force. The side wall pressure force in flow direction can be obtained with a mean pressure value:

wall pressure force $= [(mean\ pressure)(wall\ area)]\ sin\ \theta$

but $dA = (wall\ area)\ sin\ \theta$; and thus

$$\text{wall pressure force } = \left(p + \frac{dp}{2}\right) dA \qquad (4.8)$$

$$\sum \mathbf{F} = pA + \left(p + \frac{dp}{2}\right) dA - (p + dp)(A + dA) \qquad (4.9)$$

$$pA + \left(p + \frac{dp}{2}\right) dA - (p + dp)(A + dA) = \rho VA[(V + dV) - V] \qquad (4.10)$$

Simplifying and ignoring high orders

$$\boldsymbol{dp + \rho V dV = 0} \qquad (4.11)$$

- **Energy equation**

$$\iint_{cs} e\ \rho\ (\boldsymbol{V}.\hat{n})\ dA = 0 \qquad (4.12)$$

$$\iint_{cs} [\delta q - \delta w_s + d(u + pv + k.e. + p.e)]\ \rho\ (\boldsymbol{V}.\hat{n})\ dA = 0 \qquad (4.13)$$

The specific energy e is stand for internal, flow, kinetic and potential energies, since there is no heat and work transfer. Then from S.F.E.E.;

$$\delta q + \left(pv + u + \frac{V^2}{2} + gz\right) = \delta w_s + \left((p + dp)(v + dv) + (u + du) + \frac{(V + dV)^2}{2} + g(z + dz)\right)$$

$$0 = \left(pdv + vdp + du + \frac{2VdV}{2}\right) \qquad (4.14)$$

$$0 = dh + \frac{dV^2}{2} \qquad (4.15)$$

Substitute from thermodynamics relations

$$\delta q = dW_s + du = pdv + du = dh - vdp = 0$$

$$dh = vdp$$

$$dp + \rho V dV = 0 \qquad (4.16)$$

This is the energy equation which is similar to equation (4.11).

4.2 Stagnation concept and relations

If you had a thermometer and pressure gage, they would indicate the temperature and pressure corresponding to the *static* state of the fluid, as you move with flow velocity. Thus *the static properties are those that would be measured if you moved with the fluid*.

Stagnation state defined as that thermodynamic state which would exist if the fluid were brought to zero velocity and zero potential. To yield a consistent reference state, we must qualify how this *stagnation process* should be accomplished. The stagnation state must be reached

1. Without any energy exchange ($Q = W = 0$)
2. Without friction losses.

From (1), change of entropy due to energy exchange is zero, i.e. $ds_{ext} = 0$; and from (2), change of entropy due to friction is zero, i.e. $ds_{int} = 0$. Thus *the stagnation process is isentropic*!

Consider fluid that is flowing and has the static properties shown as (a) in Figure 4.3. At location (b) the fluid has been brought to zero velocity and zero potential under the foregoing restrictions. If we apply the energy equation to the control volume indicated for steady one-dimensional flow, we have.

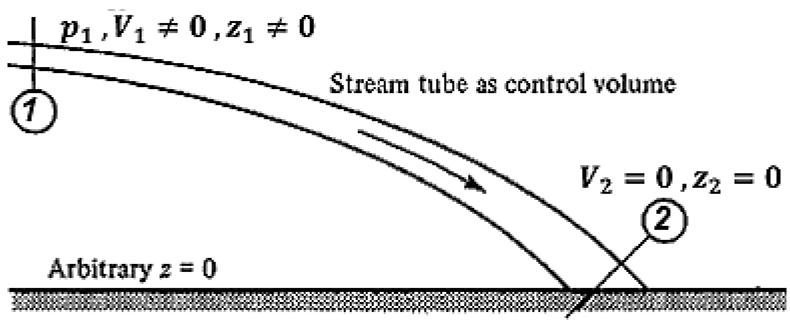


Figure 3.1 Stagnation Process

$$q + \left(h_1 + \frac{V_1^2}{2} + gz_1\right) = w_s + \left(h_2 + \frac{V_2^2}{2} + gz_2\right)$$

$$h_1 + \frac{V_1^2}{2} + gz_1 = h_2 \qquad (4.17)$$

Since condition (2) represents the *stagnation state* corresponding to the *static state* (1). Thus we call h_2 the *stagnation* or *total enthalpy* corresponding to state (1) and designate it as h_{to}. Thus

$$h_{to} = h_1 + \frac{V_1^2}{2} + gz_1$$

Or for any state, we have in general,

$$h_o = h + \frac{V^2}{2} + gz \qquad (4.18)$$

This is an important relation that is *always* valid. When dealing with gases, potential energy changes are usually neglected, and we write.

$$h_o = h + \frac{V^2}{2} \qquad (4.19)$$

The one-dimension S.F.E.E. becomes:

$$h_{o1} + q = h_{o2} + w_s \qquad (4.20a)$$

$$h_{o1} = h_{o2} \quad or \quad dh_o = 0 \qquad (4.20b)$$

Equation (4.20) shows that for any adiabatic, no-work, steady, one-dimensional flow system, the stagnation enthalpy remains constant, *irrespective of the losses*.

One must realize that when the frame of reference is changed, stagnation conditions change, although the static conditions remain the same. Consider still air with Earth as a reference frame. In this case, since the velocity is zero the static and stagnation conditions are the same. For gases we eliminate potential term

$$c_p = \frac{\gamma R}{\gamma - 1}, \qquad h = c_p T$$

$$h_o = h + \frac{V^2}{2} = h + \frac{M^2 \gamma R T}{2} = h + M^2 \frac{\gamma - 1}{2} c_p T$$

$$h_o = h \left(1 + M^2 \frac{\gamma - 1}{2}\right) \qquad (4.21)$$

$$T_o = T \left(1 + M^2 \frac{\gamma - 1}{2}\right) \qquad (4.22)$$

The stagnation process is isentropic. Thus γ is used as the exponent in the relations between any two points on the same isentropic streamline. Let point 1 refers to the static conditions, and point 2, the stagnation conditions. Then,

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\gamma/(\gamma-1)}$$

$$p_o = p\left(1 + M^2\frac{\gamma-1}{2}\right)^{\gamma/(\gamma-1)} \quad (4.23)$$

$$\rho_o = \rho\left(1 + M^2\frac{\gamma-1}{2}\right)^{1/(\gamma-1)} \quad (4.24)$$

Example 4.1 Air flows with a velocity of $243.84\ m/s$ and has a pressure of $206.843\ kN/m^2$ and temperature of $60.2\ °C$. Determine the stagnation pressure.

Solution

$$a = \sqrt{\gamma RT} = \sqrt{1.4 * 287 * (60.2 + 273)} = 365.9\ m/s$$

$$M = \frac{V}{a} = \frac{243.84}{365.9} = 0.666$$

$$p_o = p\left(1 + M^2\frac{\gamma-1}{2}\right)^{\gamma/(\gamma-1)} = 206.843\left(1 + 0.666^2\frac{1.4-1}{2}\right)^{(1.4/1.4-1)}$$

$$= 278.506\ kN/m^2$$

Example 4.2 Hydrogen, $\gamma_{Hy} = 1.405$, has a static temperature of $25°C$ and a stagnation temperature of $250°C$. What is the Mach number?

Solution

$$T_o = T\left(1 + M^2\frac{\gamma-1}{2}\right)$$

$$(250 + 273) = (25 + 273)\left(1 + M^2\frac{1.405-1}{2}\right)$$

$$523 = 293\,(1 + 0.2025\,M^2) \rightarrow M^2 = 3.8765 \rightarrow M = 1.969$$

Chapter Five/Subsonic and Supersonic Flow through a Varying Area Channels

5.1 Isentropic Flow in varying Area ducts

For isentropic flow, from continuity

$$\frac{d\rho}{\rho}+\frac{dA}{A}+\frac{dV}{V}=0 \qquad (4.5)$$

and from momentum equations

$$dp+\rho V dV=0 \qquad (4.11)$$

$$dV=-\frac{dp}{\rho V}$$

Substitute into momentum eq.

$$\frac{d\rho}{\rho}+\frac{dA}{A}-\frac{dp}{\rho V^2}=0 \qquad (5.1a)$$

$$dp-\rho V^2\left(\frac{d\rho}{\rho}+\frac{dA}{A}\right)=0 \qquad (5.1b)$$

From definition of sonic velocity, eq.3.4

$$a^2=\left(\frac{\partial p}{\partial \rho}\right)_{ise}=\left(\frac{dp}{d\rho}\right)_{ise}\Rightarrow d\rho=\frac{dp}{a^2}$$

$$dp-\rho V^2\left(\frac{dp}{\rho a^2}+\frac{dA}{A}\right)=0$$

$$dp-M^2 dp=\rho V^2\frac{dA}{A}$$

$$dp=\rho V^2\left(\frac{1}{(1-M^2)}\right)\frac{dA}{A} \qquad (5.2a)$$

$$p=\rho RT=\frac{\rho}{\gamma}a^2$$

$$\frac{dp}{p} = \left(\frac{\gamma M^2}{(1 - M^2)}\right)\frac{dA}{A} \qquad (5.2b)$$

Also from eq. 5.1. after substitute for $dp = a^2 dp$ from definition of sonic velocity

$$\frac{d\rho}{\rho} + \frac{dA}{A} - \frac{dp}{\rho V^2} = 0$$

$$\frac{d\rho}{\rho} + \frac{dA}{A} - \frac{1}{M^2}\frac{d\rho}{\rho}$$

$$\frac{d\rho}{\rho} = \frac{M^2}{(1 - M^2)}\left(\frac{dA}{A}\right) \qquad (5.3)$$

Substitute eq.5.3 into continuity eq.4.5. gives

$$\frac{M^2}{(1 - M^2)}\frac{dA}{A} + \frac{dA}{A} + \frac{dV}{V} = 0$$

$$\frac{dV}{V} = -\left(\frac{1}{1 - M^2}\right)\left(\frac{dA}{A}\right) \qquad (5.4)$$

Let us consider what is happening to fluid properties as it flows through a variable-area duct.
For subsonic flow, $M < 1$, then $(1 - M^2)\ is + ve$.

When dA is negative (area is decreasing), then dp is negative (pressure decreases) and $d\rho$ is negative (density decreases) and dV is positive (velocity increases) and vice versa.

For supersonic flow, $M > 1$, then $(1 - M^2)\ is - ve$.

When dA is negative (area is decreasing), then dp is positive (pressure increases) and $d\rho$ is positive (density increases) and dV is negative (velocity decreases) and vice versa.

We summarize the above by saying that *as the pressure decreases*, the following variations occur:

		Subsonic ($M < 1$)	Supersonic ($M > 1$)
Area	A	Decreases	Increases
Density	ρ	Decreases	Decreases
Velocity	V	Increases	Increases

Table 5.1: Variation of area, density and velocity with Much number as the pressure decreases

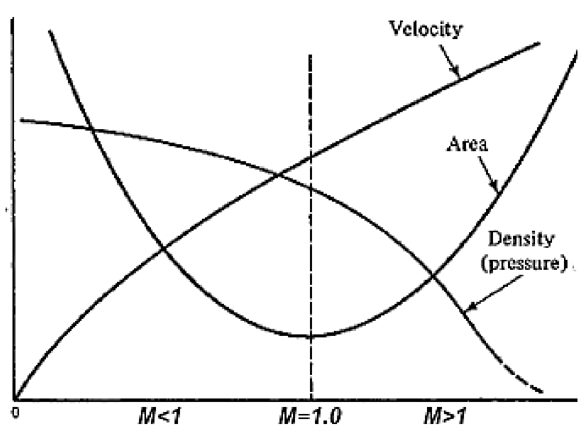


Figure 5.1: Property variation with Mach number

Combines equations (5.4) and (5.3) to eliminate the term dA/A with the following result:

$$\frac{d\rho}{\rho} = -M^2 \left(\frac{dV}{V}\right) \tag{5.5}$$

From this equation we see that:

At **low Mach** numbers, density variations will be quite small. This means that the density is nearly constant ($d\rho = 0$) in the low subsonic regime ($M \leq 0.3$) and the velocity changes compensate for area changes.

At a **Mach** number equal to **unity**, we reach a situation where density changes and velocity changes compensate for one another and thus no change in area is required ($dA = 0$).

At **supersonic** flow, the density decreases so rapidly that the accompanying velocity change cannot accommodate the flow and thus the area must increase.

A ***nozzle*** is a device that converts enthalpy (or pressure energy for the case of an incompressible fluid) into kinetic energy. From Figure 5.1 we see that an increase in velocity is accompanied by either an increase or decrease in area, depending on the Mach number. Figure 5.2 shows what these devices look like in the subsonic and supersonic flow regimes.

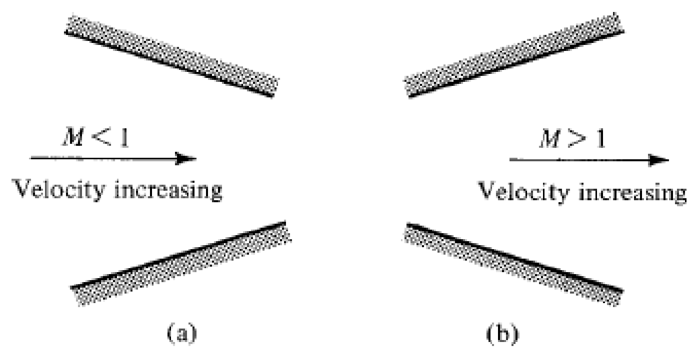


Figure 5.2 Nozzle configurations.

A ***diffuser*** is a device that converts kinetic energy into enthalpy (or pressure energy for the case of incompressible fluids). Figure 5.3 shows what these devices look like in the subsonic and supersonic regimes. Thus we see that the same piece of equipment can operate as either a nozzle or a diffuser, depending on the flow regime.

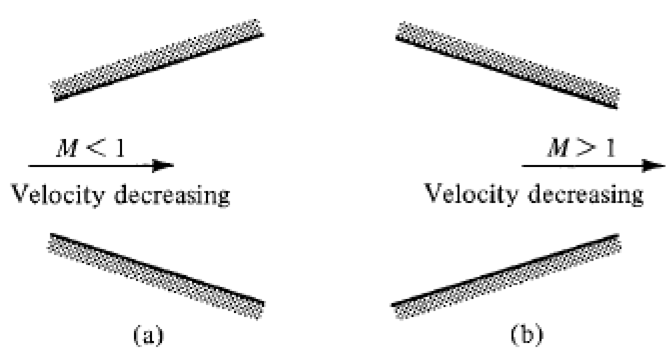


Figure 5.3 Diffuser configurations.

Notice that a device is called a nozzle or a diffuser because of *what it does*, not what it looks like.

Further consideration of Figures 5.1 and 5.2 leads to some interesting conclusions. If one attached a converging section (see Figure 5.2*a*) to a high-pressure supply, one could never attain a flow greater than Mach 1, regardless of the pressure difference available. On the other hand, if we made a converging–diverging device (combination of Figure 5.2*a* and *b*), we see a means of accelerating the fluid into the supersonic regime, provided that the proper pressure difference exists between inlet and exit plane.

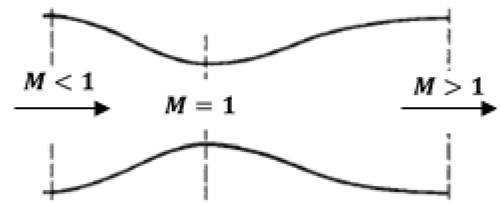


Figure 5.4: Convergent-divergent nozzle

5.2 The (∗) Reference Concept

Concept of a stagnation reference state was introduced which is an *isentropic process*. It will be convenient to introduce another reference condition since the stagnation state is not a feasible reference when dealing with area changes. (Why?)

The new reference state with a superscript (∗) and define it as "that thermodynamic state which would exist if the fluid reached a Mach number of unity *by some particular process*". There are many processes by which we could reach Mach 1.0 from any given starting point, and they would each lead to a different thermodynamic state.

For isentropic flow process, adiabatic frictionless, flow the stagnation properties for all points are the same as well as the (*) properties are the same.

For actual flow process, each point in the flow has its own stagnation and (*) properties.

Consider a steady, one-dimensional flow of a perfect gas with no heat or work transfer and negligible potential changes but with friction. Figure 5.5 shows a T –s diagram indicating two points in such a flow system. Above each point is shown its stagnation reference state, and below its reference state (∗).

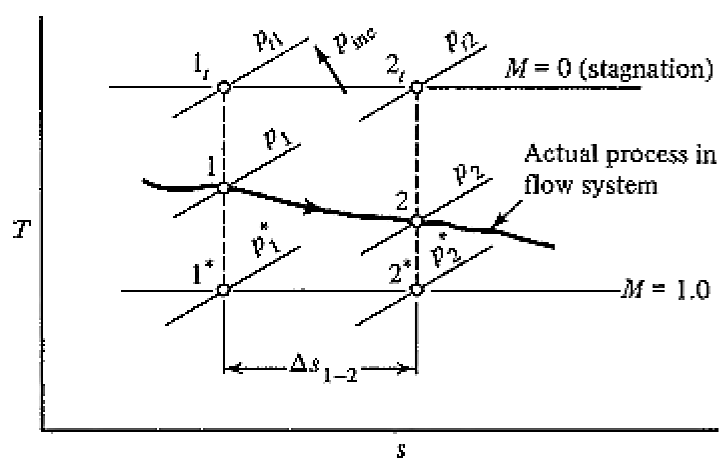


Figure 5.5 Isentropic * reference states.

Note that the stagnation temperatures are the same and lie on a horizontal line, but the stagnation pressures are different, and also (∗) reference points will lie on another horizontal line (since no heat is added).

Between (∗) reference state and the stagnation reference state lie all points in the subsonic regime. Below the (∗) reference state lie all points in the supersonic regime.

5.3 Isentropic Table

Mass flow rate at flow cross sectional area A can be expressed in terms of stagnation pressure and temperature

$\dot{m} = \rho A\, V = const$ *continuty equation*

$p = \rho RT$ *state equation*

$a = \sqrt{\gamma RT}$ *sonic speed*

$M = V/a$ *Much number*

For perfect gas with constant specific heat

$$\dot{m} = \frac{p}{RT} AM\sqrt{\gamma RT} = \frac{p}{R\sqrt{T}} AM\sqrt{\gamma R} \qquad (5.6)$$

Substitute for p and T from

$$T_o = T\left(1 + M^2\frac{\gamma - 1}{2}\right) \qquad (4.26)$$

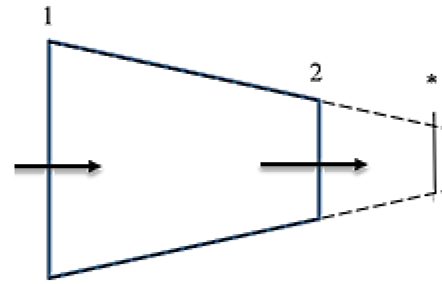
$$p_o = p\left(1 + M^2\frac{\gamma - 1}{2}\right)^{\gamma/(\gamma-1)} \qquad (4.28)$$

$$\dot{m} = \frac{p_o}{R\sqrt{T_o}} AM\sqrt{\gamma R}\left(1 + M^2\frac{\gamma - 1}{2}\right)^{-(\gamma+1)/2(\gamma-1)} \qquad (5.7)$$

$$\dot{m} = \frac{p_o A}{R\sqrt{T_o}} f(\gamma, M) \qquad (5.8)$$

$$f(\gamma, M) = \frac{M\sqrt{\gamma}}{\left(1 + M^2\frac{\gamma - 1}{2}\right)^{(\gamma+1)/2(\gamma-1)}} \qquad (5.9)$$

For isentropic flow where p_o and T_o are constant, cross section A can be related directly to Mach number. Select flow cross section area where $M = 1$ as a



reference area A^* . For steady flow, the mass flow rate at area A is equal to the mass flow rate at area A^*.

$\dot{m} = \dot{m}^*$

$$\frac{p_o A}{R\sqrt{T_o}} f(\gamma, M) = \frac{p_o A^*}{R\sqrt{T_o}} f(\gamma) \qquad (5.10)$$

$$\frac{A}{A^*} = g(\gamma, M)$$

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} \qquad (5.11)$$

The result of equation (5.11) is plotted in figure (5.6) for $\gamma = 1.4$. For each value of A/A^* there are two possible isentropic solution, one subsonic and the other supersonic. The minimum area or throat area occurs at $M = 1$. This agree well with the result of eq 5.6 that illustrated in figure 5.2. and 5.3.

A convergent-divergent nozzle is required to accelerate a slowly moving stream to supersonic velocities.

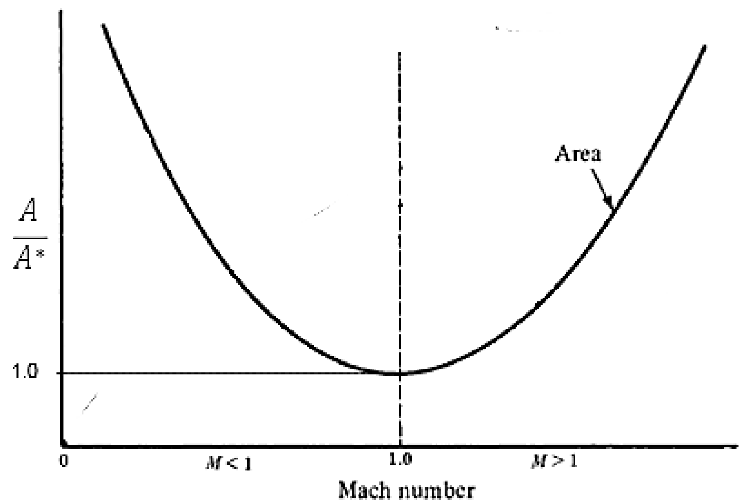


Figure 5.6 Area ratio variation with Mach number

Example: 5.1

An airstream flows in a converging duct from cross section area A_1 of $50\ cm^2$ to a cross-sectional area A_2 of $40\ cm^2$. If $T_1 = 300\ K$, $p_1 = 100\ kPa$ and $V_1 = 100\ m/s$. Find M_2, p_2 and T_2 . Assume steady one-dimensional isentropic flow.

Solution:

Over the temperature range, air behaves as perfect gas with $\gamma = 1.4$.

$$M_1 = \frac{V_1}{a} = \frac{V_1}{\sqrt{\gamma RT}} = \frac{100}{\sqrt{1.4 * 0.287 * 300}} = 0.288$$

At $M_1 = 0.288$ from isentropic flow table with $\gamma = 1.4$

$$\frac{A_1}{A^*} = 2.11$$
But
$$\frac{A_2}{A_1} = \frac{40}{50} = 0.80$$
So that
$$\frac{A_2}{A^*} = \frac{A_1}{A^*} * \frac{A_2}{A_1} = 1.689$$
From isentropic flow table , $M_2 = 0.372$

For isentropic flow, (no shaft work, potential energy is neglected for a gas), p_t and T_t are constant. At $M = 0.288$ from isentropic flow table :
$$\frac{p_1}{p_{o1}} = 0.944 \rightarrow p_{t1} = \frac{100}{0.944} = 105.9\ kPa = p_{t1}$$
$$\frac{T_1}{T_{o1}} = 0.984 \rightarrow T_{t1} = \frac{300}{0.984} = 304.9\ K$$
At $M_2 = 0.372$
$$\frac{p_2}{p_{o1}} = 0.909 \rightarrow p_2 = 0.909 * 105.9 = 96.3\ \ kPa$$
$$\frac{T_2}{T_{o1}} = 0.973 \rightarrow T_2 = 0.973 * 304.9 = 296.7\ K$$

Chapter Six/Isentropic Flow in Converging Nozzles

6.1 performance of Converging Nozzle

Two types of nozzles are considered: a converging-only nozzle and a converging–diverging nozzle. A assume a fluid stored in a large reservoir, at 6 bar and 60 ℃ , is to be discharge through a converging nozzle into an extremely large receiver where the back pressure can be regulated. We can neglect frictional effects, as they are very small in a converging section.

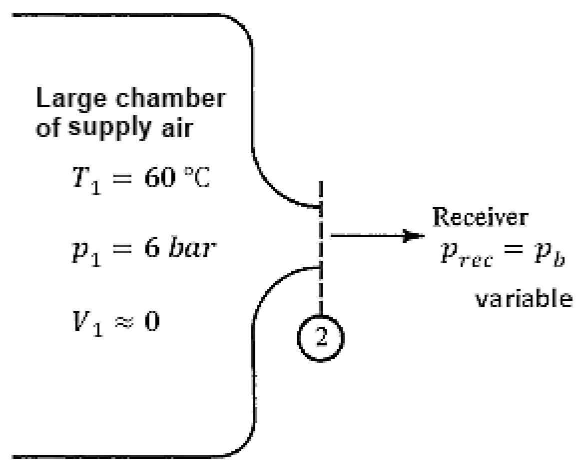


Figure 6.1: Converging-only nozzle.

If the receiver (back) pressure is set at 6 bar, no flow results. Once the receiver pressure is lowered below 6 bar, air will flow from the supply tank. Since the supply tank has a large cross section relative to the nozzle outlet area, the velocities in the tank may be neglected. Thus $T_1 \approx T_{o1}$ and $p_1 \approx p_{o1}$ (stagnation properties). There is no shaft work and we assume no heat transfer and no friction losses, i.e. the flow is isentropic.

We identify section 2 as the nozzle outlet. Then from energy equation

$$h_{o1} + \delta q = h_{o2} + \delta w_s$$

$$h_{o1} = h_{o2} \ \rightarrow \ c_p T_{o1} = c_p T_{o2}$$

And for perfect gas where specific heats are assumed constant

$$T_{o1} = T_{o2}$$

It is important to recognize that the receiver pressure is controlling the flow. The velocity will increase and the pressure will decrease as we progress through the nozzle until the pressure at the nozzle outlet equals that of the receiver. This will always be true *as long as* the nozzle outlet can "sense" the receiver pressure.

Example: Let us assume

For receiver $p_b = 4.812\ bar$

$p_2 = p_b = 4.812\ bar$

For reservoir $p_{o1} = p_1 = 6.0\ bar \quad and\ T_o = T_1 = 60\ ℃$

$p_{o2} = p_{o1} = 6.0\ bar\ \ and\ \ T_{o2} = T_{o1} = 60\ ℃$ for isentropic flow

$$\frac{p_2}{p_{o2}} = \frac{4.812}{6.0} = 0.802$$

From isentropic table corresponding to $p/p_o = 0.802$

$M_2 = 0.57$ and $T/T_o = 0.939$

$\therefore T_2 = 0.939 * (273 + 60) = 312.687\ K$

$a_2 = \sqrt{\gamma RT} = \sqrt{1.4 * 287 * 312.687} = 354.5\ m/s$

$V_2 = M_2 * a_2 = 0.57 * 354.5 = 202\ m/s$

Figure 6.2 shows this process on a *T* –*s* diagram as an isentropic expansion. If the pressure in the receiver were lowered further, the air would expand to this lower pressure and the Mach number and velocity would increase. Assume that the receiver pressure is lowered to $3.1692\ bar$. *Show* that

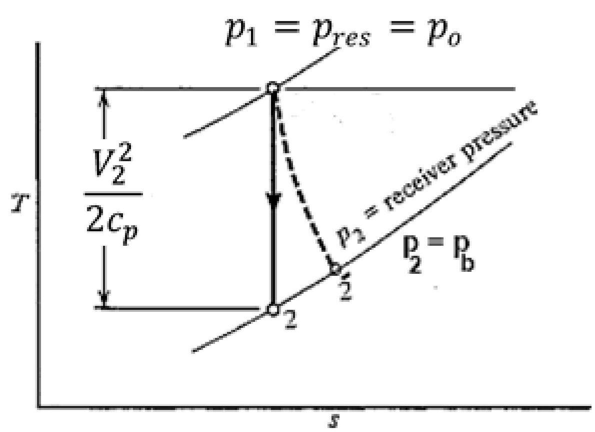


Figure 6.2 *T–s* diagram for converging-only nozzle

$$\frac{p_2}{p_{o2}} = \frac{3.16968}{6.0} = 0.52828$$

This gives:

$M_2 = 1.0\ \ and\ \ T/T_o = 0.8333$

$T_2 = 0.8333 * (273 + 60) = 277.4889\ K$

$a_2 = \sqrt{\gamma RT} = \sqrt{1.4 * 287 * 277.4889} = 333.91\ m/s$

$V_2 = M_2 * a_2 = 1.0 * 333.91 = 333.91\ m/s$

$T^* = T_2 = 277.4889\ K$ and $p^* = p_2 = 3.1692\ bar$ are *critical* properties

Notice that the air velocity coming out of the nozzle is exactly sonic. The velocity of signal waves is equal to the velocity of sound relative to the fluid into which the wave is propagating. If the fluid at cross section is moving at sonic velocity, the absolute velocity of signal wave at this section is zero and it cannot travel past this cross section.

If we now drop the receiver pressure below this *critical pressure* ($3.1692\ bar$), see figure (6.3), the nozzle has no way of adjusting to these conditions. That's

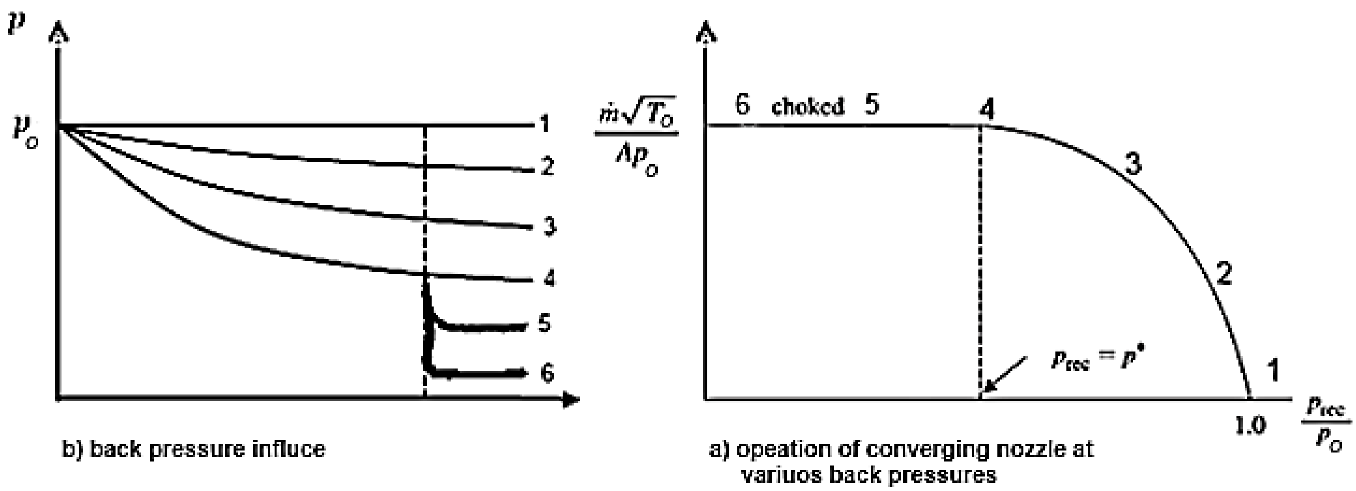


Figure 6.3

because fluid velocity will become supersonic and signal waves (sonic velocity) are unable to propagate from the back pressure region to the reservoir.

Assume that the nozzle outlet pressure could continue to drop along with the receiver. This would mean that $p_2 / p_{o2} < 0.5283$, which corresponds to a supersonic velocity (point 4).We know that if the flow is to go supersonic, the area must reach a minimum and then increase. Thus for a converging-*only* nozzle, the flow is governed by the receiver pressure until sonic velocity is reached at the nozzle outlet and *further reduction of the receiver pressure will have no effect on the flow conditions inside the nozzle*. Under these conditions, the nozzle is said to be ***choked*** and the nozzle outlet pressure remains at the *critical pressure*. Expansion to the receiver pressure takes place *outside* the nozzle (points 5 and 6).

The analysis above assumes that conditions within the supply tank remain constant. One should realize that the choked flow rate can change if, for example, the supply pressure or temperature is changed or the size of the throat (exit hole) is changed.

The pressure ratio below which the nozzle is chocked can be calculated for isentropic flow through the nozzle. For perfect gas with constant specific heats,

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

$$\frac{p_r}{p_b} = \left(1 + \frac{\gamma - 1}{2} (1)^2\right)^{\gamma/(\gamma-1)} = 0.5283 \qquad at\ \gamma = 1.4$$

Example 6.1 Air is allowed to flow from a large reservoir through a convergent nozzle with an exit area of $50\,cm^2$. The reservoir is large enough so that negligible changes in reservoir pressure and temperature occur as fluid is exhausted through the nozzle. Assume isentropic, steady flow in the nozzle, with $p_{res} = 500\ kPa$ and $T_{res} = 500\ K$. Assume also that air behaves as a perfect gas with constant specific heats, $\gamma = 1.4$. Determine the mass flow through the nozzle for back pressures $125, 250$, and $375\ kPa$.

At $M_e = 1$ and $\gamma = 1.4$ the critical pressure ratio is 0.5283; therefore for all back pressures below;

$$p_{exit} = p_r * \frac{p}{p_o} = 500 * 0.5283 = 264.15\ kPa$$

The nozzle is choked. Under these conditions, the Mach number at the exit plane is unit and the pressure at exit plane is $264.15\ kPa$ and the temperature at exit plane

$$T_{exit} = T_o * \frac{T}{T_t} = 500 * 0.8333 = 416.7\ K$$

The nozzle is chocked for back pressures of $0, 125\ and\ 250\ kPa$ and the mass flow rate is;

$$\dot{m} = \rho A\ V = \frac{p_e}{RT_e} AM_e\sqrt{\gamma RT_e} = \frac{264.15*50\times10^{-4}*1}{0.287*416.7}\sqrt{1.4*0.287*416.7}$$

$$= 4.519\ kg/s$$

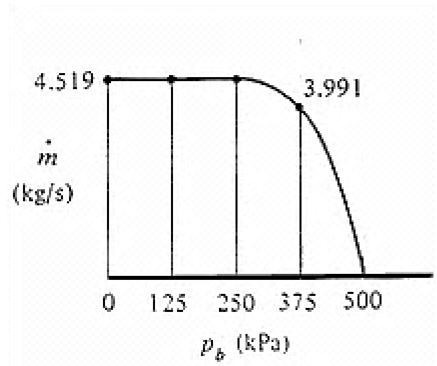


Figure 6.4

For back pressures of $370\ kPa$ the nozzle is not choked and the exit plane pressure equals to back pressure;

$$\frac{p}{p_o} = \frac{375}{500} = 0.75$$

From isentropic table at , $\gamma = 1.4$, $M_e = 0.654$, and $T/T_o = 0.921$

$$T_e = T_o * T/T_o = 500 * 0.921 = 460.5\ K$$

$$\dot{m} = \frac{375 * 50 \times 10^{-4} * 1}{0.287 * 460.57}\sqrt{1.4 * 287 * 460.5} * 0.654$$

$$= 3.991\ kg/s$$

Example 6.2 Nitrogen is stored in a tank $2\ m^3$ in volume at a pressure of $3\ MPa$ and a temperature of $300\ K$. The gas is discharge through a converging nozzle with an exit area of $12\ m^2$. For back pressure of $101\ kPa$, find the time for the tank pressure to drop to $300\ kPa$. Assume isentropic nozzle flow with nitrogen behaves as a perfect gas with $\gamma = 1.4$ and $R = 0.2968\ kJ/kg\,.K$. Assume quasi-steady flow through the nozzle with the steady flow equation applicable at each instant of time assume also that T is constant too

---

Solution; As the reservoir pressure drops from $3\,MPa$ to $300\,kPa$, the ratio $p_b/p_o = 101/3000 = 0.03367$ and $p_b/p_o = 101/300 = 0.3367$ remains below critical pressure ratio (0.5263) and $M_{exit} = 1$.

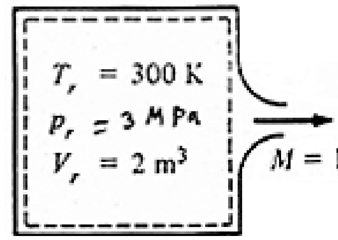


Figure 6.5

$$T_e = T_o * T/T_o = 300 * 0.8333 = 250\,K$$

$$\dot{m} = \rho A\,V = \frac{p_e}{RT_e} A M_e \sqrt{\gamma R T_e}$$

$$\dot{m} = \frac{(0.5283 p_{res}) * 12 \times 10^{-4} * 1}{296.8 * 250}\sqrt{1.4 * 296.8 * 250}$$

$$= 2.754 p_o \times 10^{-6}\,kg/s = \quad where\ p_o\ is\ in\ Pascals$$

From conservation of mass

$$\frac{\partial}{\partial t}\iiint_{cv} \rho\, d\Upsilon + \iint_{cs} \rho\,(\boldsymbol{V}.\hat{n})\,dA = 0$$

The mass inside the tank at any time is m;

$$\iiint_{cv} \rho\, d\Upsilon = \frac{p_{res}\Upsilon_{res}}{RT_{res}} \quad and \quad \iint_{cs} \rho\,(\boldsymbol{V}.\hat{n})\,dA = 2.754\,p_{res} \times 10^{-6}\,kg/s$$

The mass coming out of tank exit at any time

$$\frac{\partial}{\partial t}\left(\frac{p_{res}\Upsilon_{res}}{RT_{res}}\right) + 2.754\,p_{res} \times 10^{-6} = 0$$

$$\frac{\Upsilon_{res}}{RT_{res}}\frac{dp_{res}}{dt} + 2.754\,p_{res} \times 10^{-6} = 0$$

$$\int dt = -\frac{1}{2.754 \times 10^{-6}} * \frac{\Upsilon_{res}}{RT_{res}}\int \frac{dp_{res}}{p_{res}}$$

$$\Delta t = -\frac{2}{0.2968 * 300 * 2.754 \times 10^{-3}}\int_{3000}^{300}\frac{dp_{res}}{p_{res}} \qquad p_{res}\ is\ in\ kN/m^2$$

$$\Delta t = 8.156 \ln 10 = 18.78\,seconds$$

Chapter Seven/Isentropic Flow in Converging–Diverging Nozzles

7.1 Converging–Diverging Nozzle

Let us examine the converging–diverging nozzle (sometimes called a (*DE Laval nozzle*), shown in Figures (7.1).We identify the *throat* (or section of minimum area) as 2 and the exit section as 3. The distinguishing physical characteristic of this type of nozzle is the *area ratio*, meaning the ratio of the exit area to the throat area.

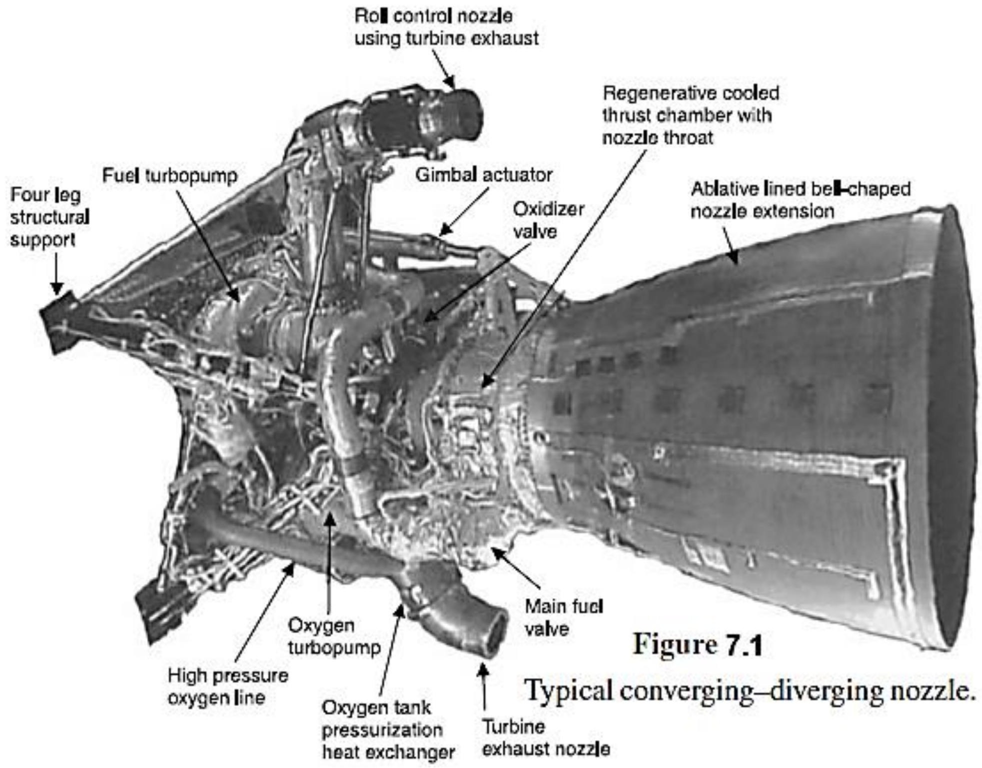


Figure 7.1 Typical converging–diverging nozzle.

Fluid stored in a large reservoir is to be discharge through a converging-diverging nozzle. It is desired to determine mass flow and pressure distribution in the nozzle over a range of values of p_b/p_r .the reservoir pressure is maintain constant, with one-dimensional isentropic flow in the nozzle.

Figure 7.2 shows the pressure distribution in the nozzle for different values of back pressure p_b.

For p_b equal to p_r (curve 1) there is no flow in the nozzle, and pressure is constant with x (nozzle length).

For p_b slightly less than p_r (curve 2), flow induced through the nozzle with

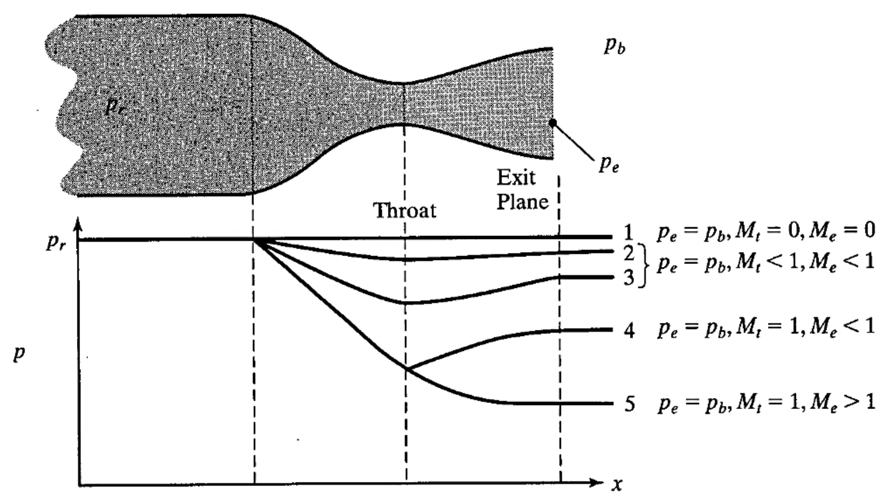


Figure 7.2 Pressure Distributions for Isentropic Flow in a C–D Nozzle

subsonic velocities in both converging and diverging sections of the nozzle. Eq. (5.4), $dp = \rho V^2[1/(1-M^2)]\,dA/A$, tells us that for subsonic flow pressure decreases in the converging section and increases in the diverging section.

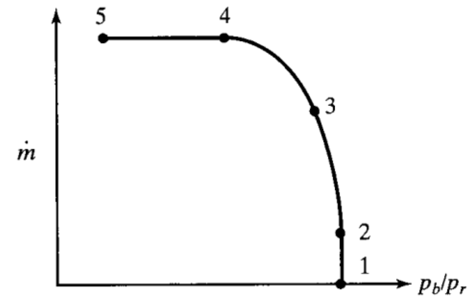


Figure 7.3 Mass-Flow Rate versus Pressure Ratio for Isentropic Flow in a C–D Nozzle

As the back pressure is decreased more and more flow is induced in the nozzle (curve 3) until eventually sonic flow occurs in the throat (curve 4). And the pressure ratio is called the first critical point. Nozzle is choked and mass flow rate becomes a maximum.

With receiver (back) pressures above the first critical, the nozzle operates as a venturi and we never reach sonic velocity in the throat. An example of this mode of operation is shown as curve "3" in Figure 7.2b. The nozzle is no longer choked and the flow rate is less than the maximum.

Further decrease in back pressure cannot be sensed upstream of the throat ; so for all back pressures below that of curve 4 the reservoir continues to send out the same flow rate as curve 4, and the pressure distribution nozzle up to the throat remains the same. For all back pressures below that of curve 4 the converging-diverging nozzle is choked. Note that for the same reservoir pressure, a converging-diverging nozzle is choked at a greater back pressure than a converging nozzle.

There are two possible isentropic solutions for a given area ratio A/A*, one subsonic and the other supersonic. For a throat Mach number of 1, isentropic flow can either decelerate to a subsonic exit velocity or continue to accelerate to a supersonic exit velocity. Curve 4 corresponds to the case of subsonic flow at the nozzle exit plane; curve 5 corresponds to supersonic flow at the exit plane. Thus, if the back pressure is lowered to that of curve 5, pressure decreases in both converging and diverging portions of the nozzle, with supersonic flow at the exit plane. And the pressure ratio is called the third critical point.

For back pressures between those of curves 4 and 5 i.e. between the first and third critical points, the flow is not isentropic and one-dimensional isentropic solutions to the equations of motion are not possible. These flows involve shock waves, which are irreversible processes, which are compression waves that will occur in either the diverging portion of the nozzle or after the exit

If the receiver (back) pressure is below the third critical point (curve 5) , the nozzle operates *internally* as though it were at the design condition but expansion waves occur *outside* the nozzle. These operating modes will be discussed in detail later.

Figure (7.3) shows the variation of mass flow rate with back pressure p_b/p_r for data of figure (7.2).

The objective of making a converging–diverging nozzle is to obtain supersonic flow. Let us first examine the *design operating condition* for this nozzle. For the nozzle is to operate as desired, the flow will be subsonic from 1 to 2, sonic at 2, and supersonic from 2 to 3. To discover the conditions that exist at the exit (under design operation), we seek the ratio A_3/A_3^*:

Since velocity is sonic at throat ($M_2 = 1$), then $A_2^* = A_2$ and from eq. (5.11) the relation between any two sections for isentropic flow

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} \qquad (5.11)$$

Then

$$\frac{A_2^*}{A_3^*} = \frac{1}{1}\left(\frac{(\gamma + 1)/2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = 1 \qquad (7.1)$$

So

$$A_3^* = A_2^* = A_2 \qquad (7.2)$$

$$\frac{A_3}{A_3^*} = \frac{A_3}{A_2} * \frac{A_2}{A_2^*} * \frac{A_2^*}{A_3^*} = \frac{A_3}{A_2}$$

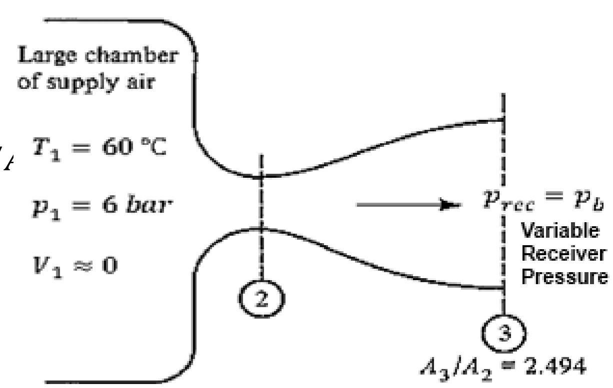


Figure 7.4: Converging–diverging nozzle.

Example 7.1 A converging–diverging nozzle with A_3/A
temperature of 6 bar and 60 ℃. Find back pressure.

Solution

1. From isentropic table at $A_3/A^* = 2.494$ in the *supersonic* section of the isentropic table and see that

$M_3 = 2.44$

$p_3/p_o = 0.0643$

$T_3/T_o = 0.4565$, Thus

$$p_3 = \frac{p_3}{p_o} * p_o = 0.0643 * 6.0 = 0.3858\ bar$$

And to operate the nozzle at this *design condition* the receiver pressure *must be* at $0.3858\ bar$. The pressure variation through the nozzle for this case is shown as curve "5" in Figure 7.3. From the temperature ratio T/T_o we can easily compute $T_3, a_3\ and\ V_3$.

2. Also we can find $A/A^* = 2.494$ in the subsonic section of the isentropic table. (Recall that these two answers come from the solution of a quadratic equation.) For this case

$M_3 = 0.24$

$p_3/p_o = 0.9607$

$T_3/T_o = 0.9886$, Thus

$$p_3 = \frac{p_3}{p_o} * p_o = 0.9607 * 6.0 = 5.7642\ bar$$

And to operate at this condition the receiver pressure *must be* at $5.7642\ bar$. With this receiver pressure the flow is subsonic from 1 to 2, sonic at 2, and *subsonic* again from 2 to 3. The converging-diverging is nowhere near its design condition and is really operating as a *venturi tube*; that is, the converging section is operating as a nozzle and the diverging section is operating as a diffuser. The pressure variation through the nozzle for this case is shown as curve "4" in Figure (7.2)

8.2. Nozzle performance

The most important parameters in nozzle performance are area ratio A_e/A_{th} and Mach number M. The area ratio for an isentropic nozzle can be expressed in terms of Mach numbers for any points x and y within the nozzle along its axis. Since $\rho VA = C$; then

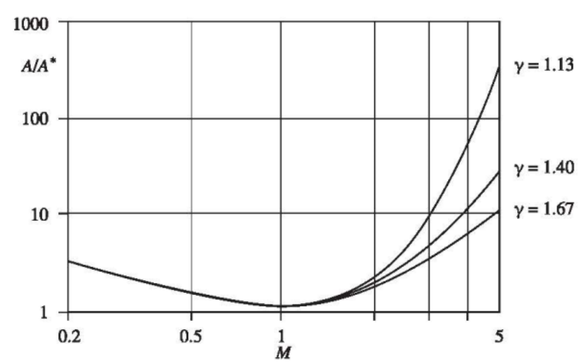


Figure 7.5: A/A* area ratio versus Mach number for various value of γ

$$\frac{A_y}{A_x} = \frac{\rho_x V_x}{\rho_y V_y} = \frac{p_x M_x\sqrt{\gamma R T_x}}{RT_x}.\frac{RT_y}{p_y M_y\sqrt{\gamma R T_y}} = \frac{p_x M_x}{\sqrt{T_x}}.\frac{\sqrt{T_y}}{p_y M_y}$$

$$\frac{A_y}{A_x} = \frac{M_x}{M_y}\sqrt{\left\{\frac{1+[(\gamma-1)/2]M_y^2}{1+[(\gamma-1)/2]M_x^2}\right\}^{(\gamma+1)/(\gamma-1)}} \qquad (7.4)$$

$$\frac{A}{A^*} = \frac{A}{A_{th}} = \frac{1}{M}\sqrt{\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{(\gamma+1)/(\gamma-1)}} \qquad (5.11)$$

Relation of eq. (5.11) is plotted in Figure (7.5).

From Equation (4.16) the *nozzle exit velocity* V_2 can be found. From s.f.e.e without heat and work exchanging and ignoring potential energy, we have:

$$0 = dh + \frac{dV^2}{2} \qquad (4.16)$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$V_2 = \sqrt{2(h_1 - h_2) + V_1^2} \qquad (7.5)$$

This relation applies to ideal and non-ideal rocket units. For constant γ this expression can be rewritten while the subscripts 1 and 2 apply to nozzle inlet and exit conditions, respectively and since the flow is assumed isentropic, then

$$V_2 = \sqrt{2c_p(T_1 - T_2) + V_1^2} \qquad (7.6)$$

$$V_2 = \sqrt{\frac{2\gamma}{\gamma - 1} RT_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\gamma - 1/\gamma}\right] + V_1^2} \qquad (7.7)$$

This equation also holds for any two points within the nozzle. When the chamber cross section is large compared to the nozzle section, the chamber velocity is comparatively small, and the term V_1^2 can be neglected. The chamber temperature T_1 is equal to the nozzle inlet temperature; for an isentropic nozzle flow process it is also equal to the stagnation temperature

$$V_2 = \sqrt{\frac{2\gamma}{\gamma - 1} RT_o \left[1 - \left(\frac{p_2}{p_o}\right)^{\gamma - 1/\gamma}\right]} \qquad (7.8)$$

Example 7.2 A converging-diverging nozzle is designed to operate isentropically with an exit Mach number of 1.5. The nozzle is supplied from an air reservoir in which The pressure is $500\ kPa$; the temperature is $500\ K$. The nozzle throat area is $5\ cm^2$. Assume air to behave as a perfect gas, with $\gamma = 1.4$ and $R = 0.2870\ kJ/kg.K$.

a) Determine the ratio of exit area to throat area.
b) Find the range of back pressure over which the nozzle is choked.
c) Determine the mass flow rate for a back pressure of $450 kPa$.
d) Determine the mass flow rate for a back pressure of $0\ kPa$.

Solution

a) To produce a supersonic Mach number of 1.5 at the nozzle exit, the Mach number at the throat must be 1. Therefore, the throat area is equal to A^* .From isentropic table for $M = 1.5$, $A/A^* = 1.176$. So the ratio of exit area to throat area to produce Mach 1.5 is 1.176. or $A_e = 5.88\ cm^2$.

b) For all back pressures below that corresponding to (curve 4) of Figure 7.2, the nozzle is choked. For (curve 4), sonic flow is attained at the throat, followed by subsonic deceleration. The subsonic solution for $A/A^* = 1.176$ is found from isentropic table, $M = 0.61$. At this Mach number, $p/p_o = 0.778$. Therefore, the greatest back pressure at which the nozzle is choked is $p_b = 0.778(500\ kPa) = 389\ kPa$. In other words, over the range $0 < p_b < 389\ kPa$, the nozzle is choked.

c) For a back pressure of $450\ kPa$, the nozzle is not choked; subsonic flow occurs throughout the nozzle. For this condition, the exit-plane pressure is equal to the back pressure. From isentropic, for $p/p_o = 0.90$, $M = 0.39$ and $T/T_t = 0.971$. Exit-plane pressure p_e and temperature T_e are respectively, $450\ kPa$ and $485.5\ K$.

$$\dot{m} = \rho_e A_e V_e$$

$$\dot{m} = \frac{p_e}{RT_e} A M_e \sqrt{\gamma R T_e}$$

$$\dot{m} = \left[\frac{450}{0.287 * 485.5}\right] * 5.88 \times 10^{-4} * 0.39\sqrt{1.4 * 287 * 485.5}$$

$$= 3.230 * 5.88 \times 10^{-4} * 0.39 * 441.7 = 0.327\ \ kg/s$$

d) For back pressure of $0\ kPa$, the nozzle is choked, with the exit–plane pressure not equal to the back pressure . For this condition the Mach number at the throat is 1, with the throat pressure and temperature equal respectively to $264.2\ kpa$ and $416.7\ K$.

Figure 7.6

$$\dot{m} = \rho_{th} A_{th} V_{th}$$

$$\dot{m} = \left[\frac{264.2}{0.287 * 416.7}\right] * 5.0 \times 10^{-4} * 1 * \sqrt{1.4 * 287 * 416.7}$$

$$= 0.452\ \ kg/s$$

The results of this example is plotted in figure (7.6)

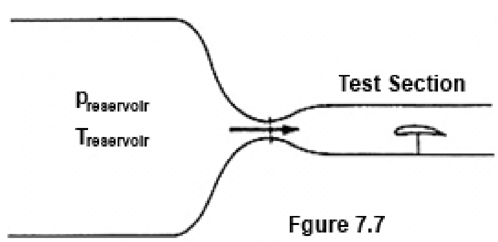
Example 7.3 A nozzle is to be designed for a supersonic helium wind tunnel. Test section specifications are as flow: Diameter, $10\ cm$, Mach number 3.0, Static pressure $12.1\ kPa$ at $15\ km$ altitude and Static temperature, $216.7\ K$ at this altitude. Determine the mass flow that must be provided, the nozzle throat area and the reservoir temperature and pressure. Assume isentropic flow in the nozzle at the design condition, and neglect boundary layer effects (Figure 7.7). Assume that helium behaves as a perfect gas, with $\gamma = 1.667$ (constant) and $R = 2.077\ kJ/kgK$.

Solution:

Test section mass flow rate

$$\dot{m} = \rho V A$$

$$\dot{m} = \frac{p_s}{RT_s}\left(\frac{\pi}{4}D^2\right) M_s \sqrt{\gamma R T_s}$$



Fgure 7.7

$$\dot{m} = \left[\frac{12.1}{2.077 * 216.7}\right] * \frac{\pi}{4} 0.01 * 1 * \sqrt{1.667 * 2077 * 216.7} = 0.5487 \ kg/s$$

From gas dynamics tables for isentropic flow, at $M = 3.0$;

$A_s/A^* = 3.0$

$$A^* = throat\ area = A_s \div \frac{A_s}{A^*} = \frac{\frac{\pi}{4}0.01}{3.0} = \frac{0.007854}{3} = 0.002.618 \times 10^{-3}\ m^2$$

$p_s/p_o = 0.03125$

$$p_r = p_t = p_s \div \frac{p_s}{p_o} = \frac{12.1}{0.03125} = 387.2\ kN/m^2$$

$T_s/T_o = 0.250$

$$T_r = T_o = T_s \div \frac{T_s}{T_o} = \frac{216.7}{0.250} = 866.8\ K$$

Example 7.4 A converging–diverging nozzle with an area ratio of 3.0 exhausts into a receiver where the pressure is 1 bar. The nozzle is supplied by air at 22°C from a large chamber. At what pressure should the air in the chamber be for the nozzle to operate at its design condition ? What will the outlet velocity be?

Solution

$$\frac{A_3}{A_3^*} = \frac{A_3}{A_2} = 3.0$$

From isentropic table

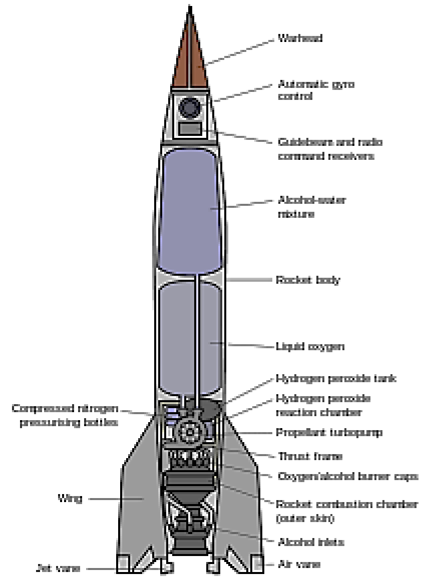
$$M_3 = 2.64,\ \ \frac{p_3}{p_o} = 0.0471,\ \ \frac{T_3}{T_o} = 0.4177$$

$$p_1 = p_o = p_3 \div \frac{p_3}{p_o} = \frac{1}{0.0471} = 21.2\ bar$$

$$T_3 = \frac{T_3}{T_o} * T_o = 0.4177 * (22 + 273) = 123.2\ K$$

$$V_3 = M_3 * a_3 = 2.64 * \sqrt{1.4 * 287 * 123.2} = 587\ m/s$$

Chapter Eight /Thrust of Rocket Engine



Some say that the first recorded use of a rocket in battle was by the Chinese in 1232 against the Mongol hordes. Rocket technology first became known to Europeans following their use by the Mongols, Genghis Khan and Ögedei Khan, when they conquered parts of Russia, Eastern, and Central Europe. The first iron-cased and metal-cylinder rocket artillery, made from iron tubes, were developed by the weapon suppliers of Tipu Sultan, an Indian ruler of the Kingdom of Mysore, and his father Hyder Ali, in the 1780s.

In 1903, high school mathematics teacher Konstantin Tsiolkovsky (1857–1935) published Исследование мировых пространств реактивными приборами (The Exploration of Cosmic Space by Means of Reaction Devices), the first serious scientific work on space travel.

In 1912, Robert Esnault-Pelterie published a lecture on rocket theory and interplanetary travel. Robert Goddard began a serious analysis of rockets in 1912, concluding that conventional solid-fuel rockets needed to be improved in three ways. In 1920, Goddard published these ideas and experimental results in A Method of Reaching Extreme Altitudes. Modern rockets were born when Goddard attached a supersonic (de Laval) nozzle to a liquid-fueled rocket engine's combustion chamber.

Some of the first successful American rockets were the JATO (jet-assisted take-off) units used during the war (solid in 1941 and liquid in 1942). Also famous was the V-2 rocket developed by Wernher von Braun in Germany. This first flew in 1942 and had a liquid propulsion system that developed 56,000 pounds of thrust. The first rocket-propelled aircraft was the German ME-163.

8.1 Thrust of rocket engine

Select a control volume as shown in figure 8.1. The forces acted on this control volume are thrust and the unbalance pressure force acting on the exit plane. (Other forces such gravity, friction …etc. are ignored) Applying eq. 4.6

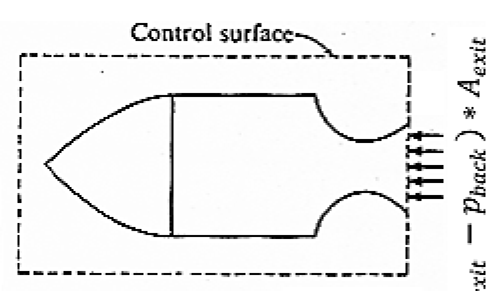


Figure 8.1

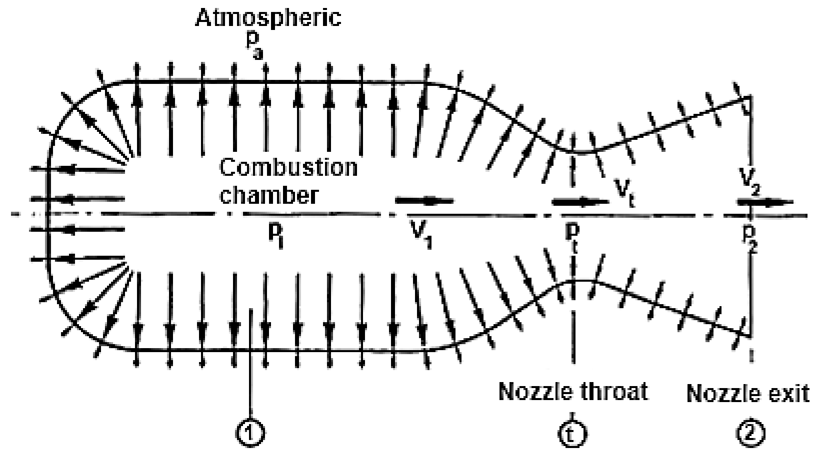


Fig. 8.2. Pressure balance on chamber and nozzle wall: internal gas pressure is highest inside chamber and decreases steadily in nozzle, while external atmospheric pressure is uniform

$$\sum \mathbf{F} = \iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (4.6)$$

$(Thrust = rate\ of\ change\ momentum)$

$$Thrust = \iint_{cs} \mathrm{V}_x\,\rho\,(V_x)\,dA \qquad (8.1)$$

This force is the thrust obtained for any true rocket propulsion engine. It assumes a uniform exhaust velocity that does not vary across the area of the jet. The preceding equation shows that the thrust is proportional to the propellant flow rate and the exhaust velocity. The surrounding fluid (usually air) has an influence on the thrust.

Figure (8.2) shows schematically the external pressure acting uniformly on the outer surface of a rocket chamber and the gas pressures on the inside of a typical rocket engine. The size of the arrows indicates the relative magnitude of the pressure forces. The axial thrust can be determined by integrating all the pressures acting on areas that can be projected on a plane normal to the nozzle axis. The radially outward acting forces are appreciable but do not contribute to the axial thrust, because the rocket is axially symmetrical.

By inspection it can be seen that at the exit area A_2 of the engine's gas exhaust there is an unbalance of the external environmental or atmospheric pressure $\mathrm{p_a}$and the local pressure p_2 of the hot gas jet at the exit plane of the nozzle. Thus, for a steadily operating rocket engine flying in a homogeneous atmosphere (neglecting localized boundary layer effects), the thrust is equal to

$$F = \dot{m}V_e + (p_e - p_a)A_e \qquad (8.2a)$$

$$F = \rho_e A_e V_e^2 + (p_e - p_a)A_e \qquad (8.2b)$$

The thrust acting on the vehicle is composed of two terms. The first term, the ***momentum thrust***, is the product of the propellant mass flow rate, $\dot{m}$, and the exhaust velocity relative to the vehicle, V_e . The second term, the ***pressure thrust***, consists of the product of the cross-sectional area of the exhaust jet leaving the vehicle and the difference between the e xhaust pressure and the fluid pressure. Equation (8.2) gives values of the thrust variations of rockets with altitude.

If the exhaust pressure is less than the surrounding fluid pressure, the pressure thrust is negative. Because this condition gives a low thrust and is undesirable, the rocket exhaust nozzle is usually so designed that the exhaust pressure is equal to or slightly higher than the fluid pressure.

When the fluid pressure is equal to the exhaust pressure, the pressure thrust term is zero, and the thrust is expressed as

$$F = \dot{m}V_2 \qquad (8.3)$$

This condition gives a maximum thrust for a given propellant and chamber pressure. The rocket nozzle design, which permits the expansion of the propellant products to the pressure that is exactly equal to the pressure of the surrounding fluid, is referred to as the rocket nozzle with ***optimum expansion ratio***. When expanding into a vacuum, $p_a = 0$, and the thrust is then simply

$$F = \rho_e A_e V_e^2 + p_e A_e \qquad (8.4)$$

The supersonic convergent – divergent nozzle is used in rockets. The ratio between the inlet and exit pressures in all rockets is sufficiently large to induce supersonic flow. Only if the chamber pressure drops below approximately $2.17\ atm$ then there is a danger of not producing supersonic flow in the divergent portion of the nozzle when operating at sea level.

We know that the velocity of sound is equal to the velocity of propagation of a pressure wave within the medium, sound being a pressure wave. If, therefore, sonic velocity is reached at any one point within a steady flow system, it is impossible for a pressure disturbance to travel upstream past the location of sonic or supersonic velocity. Therefore, any partial obstruction or disturbance of the flow downstream of the nozzle throat section has no influence on the flow at the throat section or upstream of the throat section, provided that this disturbance does not raise the downstream pressure above its critical value.

It is not possible to increase the throat velocity or the flow rate in the nozzle by lowering the exit pressure or evacuating the exhaust section.

The *flow* through the critical section of a supersonic nozzle is calculated from

$$\dot{m} = \rho_t A_t V_t = \frac{p_t}{RT_t} A_t \sqrt{\gamma R T_t}$$

$$= \frac{p_o}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \frac{\left(\frac{\gamma+1}{2}\right)}{RT_o} A_t \sqrt{\frac{\gamma R T_o}{\left(\frac{\gamma+1}{2}\right)}} = p_o A_t \sqrt{\frac{\gamma}{RT_o} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \qquad (8.5)$$

The mass flow through a rocket nozzle is therefore proportional to the throat area $A,$ and the upstream pressure $p_o,$ inversely proportional to the square root of the absolute nozzle inlet temperature T_o, and a function of the gas properties.

For a supersonic nozzle the *ratio between the throat area and any downstream area* at which the pressure p_x prevails can be expressed as a function of the pressure ratio and the specific heat ratio as follows,

$$\frac{A_{th}}{A_x} = \frac{\rho_x V_x}{\rho_t V_{th}} = \left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)} \left(\frac{p_x}{p_{th}}\right)^{1/\gamma} \sqrt{\frac{\gamma+1}{\gamma-1}\left[1-\left(\frac{p_x}{p_{th}}\right)^{(\gamma-1)/\gamma}\right]} \qquad (8.6)$$

For an ideal rocket with γ being constant throughout the expansion process, the exit velocity is;

$$V_e = \sqrt{2c_p(T_o - T_e)} = \sqrt{\frac{2\gamma}{\gamma-1} RT_o \left[1-\left(\frac{p_e}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} \qquad (8.7a)$$

Eq. (8.2) is general and applies to all rockets. It can be written as;

$$F = A_t\, p_o \sqrt{\frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{p_e}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} + (p_e - p_a)A_e \qquad (8.7b)$$

This equation shows that the thrust is proportional to the throat area A_{throat} and the nozzle inlet pressure p_o and is a function of the pressure ratio across the nozzle p_e/p_o, the specific heat ratio γ, and the pressure thrust. It is called the ***ideal thrust equation***.

An ***under-expanding nozzle*** discharges the fluid at a pressure greater than the external pressure because the exit area is too small. The expansion of the fluid is therefore incomplete within the nozzle and continues outside. The nozzle exit pressure is higher than the local atmospheric pressure.

In an ***over-expanding nozzle*** the fluid is expanded to a lower pressure than the external pressure; it has an exit area that is too large.

When a supersonic nozzle is operating in the ***under- or overexpanded*** regimes, with flow in the nozzle independent of back pressure, the exit velocity is unaffected by back pressure. Thus, over this range of back pressures, Eq. (8.2) shows that the greater thrusts are developed in the underexpanded case, and the lesser in the overexpanded case.

For back pressures greater than the upper limit indicated, a normal shock appears in the diverging portion of the nozzle, the exit velocity becoming subsonic, and this analysis no longer applies.

For jet turbine engine, for simplicity we shall assume here that the mass flow $\dot{m}$ is constant (i.e. that the fuel flow is negligible), the *net thrust F* due to the rate of change of momentum is

----

$$F = m(V_e - V_a) \qquad (8.8a)$$

where V_a is speed of air that enters aircraft intakes which is equal to the aircraft speed for steady level flight. mV_e is called the *gross momentum thrust* and mV_a the *intake momentum drag.* When the exhaust gases are not expanded completely to P_{atm} *in* the propulsive duct (which is a duct ends with a nozzle), the pressure p_e in the plane of the exit will be greater than P_{atm} and there will be an additional pressure thrust exerted over the jet exit area A· The net thrust is then the sum of the *momentum thrust* and the *pressure thrust,* namely

$$F = m(V_e - V_a) + (p_e - p_a)A_e \qquad (8.8b)$$

For design condition, i.e. maximum V_e, the exhaust gases must expanded completely to P_{atm}

8.2 characteristics of rocket engine

Thrust coefficient, C_F: is defined as the thrust divided by the chamber pressure p_o and the throat area A_t.

$$C_F = \frac{F}{A_t\, p_o} = \sqrt{\frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{p_e}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \frac{(p_e - p_a)}{p_o}\frac{A_e}{A_t} \qquad (8.9)$$

For any fixed pressure ratio (p_e/p_o) the thrust coefficient C_F *has a maximum value when* $p_e = p_a$. This value is known as the ***optimum thrust coefficient.*** The use of the thrust coefficient permits a simplification of Equation (8.2)

$$F = C_F A_t\, p_o \qquad (8.10)$$

Thrust power output of the propulsive device is the actual rate of doing useful propulsion work and is designated as p_T

$$p_T = F * V_{rocket} \qquad (8.11)$$

Total impulse, I_t is the thrust force F (which can vary with time) integrated over the burning time.

$$I_t = \int_0^t F dt \qquad N.s \qquad (8.12)$$

For constant thrust and negligible start and stop transients this reduce to

$$I_t = F.t \qquad N.s \qquad (8.13)$$

Specific impulse, I_s is the total impulse per unit weight of propellant consumption, $\dot{w}$. The units are sec

$$I_s = \frac{\int_0^t F dt}{\int_0^t \dot{w} dt} \qquad s \qquad (814)$$

For constant thrust and propellant flow

$$I_s = \frac{F}{\dot{w}} \qquad s \qquad (8.15)$$

***Effective exhaust velocity,* c**: is the average equivalent velocity at which propellant ejects from rocket nozzle, the units are m/s.

$$c = gI_s = \frac{F}{\dot{m}} \qquad m/s \qquad (8.16)$$

Specific propellant consumption the required propellant weight to produce a unit thrust in an equivalent rocket. The units are $kg/N.sec$

$$specific\ propellant\ consumption = \frac{1}{I_s} = \frac{\dot{w}}{F} = \frac{g\dot{m}}{F} \qquad 1/s \qquad (8.17)$$

For other engines the *specific propellant consumption* in common is based on the power output with units $kg/kW.hr$.

Mass ratio, which is define as the ratio of final rocket mass to the initial rocket mass.

$$m.r = \frac{m_{final}}{m_o} = \frac{m_{final}}{m_{final} + m_{prop}}$$

where m_{prop} is useful propellant weight.

Equation (8.2) shows that the thrust of a rocket unit is independent of flight velocity in opposite to jet turbine engine. Because changes in the fluid pressure (p_o and p_e) affect the pressure thrust as well as p_a, a variation of the rocket thrust with altitude is to be expected. As the atmospheric pressure decreases with increasing altitudes, the thrust and therefore also the specific impulse will increase if the vehicle is propelled at a higher altitude. The change in pressure thrust due to altitude changes can amount to $10\ to\ 30\%$ of the overall thrust, as shown for a typical rocket engine in Figure (8.3).

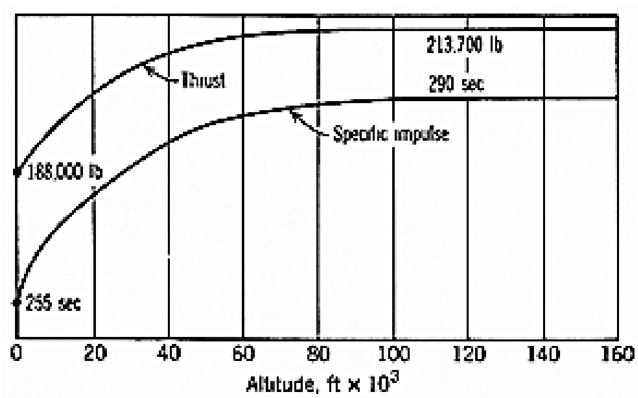


Figure 8.3 Altitude performance of the H-1 liquid propellant rocket engine used in the Thor launch vehicle.

Example 8.1: A rocket projectile has the following characteristics:

Initial mass	$200\ kg$
Mass after rocket operation	$130\ kg$
Payload, non propersive structure, etc.	$110\ kg$
Rocket operating duration	$3.0\ sec$
Average specific impulse of propellent	$240\ N.sec^3/kg.m$

Determine mass ratio, propellant mass fraction, propellant flow rate, thrust, thrust-to-weight ratio, acceleration of vehicle, effective exhaust velocity, total impulse, and the impulse-to-weight ratio.

***Solution*:**

Mass ratio of vehicle

$$m.r = \frac{m_{f\text{inal}}}{m_o} = \frac{130}{200} = 0.65$$

mass ratio of rocket system

$$m.r = \frac{m_f}{m_o} = \frac{130 - 110}{200 - 110} = 0.222$$

Note that the empty and initial masses of the rocket are 20 and 90 kg respectively. Propellant mass fraction

$Propellant\ mass\ fraction = \left(m_o - m_f\right)/m_o = (90 - 20)/90 = \ 0.778$

The propellant mass is $200 - 130 = 70\ kg$.

Propellant mass flow rate is $\mathrm{m} = 70/3\ = 23.3\ \mathrm{kg/sec}$.

The thrust $\mathrm{F} = \mathrm{I_s}\dot{\mathrm{w}}\ =\ 240 *\ 23.3 *\ 9.80\ =\ 54{,}800\ \mathrm{N}$

Thrust-to-weight ratio of vehicle,

Initial value $\mathrm{F/w_o} =\ 54{,}800/(200 *\ 9.80)\ =\ 28$

Final value $\mathrm{F/w_o} = 54{,}800/(130 *\ 9.80)\ =\ 43$

Maximum acceleration of vehicle is $43 * 9.80 = 421\ m/sec^2$.

Effective exhaust velocity is $\mathrm{c}\ = \mathrm{gI_s} = 9.81 * 240 = 2352\ \mathrm{m/sec}$.

Total impulse $\mathrm{I_t} = \mathrm{I_s w} = 240 * 70 * 9.80 = 164{,}600\ \mathrm{N.sec}$.

This result can also be obtained by multiplying the thrust by the duration.

The impulse-to-weight ratio $\mathrm{I_t/w_o} =\ 54{,}870/[(200 - 110)9.80]\ =\ 187$

Example 8.2: An ideal rocket chamber is to operate at sea level using propellants whose combustion products have a specific heat ratio of 1.30. Determine the required chamber pressure and nozzle area ratio between throat and exit if the nozzle exit Mach number is 2.40. The nozzle inlet Mach number may be considered to be zero.

Solution:

For optimum expansion the exit pressure should be equal to the atmospheric pressure of $0.1013\ Mpa$. If the chamber velocity is small, the chamber pressure is equal to the total pressure and is

---

$$p_o = p\left[1 + \frac{(\gamma - 1)}{2}M^2\right]^{\gamma/(\gamma-1)}$$

$$p_o = 101.3\left[1 + \frac{(1.3 - 1)}{2}2.4^2\right]^{1.3/(1.3-1)} = 1500\ kPa$$

The area ratio

$$\frac{A_{exit}}{A_{throt}} = \frac{A_e}{A^*}$$

$$\frac{A_e}{A^*} = \frac{1}{M_e}\left(\frac{1 + [(\gamma - 1)/2]M_e^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)}$$

$$\epsilon = \frac{A_e}{A^*} = \frac{1}{2.4}\sqrt{\left(\frac{1 + [(1.3 - 1)/2]2.4^2}{(1.3 + 1)/2}\right)^{(1.3+1)/(1.3-1)}} = 2.6535$$

Or using isentropic table , at $M_e = 2.4$ for $\gamma = 1.3$ gives $A_e/A^* = 2.654$

Example 8.3 A rocket nozzle is designed to operate supersonically with a chamber pressure of 3 MPa and an ambient pressure of 101 kPa. Find the ratio between the thrust at sea level to the thrust in space (0 kPa). Assume a constant chamber pressure, with a chamber temperature of 1600 K. Assume the rocket exhaust gases to behave as a perfect gas with $\Gamma = 1.3$ and $R = 0.40\ kJ/kg.K$.

Solution

Apply the momentum equation.

$$Thrust = (p_e - p_a)A_e + \rho_e A_e V_e^2$$

The exit plane pressure and exit velocity are the same in space as at sea level.

From isentropic table at $p/p_o = 101/3000 = 0.03367$

$M = 2.81$ and $T/T_o = 0.4578$

Then $T_e = T_o * T/T_o = 1600 * 0.4578 = 732.5\ K$

The exhaust velocity is then

$$V_e = M_e * a_e = 2.81 * \sqrt{1.3 * 400 * 732.5} = 1734.2\ m/s$$

$$\rho_e = \frac{p_e}{RT_e} = \frac{101}{0.4 * 732.5} = 0.3447 \ kg/m^3$$

$$Thrust\ at\ sea\ level = \rho_e A_e V_e^2 = 0.3447 * A_e * 1734.2^2$$

$$Thrust\ at\ space = (p_e - p_a)A_e + \rho_e A_e V_e^2$$

$$= 101 \times 10^3 * A_e + 0.3447 * A_e * 1734.2^2$$

$$\frac{Thrust\ at\ sea\ level}{Thrust\ at\ space} = \frac{0.3447 * A_e * 1734.2^2}{101 \times 10^3 * A_e + 0.3447 * A_e * 1734.2^2} = 0.911$$

***Example 8.4*:** Design a nozzle for an ideal rocket that has to operate at a $25\ km$ altitude and give a $5000\ N$ thrust at a chamber pressure of $2.068\ MPa$ and a chamber temperature of $2800\ K$. Assuming $\gamma = 1.30$ and $\mathrm{R} = 355.4\ \mathrm{J/kg.K}$, determine

a) Exit velocity, temperature and area
b) Throat velocity, temperature and area
c) Area ratio

Solution.

At a 25 km altitude, the atmosphere pressure equals $25.49\ KPa$, and as $p_t = p_1$, then The pressure ratio is,

a)

$$\frac{T_e}{T_o} = \left[\frac{\mathrm{p_e}}{\mathrm{p_o}}\right]^{(\gamma-1)/\gamma} = \left[\frac{0.02549}{2.068}\right]^{0.3/1.3} = 0.3626$$

$$T_e = T_o * 0.3626 = 1015.3\ K$$

$$V_e = \sqrt{\frac{2\gamma}{\gamma - 1} RT_o \left[1 - \left(\frac{p_e}{p_o}\right)^{\gamma-1/\gamma}\right]}$$

$$= \sqrt{\frac{2.6}{0.3} * 355.4 * 2800 \left[1 - \left(\frac{0.02549}{2.068}\right)^{0.3/1.3}\right]} = 2344.618\ m/sec$$

$$\dot{m} = F/V_e = 5000/2344.618 = 2.133\ kg/sec$$

$$v_e = \frac{RT_e}{p_e} = \frac{355.4 * 1015.3}{0.2549 * 10^5} = 14.156\ m^3/kg$$

$$\rho_e = 1/v_e = 1/14.156 = 0.0706\ kg/m^3$$

$$A_e = \frac{\dot{m}}{\rho_e V_e} = \frac{2.133}{.0706 * 2344.618} = 128.859 * 10^{-4} m^2$$

b)

$$\frac{p_o}{p_t} = \left[1 + \frac{\gamma - 1}{2} M_t^2\right]^{\gamma/(\gamma-1)} = \left[\frac{2.3}{2}\right]^{1.3/0.3} = 1.832$$

$$p_t = \frac{2.068}{1.832} = 1.129\ MPa$$

$$\frac{T_o}{T_t} = \left[1 + \frac{\gamma - 1}{2} M_t^2\right] = \left[\frac{2.3}{2}\right] = 1.15$$

$$T_t = \frac{2800}{1.15} = 2434.783\ K$$

$$V_t = a_t = \sqrt{\gamma R T_t} = \sqrt{1.3 * 355.4 * 2434.783} = 1060.622\ m/sec$$

$$v_t = \frac{R T_t}{p_t} = \frac{355.4 * 2434.783}{1.129 * 10^6} = 0.76645\ m^3/kg$$

$$A_t = \frac{v_t \dot{m}}{V_t} = \frac{0.76645 * 2.133}{1060.622} = 15.414 * 10^{-4} m^2$$

$$\epsilon = A_e/A_t = 128.859/15.414 = 8.36$$

Try to use isentropic flow Table and resolve this example.

Example 8.5 A rocket operates at sea level ($p_2 = 1$ atm) with a chamber pressure of $p_1 = 2.068\ MN/m^2$, a chamber temperature of $T_1 = 2222°K$ and a propellant consumption of $\dot{m} = 1.0\ kg/s$. calculate the value of $A, v, V,$ and $M,$ in the nozzle at a section where $p_x = 1.379\ MPa$. Calculate also the ideal thrust and the ideal specific impulse. Take $\gamma = 1.30$, $c_p = 0.359$ kcal/kg.K, and R = 345.7 J/kg.K

Solution:

In an isentropic flow at a point (x). Initial specific volume

$$v_1 = \frac{R T_1}{p_1} = \frac{345.7 * 2222}{2.068 * 10^6} = 0.3714\ m^3/kg$$

The specific volume is

$$v_x = v_1 \left(\frac{p_1}{p_x}\right)^{1/\gamma} = 0.3714 \left(\frac{2.068}{1.379}\right)^{1/1.3} = 0.5072\ m^3/kg$$

The temperature is

$$T_x = T_1 \left(\frac{p_1}{p_x}\right)^{(\gamma-1)/\gamma} = 2222 \left(\frac{1.379}{2.068}\right)^{\frac{0.3}{1.3}} = 2023.6\ K$$

The velocity is

$$V_x = \sqrt{\frac{2\gamma}{\gamma-1}RT_1\left[1-\left(\frac{p_x}{p_1}\right)^{\gamma-1/\gamma}\right]} = \sqrt{\frac{2*1.3}{1.3-1}*345.7*2222\left[1-\left(\frac{1.379}{2.068}\right)^{\frac{0.3}{1.3}}\right]}$$

$$= 770.921\ m/s$$

The cross section area is

$$\dot{m}_x = \rho_x V_x A_x$$

$$A_x = \frac{\dot{m}_x v_x}{V_x} = \frac{1*0.5072}{770.921} = 6.579*10^{-4}m^2$$

And the Mach number is then

$$M_x = \frac{V_x}{\sqrt{\gamma R T_x}} = \frac{770.921}{\sqrt{1.3*345.7*2023.6}} = 0.808$$

At optimum expansion the ideal exhaust velocity V_e is equal to the effective exhaust velocity and $p_e = p_a$

$$V_e = \sqrt{\frac{2\gamma}{\gamma-1}RT_1\left[1-\left(\frac{p_e}{p_0}\right)^{\gamma-1/\gamma}\right]} = \sqrt{\frac{2.6}{0.3}*345.7*2222\left[1-\left(\frac{0.10136}{2.068}\right)^{0.3/1.3}\right]}$$

$$= 1826.979\ m/s$$

which is equal to effective exhaust velocity, and as $p_2 = p_a$, then

$$F = \dot{m}V_e = 1*1826.979 = 1826.979\ N$$

As the effective exhaust velocity is $= gI_s$, the specific impulse is;

$$I_s = c/g = 1826.979/9.81 = 186.236\ \text{sec}$$

Note: If you chose different sections pressure, you can simply plot the variation of A, v, V, and M. Figure (8.5) shows a plot of the variation of the velocity, the specific volume, the area, and Mach number, and the pressure in this nozzle.

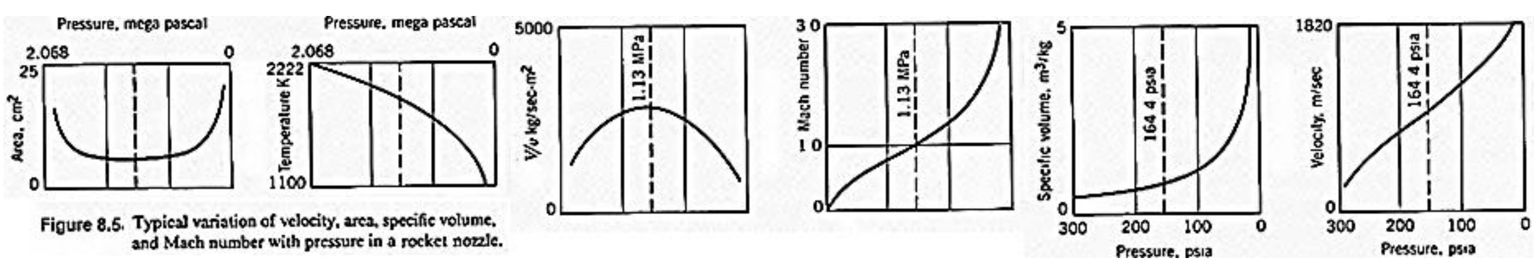


Figure 8.5. Typical variation of velocity, area, specific volume, and Mach number with pressure in a rocket nozzle.

Example 8.6 For the rocket of example 8.5 , calculate Exit temperature and Mach number, Throat area and area ratio and Gas velocity at throat.

Chapter Nine/Stationary Normal Shock Waves; part 1

9.1 Introduction

The shock process represents an abrupt change in fluid properties, in which finite variations in pressure, temperature, and density occur over a shock thickness comparable to the mean free path of the gas molecules involved. It has been established that supersonic flow adjusts to the presence of a body by means of such shock waves, whereas subsonic flow can adjust by gradual changes in flow properties. Shocks may also occur in the flow of a compressible medium through nozzles or ducts and thus may have a decisive effect on these flows. An understanding of the shock process and its ramifications is essential to a study of compressible flow.

It was pointed out previously that a series of weak compression waves can coalesce to form a finite compression shock wave. The mechanism by which this process occurs will be discussed in detail. The thermodynamics of the shock process will be reviewed, and the one-dimensional equations of continuity, momentum, and energy applied to the normal shock. Solutions of these equations will be presented to enable the working of practical engineering problems.

9.2 Formation of a Normal Shock Wave

It was shown that, when a piston in a tube is given a steady velocity to the right of magnitude dV (Figure 9.1), a sound wave travels ahead of the piston through the medium in the tube. Suppose the piston is now given a second increment of velocity dV, causing a second wave to move into the compressed gas behind the first wave. The location of the waves and the pressure distribution in the tube, after a time t_2, are shown in Figure 9.2**.** Each wave travels at the velocity of sound with respect to the gas into which it is moving. Since the second wave is moving into a gas that is already moving to the right with velocity dV, and since it is moving into a compressed gas having a slightly elevated temperature, the second wave travels with a faster absolute velocity than the first wave and gradually overtakes it. After a time t_2 (t_2 greater than t_1).

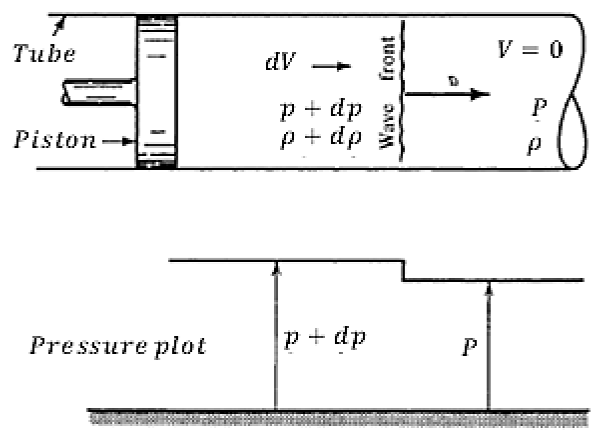


Figure 9.1 Initiation of infinitesimal pressure pulse

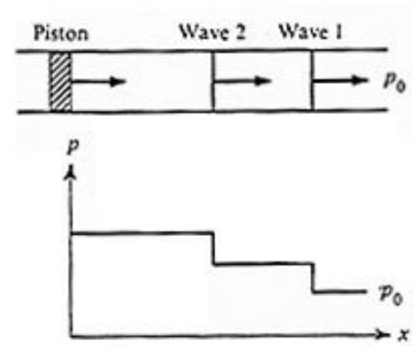


Figure 9.2

Now suppose the piston is accelerated from rest to a finite velocity increment of magnitude ΔV to the right. This finite velocity increment can be thought to consist of a large number of infinitesimal increments, each of magnitude dV. Figure (9.3) shows the velocity of the piston versus time, with the incremental velocities dV superimposed. The waves next to the piston tend to overtake those farther down the tube.

As time passes, the compression wave steepens. The tendency of the higher density parts of the wave to overtake the lower density parts is finally counteracted by heat conduction and viscous effects taking place inside the wave. The resultant constant-shape compression shock wave produced by the addition of the weak compression waves then moves through the undisturbed gas ahead of the piston. The slopes of temperature and pressure versus distance in the wave itself are very large, and so the shock can be approximated by a discontinuity (Figure 9.4).

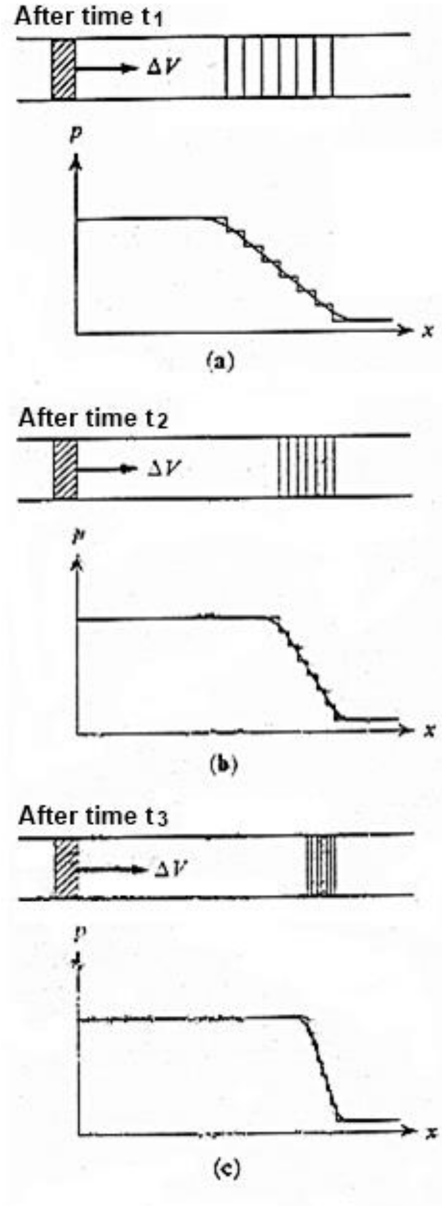


Figure 9.3

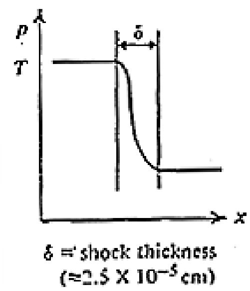


Figure 9.4

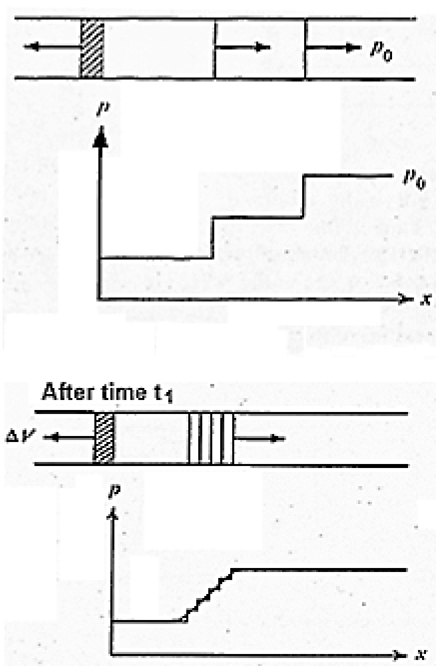


Figure 9.5

If the piston in Figure 9.5 is suddenly given an incremental velocity dV to the left, a weak expansion wave propagates to the right at the velocity of sound. When the piston is given a second increment of velocity, a second expansion wave moves into the expanded gas behind the first wave.

Again, each wave travels at the velocity of sound with respect to the gas into which it is moving. In this case, the waves and gas are moving in opposite directions. Furthermore, the second wave is traveling into a gas that has already been expanded and cooled, which lowers the sound velocity. Both effects reduce the absolute wave velocity, and cause the second wave to fall farther and farther behind the first. In this manner, expansion waves spread out; they are not able to reinforce one another (see Figure 9.6). The creation of a finite expansion shock wave is impossible.

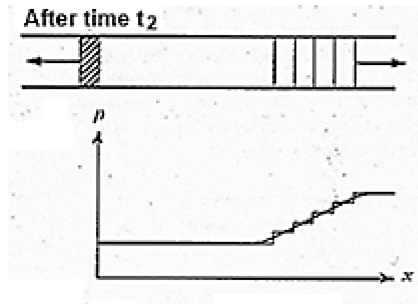


Figure 9.6

9.3 Equations of Motion for a Normal Shock Wave

A shock involves finite, rapid changes in pressure and temperature. The processes taking place inside the wave itself are extremely complex and cannot be studied on the basis of equilibrium thermodynamics. Temperature and velocity gradients inside the shock provide heat conduction and viscous, dissipation that make the shock process internally irreversible.

In a practical sense we don't focus on the interior details of the shock wave, but on the net changes in fluid properties taking place across the entire wave.

If one chooses a control volume encompassing the shock wave, the flow equations can be written without regard to the complexities of the internal processes. For this purpose, it is sufficient to note that the shock process is thermodynamically irreversible. Furthermore, with the shock temperature gradient inside the control volume, there is no external heat transfer across the control volume boundaries, so the shock process is adiabatic.

Figure 9.7 shows a standing normal shock in a section of varying area. We first establish a control volume that includes the shock region and an infinitesimal amount of fluid on each side of the shock. In this manner we deal only with the changes that occur across the shock. It is important to recognize that since the shock wave is so thin (about $10^{-6}\,m$), a control volume chosen in the manner described above is extremely thin in the *x*-direction.

This permits the following simplifications to be made without introducing error in the analysis:

1. The area on both sides of the shock may be considered to be the same.
2. There is negligible surface in contact with the wall, and thus frictional effects may be omitted.

Adiabatic $\delta q = 0$ or $ds_e = 0$

No shaft work $\delta w_s = 0$

Neglect potential $dz = 0$

Constant area $A_1 = A_2$

Neglect wall shear

Continuity

$\rho_1 V_1 = \rho_2 V_2$ (9.1)

$p = \rho RT$

$V = Ma = M\sqrt{\gamma RT}$

Then continuity equation becomes;

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} \qquad (9.2)$$

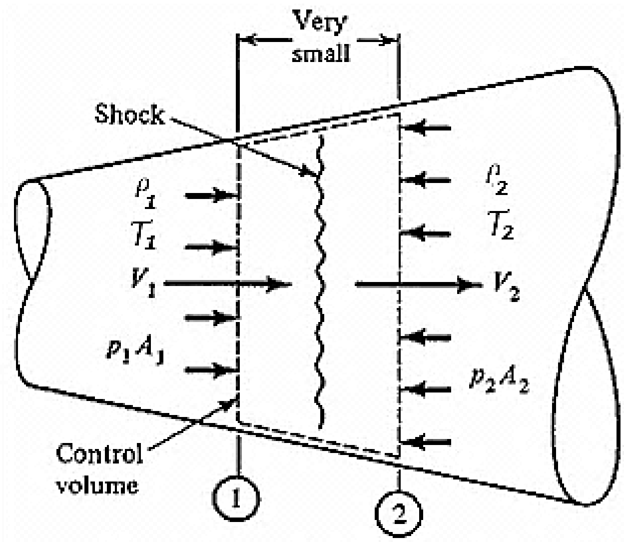


Figure 9.7 Control volume for shock analysis.

Momentum

The *x*-component of the momentum equation for steady one-dimensional flow is;

$$\sum F_x = \dot{m}\left(V_{out,x} - V_{in,x}\right) = \dot{m}(V_2 - V_1)$$

With pressure force the only external forces acting on the control volume, then

$$\sum F_x = p_1 A_1 - p_2 A_2 = (p_1 - p_2)\, A$$

Thus the momentum equation in the direction of flow becomes

$(p_1 - p_2)A = \dot{m}(V_2 - V_1) = \rho V A(V_2 - V_1)$

Canceling the area and ρV can be written as either $\rho_1 V_1$ *or* $\rho_2 V_2$, then

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \qquad (9.3)$$

$$p_1 + \frac{p_1}{RT_1} M_1^2 \gamma R T_1 = p_2 + \frac{p_2}{RT_2} M_2^2 \gamma R T_2$$

$$p_1(1 + \gamma M_1^2) = p_2(1 + \gamma M_2^2)$$

$$\frac{p_2}{p_1} = \frac{(1 + \gamma M_1^2)}{(1 + \gamma M_2^2)} \qquad (9.4)$$

Energy

$h_{o1} + \delta q = h_{o2} + \delta w_s$

$h_{o1} = h_{o2}$ i.e. $h_1 + V_1^2/2 = h_2 + V_2^2/2$, Then

$$T_{o1} = T_{o2} \qquad (9.5)$$

$\because T_o = T\left(1 + \frac{\gamma-1}{2} M^2\right)$ from stagnation properties at each point, then

$$T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right) = T_2\left(1 + \frac{\gamma - 1}{2} M_2^2\right)$$

$$\frac{T_2}{T_1} = \frac{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)}{\left(1 + \frac{\gamma - 1}{2} M_2^2\right)} \qquad (9.6)$$

Eqs. (9.4), (9.5) and (9.6) are the principle equations for a standing normal shock, in addition to the foregoing assumptions. They called the jump conditions and must be satisfied to preserve conservation of mass, momentum and energy across the shock.

In the next chapter we seek a relationship between M_1 and M_2 to solve these equations.

There are seven variables involved in these equations: $\gamma, p_1, T_1, M_1, p_2, T_2$ and M_2. Once the gas is identified, γ is known, and a given state before the shock fixes $p_1, T_1\ and\ M_1$. Thus equations (9.2), (9.4), and (9.6) are sufficient to solve for the unknowns after the shock: $p_2, T_2\ and\ M_2$.

We proceed to combine these equations above and derive an expression for M_2 in terms of the information given. First, we rewrite these equations

$$\frac{p_1 M_1}{p_2 M_2} = \sqrt{\frac{T_1}{T_2}} \qquad (9.2)$$

$$\frac{p_1}{p_2} = \frac{(1 + \gamma M_2^2)}{(1 + \gamma M_1^2)} \qquad (9.4)$$

$$\frac{T_1}{T_2} = \left(\frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2}\right) \qquad (9.6)$$

Substitue eqs (10.2) and (10.3) into eq (10.1) gives;

$$\frac{(1 + \gamma M_2^2)}{(1 + \gamma M_1^2)} \frac{M_1}{M_2} = \left(\frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2}\right)^{1/2} \qquad (9.7)$$

At this point notice that M_2 is a function of only M_1 and by inspection, it is evident that one solution to Eq. (9.7) is the trivial one, $M_1 = M_2$. This solution, involving no change in properties in a constant area flow, corresponds to isentropic flow and is not of interest for the irreversible normal shock.. Squaring both sides, cross-multiply, and arrange the result as a quadratic in M_2^2: gives:

$$\frac{M_1^2 \left(1 + \frac{\gamma - 1}{2} M_1^2\right)}{(1 + \gamma M_1^2)^2} = \frac{M_2^2 \left(1 + \frac{\gamma - 1}{2} M_2^2\right)}{(1 + \gamma M_2^2)^2}$$

$$A(M_2^2)^2 + B(M_2^2) + C = 0 \qquad (9.8)$$

$$A = \left[\left(\frac{\gamma-1}{2}\right) - \gamma^2\left(\frac{M_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)}{(1+\gamma M_1^2)^2}\right)\right] \qquad (9.9a)$$

$$B = \left[1 - 2\gamma\left(\frac{M_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)}{(1+\gamma M_1^2)^2}\right)\right] \qquad (9.9b)$$

$$C = -\left(\frac{M_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)}{(1+\gamma M_1^2)^2}\right) \qquad (9.9c)$$

Solution of the quadratic equation (9.8) is lengthy and difficult. The solution is;

$$M_2^2 = \frac{M_1^2 + 2/(\gamma-1)}{[2\gamma/((\gamma-1))]M_1^2 - 1} \qquad (9.10)$$

The result of Eq. (9.10) is plotted in Figure 9.8 for $\gamma = 1.4$.

For $M_1 > 1$, M_2 is always less tha 1, and vice versa. But when $M_1 < 1$ it is not important since there is no shock.

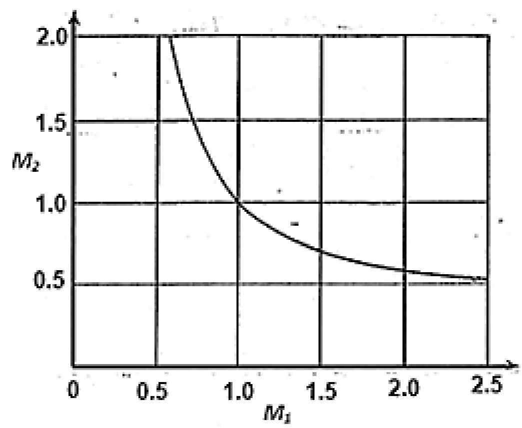


Figure 9.8

Chapter Ten/ Stationary Normal Shock Waves; part 2

10.1 Normal Shock Table

We have found that for any given fluid with a specific set of conditions entering a normal shock there is one and only one set of conditions that can result after the shock. For the perfect gas further simplifications can be made since equation (9.10) yields the exit Mach number M_2 for any given inlet Mach number M_1 and we can now eliminate M_2 from all previous equations.

Pressure ratio;

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \qquad (9.4)$$

substitute from eq. 9.10 gives

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1} M_1^2 - \frac{\gamma-1}{\gamma+1} \qquad (10.1)$$

Temperature ratio;

$$\frac{T_2}{T_1} = \frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2} \qquad (9.6)$$

substitute from eq. 10.7 gives

$$\frac{T_2}{T_1} = \frac{\{1+[(\gamma-1)/2]M_1^2\}\{[2\gamma/(\gamma-1)]M_1^2\}}{[(\gamma+1)^2/2(\gamma-1)]M_1^2} \qquad (10.2)$$

Density ratio

From state equatio

$$\frac{\rho_2}{\rho_1} = \frac{T_1}{T_2} * \frac{p_2}{p_1}$$

and from eqs. (10.1) and (10.2);

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \qquad (10.3)$$

Other interesting ratios can be developed, each as a function of only M_1. For example, since

$$p_o = p\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

$$\frac{p_{o2}}{p_{o1}} = \frac{p_2}{p_1}\left(\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2}\right)^{\gamma/(\gamma-1)} \qquad (10.4)$$

Eliminating of M_2 and substitute from eq. (10.4)

$$\frac{p_{o2}}{p_{o1}} = \frac{p_2}{p_1}\left(\frac{[(\gamma+1)/2]M_1^2}{1+[(\gamma-1)/2]M_1^2}\right)^{\gamma/(\gamma-1)} * \left[\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{\gamma/(\gamma-1)} \qquad (10.5)$$

10.2 Area ratio

For isentropic flow, the area at which the Mach number is equal to 1 was defined as A*, with this area being used as a reference. A normal shock, however, is not an isentropic process; so, for example, if a shock occurs in a channel (Figure 10.2a), flow areas downstream of the shock ($2\ to\ exit$) have $A_2^* = A_e^*$ and for the flow areas upstream the shock ($inlet\ to$ 1). have $A_i^* = A_1^*$. But $A_{i1}^* \neq A_{2e}^*$ since flow upstream the shock differs from that downstream the shock.

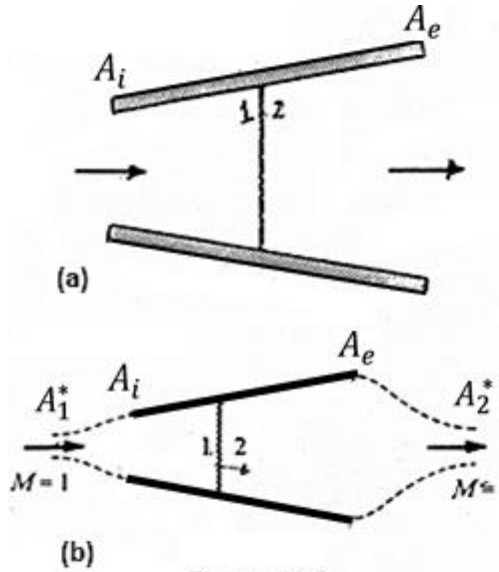


Figure 10.2

It is sometimes convenient to have a relationship between A_i^* and A_e^* . From Figure (10.2b), apply the continuity equation between A_{i1}^* and A_{e2}^*, assuming a perfect gas with constant specific heats. Since mass flow at A_{i1}^* equal mass flow at A_{e2}^* . From Eq. (8-5),

$$\dot{m} = \frac{p_o A}{R\sqrt{T_o}} f(\gamma, M) \qquad (5.8)$$

$$\dot{m} = \frac{p_{o1}A_{i1}^*}{R\sqrt{T_{o1}}} f(\gamma, M^*) = \frac{p_{o2}A_{2e}^*}{R\sqrt{T_{o2}}} f(\gamma, M^*)$$

But $M = 1$ at A_{i1}^* and A_{e2}^*. Also $T_{o1} = T_{o2}$ and γ is constant, then;

$$p_{o1}A_{i1}^* = p_{o2}A_{2e}^* \qquad (10.6a)$$

$$\frac{p_{o2}}{p_{o1}} = \frac{A_{i1}^*}{A_{2e}^*} \qquad (10.6b)$$

10.3 Entropy Change

Since flow through the shock is not isentropic, the there are a friction losses appear as increase in entropy. From the following thermodynamic relation

$$\delta q = dh - vdp$$

$$Tds = c_p dT - \frac{dp}{\rho}$$

$$\frac{ds}{R}=\frac{c_p}{R}\frac{dT}{T}-\frac{dp}{p}$$

$$\frac{\Delta s_{12}}{R}=\frac{c_p}{R}ln\frac{T_2}{T_1}-ln\frac{p_2}{p_1}$$

Subdtitute from eq.(10.4) for p_2/p_1 and (9.4) for T_2/T_1 , gives;

$$\frac{\Delta s_{12}}{R}=\frac{c_p}{R}ln\left[\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right]-ln\left\{\frac{p_{o2}}{p_{o1}}\left(\frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2}\right)^{\gamma/(\gamma-1)}\right\}$$

$$\frac{c_p}{R}=\frac{\gamma}{\gamma-1}$$

$$\frac{\Delta s_{12}}{R}=-ln\frac{p_{o2}}{p_{o1}} \qquad or \qquad \Delta s_{12}=-Rln\frac{p_{o2}}{p_{o1}} \tag{10.7}$$

As $\Delta s \geq 0$ then $p_{o1}\geq p_{o2}$ for stationary (fixed) normal shock wave.

Values of Mach number M_2 from eq. (9.10) , and for pressure ratio p_2/p_1 from eq. (10.1) and for temperature ratio T_2/T_2 from eq. (10.2), and for density ratio ρ_2/ρ_1 from eq.(10.3) and for stagnation pressure ratio p_{o2}/p_{o1} from eq.(10.4), as well as the value of the ratio (p_1/p_{o2}) are all computed in terms of M_1 and have been tabulated in normal shock table.

For an adiabatic process, stagnation pressure represents a measure of available energy of the flow in a given state. A decrease in stagnation pressure, or increase in entropy, denotes an energy dissipation or loss of available energy.

The shock phenomenon is a one-way process (i.e., irreversible). It is always a compression shock, and for a normal shock the flow is always supersonic before the shock and subsonic after the shock. One can note from the table that as M_1 increases, the pressure, temperature, and density ratios increase, indicating a stronger shock (or compression). One can also note that as M_1 increases, p_{o2}/p_{o1} decreases, which means that the entropy change increases. Thus *as the strength of the shock increases, the losses also increase*.

Velocity Change

We can also develop a relation for the velocity change across a standing normal shock for use later. Starting with the basic continuity equation;

$$\rho_1V_1=\rho_2V_2$$

$$\frac{\rho_2}{\rho_1}=\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \tag{10.3}$$

$$\frac{V_2}{V_1}=\frac{\rho_1}{\rho_2}=\frac{(\gamma-1)M_1^2+2}{(\gamma+1)M_1^2}$$

Subtract one from each side

$$\frac{V_2}{V_1} - 1 = \frac{(\gamma - 1)M_1^2 + 2}{(\gamma + 1)M_1^2} - 1$$

$$\frac{V_2 - V_1}{V_1} = \frac{(\gamma - 1)M_1^2 + 2 - (\gamma + 1)M_1^2}{(\gamma + 1)M_1^2}$$

$$\frac{V_2 - V_1}{M_1 a_1} = \frac{2(1 - M_1^2)}{(\gamma + 1)M_1^2}$$

$$\frac{V_2 - V_1}{a_1} = \left(\frac{2}{\gamma + 1}\right)\left(\frac{M_1^2 - 1}{M_1}\right) \qquad (10.4)$$

Example 10.1 An airstream with a velocity of $500\ m/s$, a static pressure of $50\ kPa$, and a static temperature of $250\ K$ undergoes a normal shock. Determine the air velocity and the static and stagnation conditions after the wave.

Solution

The Mach number of the airstream, M_1, is given by

$$M_1 = \frac{V_1}{\sqrt{\gamma R T_1}} = \frac{500}{\sqrt{1.4 * 287 * 250}} = 1.578$$

From table B

$T_2/T_2 = 1.373$,$p_2/p_1 = 2.739$, $\rho_2/\rho_1 = 1.995$, $p_{t2}/p_{t1} = 0.9033$ and $M_2 = 0.675$

From continuity equation

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2}$$

$$V_2 = \frac{V_1}{\rho_2/\rho_1} = \frac{500}{1.995} = 250.6\ m/s$$

$$p_2 = 50 * 2.739 = 137.0\ kN/m^2$$

$$T_2 = 250 * 1.373 = 343.3\ K$$

$$T_{o1} = T_1\left(1 + \frac{\gamma - 1}{2}M^2\right) = 250\left[1 + \frac{1.4 - 1}{2}(1.578)^2\right] = 374.5\ K$$

$$p_{o1} = p_1\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)} = 50\left[1 + \frac{1.4 - 1}{2}(1.578)^2\right]^{1.4/(1.4-1)}$$

$$= 205.7\ kN/m^2$$

Or, for stationary (fixed) normal shock $T_{o1} = T_{o2}$, and from table A;

$$\frac{T_1}{T_{o1}} = 0.6670 \ and \ \frac{P_1}{P_{o1}} = 0.2450 \ K$$

$$T_{o1} = \frac{T_1}{T_1/T_{o1}} = \frac{250}{0.667} = 374.8 \ K$$

$$p_{o1} = \frac{p_1}{p_1/p_{o1}} = \frac{50}{0.2450} = 205.8 \ kN/m^2$$

$$p_{o2} = p_{o1} * p_{o2}/p_{o1} = 205.8 * 0.9033 = 185.9 \ kN/m^2$$

Example 10.2 An airstream at Mach 2.0, with pressure of 100 kPa and temperature of 270 K, enters a diverging channel, with a ratio of exit area to inlet area of 3.0 (see Figure 10.3). Determine the back pressure necessary to produce a normal shock in the channel at an area equal to twice the inlet area. Assume one-dimensional steady flow, with the air behaving as a perfect gas with constant specific heats; assume isentropic flow except for the normal shock.

Solution

At $M = 2.0$, from table A with $\gamma = 1.4$;

$$\frac{A_i}{A_{i1}^*} = 1.688$$

Therefore,

$$\frac{A_1}{A_{i1}^*} = \frac{A_1}{A_i} * \frac{A_i}{A_{i1}^*} = 2.0 * 1.688 = 3.376$$

Then from table A at $A/A^* = 3.376$ we have $M_1 = 2.762$.

With the shock Mach number determined, ratios of properties across the shock can be found from normal shock table;

$$\frac{p_{o2}}{p_{o1}} = 0.4021 = \frac{A_{i1}^*}{A_{2e}^*}$$

$$\frac{A_e}{A_{2e}^*} = \frac{A_e}{A_i} * \frac{A_i}{A_{i1}^*} * \frac{A_{i1}^*}{A_{2e}^*} = 3.0 * 1.688 * 0.4021 = 2.043$$

Flow after the shock is subsonic, so that, from table A, the Mach number at exit,

$M_e = 0.299$. We can now solve for exit, p_e;

$$\frac{p_e}{p_i} = \frac{p_e}{p_{t2}} * \frac{p_{t2}}{p_{t1}} * \frac{p_{t1}}{p_i} = 0.9399 * 0.4021 * \frac{1}{0.1278} = 2.957$$

$$\therefore p_e = p_i * \frac{p_e}{p_i} = 100 * 2.957 = 295.7 \ kPa = p_{back}$$

With subsonic flow at the channel exit, the channel back pressure is equal to the exit plane pressure.

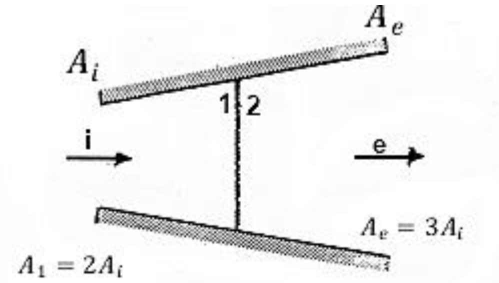


Figure 10.3

Example 10.3 Helium with $\gamma = 1.67$ is flowing at a Mach number of 1.80 and enters a normal shock. Determine the pressure ratio across the shock.

Solution

Since normal shock table does not include $\gamma = 1.67$, we use equation (10.7) to find the Mach number after the shock and (10.2) to obtain the pressure ratio.

$$M_2^2 = \frac{M_1^2 + 2/(\gamma - 1)}{\left[2\gamma/\left((\gamma - 1)\right)\right]M_1^2 - 1} \qquad (10.7)$$

$$M_2^2 = \frac{(1.8)^2 + 2/(1.67 - 1)}{\left[21.67/\left((1.67 - 1)\right)\right](1.8)^2 - 1} = 0.411$$

$$M_2 = 0.641$$

$$\frac{p_1}{p_2} = \frac{(1 + \gamma M_2^2)}{(1 + \gamma M_1^2)} \qquad (10.7)$$

$$\frac{p_2}{p_1} = \frac{(1 + 1.67\,(1.8)^2)}{(1 + 1.67\,(0.411)^2)} = 3.80$$

Example 10.4 A rocket exhaust nozzle has a ratio of exit to throat areas of 4.0. The exhaust gases are generated in a combustion chamber with stagnation pressure equal to 3.0 MPa. and stagnation temperature equal to $1500\,K$ (see figure 10.4). Assume the exhaust-gas mixture to behave as a perfect gas with $\gamma = 1.3$ and $molecular\ mass = 20$.

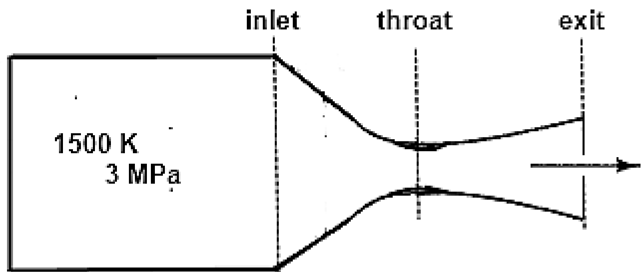


Figure 10.4

Determine the rocket exhaust velocity for isentropic nozzle flow and for the case where a normal shock is located just inside the nozzle exit plane.

Solution

For isentropic flow in the exhaust nozzle, with $A_e/A^* = 4.0$, from isentropic Table (at $\gamma = 1.3$). $M_e = 2.77$, $T_e/T_o = 0.4643$

$$T_e = T_o * T_e/T_o = 1500 * 0.4643 = 696.5\ K$$

$$R = \frac{\bar{R}}{\bar{M}} = \frac{8.3143}{20} = 415.7\ J/kg.K$$

$$V_e = M_e\sqrt{\gamma R T_e} = 2.77 * \sqrt{1.3 * 415.7 * 696.5} = 699\ m/s$$

Consider next the case of a normal shock at the nozzle exit plane. With isentropic flow up to the shock wave, $M_1 = 2.77$ *and* $T_{o2} = T_{o1} = 1500\ K$.

From normal shock table ($\gamma = 1.3$), *at* $M_1 = 2.77$ *gives;* $M_2 = 0.4680$.
From isentropic table ($\gamma = 1.3$), *at* $M_2 = 0.4680$ *gives;* $T_2/T_{o2} = 0.9681$
$T_2 = T_e = T_{o2} * T_2/T_{o2} = 1500 * 0.9681 = 1452\ K$
$V_2 = V_e = M_2\sqrt{\gamma R T_2} = 0.4680 * \sqrt{1.3 * 415.7 * 1452} = 414.6\ m/s$

Example 10.5 Fluid is air and can be treated as a perfect gas. If the conditions before the shock are: $M_1 = 2.0$, $p_1 = 138\ kPa$, and $T_1 = 278\ K$. Determine the conditions after the shock and the entropy change across the shock.

solution
First we compute p_{o1} with the aid of the isentropic table. From isentropic table at $M_1 = 2.0$ we have $p_1/p_{o1} = 0.1278$.

$$p_{o1} = p_1 * p_{o1}/p_1 = \frac{1}{0.1278} * 138 = 1079.812\ kPa$$

Now from the normal-shock table, Table B, opposite $M_1 = 2.0$, we find
$M_2 = 0.57735$, $p_2/p_1 = 4.500$, $T_2/T_1 = 1.6875$, $p_{o2}/p_{o1} = 0.72087$
Thus
$p_2 = p_1 * p_2/p_1 = 138 * 4.500 = 621\ kPa$
$T_2 = T_1 * T_2/T_1 = 278 * 1.6875 = 469.125\ K$
$p_{o2} = p_{o1} * p_{o2}/p_{o1} = 1079.812 * 0.72087 = 778.404\ kPa$
Also p_{o2} can be computed with the aid of the isentropic table $M_2 = 0.57735$, $p_2/p_{o2} = 0.7978$

$$p_{o2} = p_2 * p_2/p_{o2} = 621 * \frac{1}{0.7978} = 778.4\ kPa$$

To compute the entropy change, we use equation (8.19):

$$\Delta s_{12} = -Rln\frac{p_{o2}}{p_{o1}}$$

$$\Delta s_{12} = -287\ ln\frac{778.4}{1079.812} = 0.087\ J/kg.K$$

Example 10.6 Air has a temperature and pressure of 300 K and 2 bar abs., respectively. It is flowing with a velocity of $868\ m/s$ and enters a normal shock. Determine the density before and after the shock.

Solution

$$\rho_1 = \frac{p_1}{RT_1} = \frac{2 \times 10^5}{287 * 300} = 2.32\ kg/m^2$$

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 * 287 * 300} = 347\ \text{m/s}$$

$$M_1 = \frac{V_1}{a_1} = \frac{868}{347} = 2.50$$

From shock Table B; at $M_1 = 2.50$, gives; $p_2/p_1 = 7.125$, and $T_2/T_1 = 2.138$

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} * \frac{T_1}{T_2} = 7.125 * \frac{1}{2.138} = 3.333$$

$$\rho_2 = \rho_1 * \frac{\rho_2}{\rho_1} = 2.32 * 3.333 = 7.73\ kg/m^2$$

Example 10.7 Oxygen enters the converging section shown in the figure (10.5), and a normal shock occurs at the exit. The entering Mach number is 2.8 and the area ratio $A_1/A_2 = 1.7$. Compute the overall static temperature at exit if the inlet temperature is $300\ K$. Neglect all frictional losses.

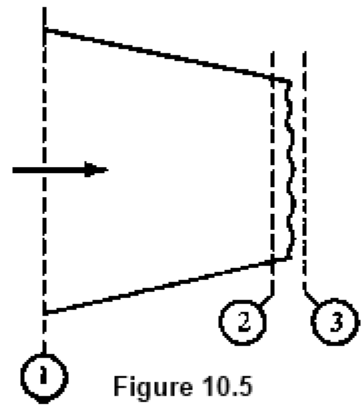


Figure 10.5

Solution

From isentropic flow isentropic table at $M_1 = 2.8$,

$p_1/p_{o1} = 0.3685$, $T_1/T_{o1} = 0.3894$, $A_1/A^* = 3.5$

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1} * \frac{A_1}{A_1^*} * \frac{A_1^*}{A_2^*} = \frac{1}{1.7} * 3.5 * 1 = 2.06$$

From same table at $A_2/A_2^* = 2.06$ we get $M_2 = 2.23$ and $T_2/T_{o2} = 0.5014$

From normal shock wave normal shock table at $M_2 = 2.23$

$M_3 = 0.5431,\quad T_3/T_2 = 1.883$

$$\frac{T_3}{T_1} = \frac{T_3}{T_2} * \frac{T_2}{T_{o2}} * \frac{T_{o2}}{T_{o1}} * \frac{T_{o1}}{T_1} = 1.883 * 0.5014 * 1 * \frac{1}{0.3894} = 2.43$$

$T_3 = T_e = T_1 * T_3/T_1 = 300 * 2.43 = 729\ K$

Chapter Eleven/Normal shock in converging–diverging nozzles

We have discussed the isentropic operations of a converging–diverging nozzle. This type of nozzle is physically distinguished by its ***area ratio***, the ratio of the exit area to the throat area. Furthermore, its flow conditions are determined by the ***operating pressure ratio***, the ratio of the receiver (back) pressure to the inlet stagnation (reservoir) pressure ($p_b/p_{reservoir}$). From figure (11.1) we identified two significant critical pressure ratios.

With $p_b = p_r$, there is no flow in the nozzle (curve 1) from figure (11.1a). As p_b is reduced below p_r, subsonic flow is induced through the nozzle, with pressure decreasing to the throat, and then increasing in the diverging portion of the nozzle (curve 2 and 3).. For any pressure ratio above $p_{b,a}/p_r$, for curve (a), the nozzle is not choked and has subsonic flow throughout (typical venturi operation). When the back pressure is lowered to that of curve a, sonic flow occurs at the nozzle throat. Pressure ratio $p_{b,a}/p_r$ is called the ***first critical point*** which represents flow that is subsonic in both the convergent and divergent sections but is choked with a Mach number of 1.0 in the throat. ((**chocked means flow maximum and fixed**))

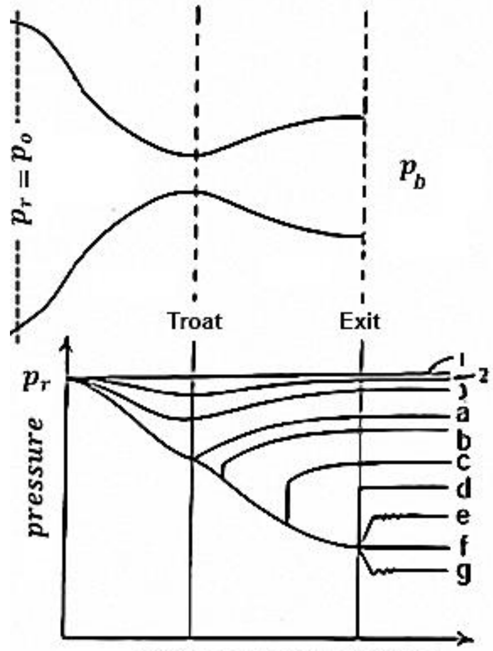


Figure 11.1a

When the back pressure is lowered to that of curve f, subsonic flow exits in the converging section, and sonic flow exits in the throat and it is choked where $M = 1.0$. A supersonic flow exists in the entire diverging section. This is the ***third critical point*** which represents the design operation condition.

The first and third critical points are the only operating points that have;

(1) Isentropic flow throughout the nozzle, and

(2) A Mach number of 1 at the throat, and

(3) Exit pressure equal to receiver (surrounding) pressure.

Remember that with subsonic flow at the exit, $p_e = p_b$, and p_b is back or receiver pressure.

Imposing a pressure ratio slightly below that of the first critical point presents a problem in that there is no way that *isentropic* flow can meet the boundary condition of pressure equilibrium at the exit. However, there is nothing to prevent a *non-isentropic* flow adjustment from occurring within the nozzle. This internal adjustment takes the form of a standing ***normal shock***, which we now know involves an entropy change (losses).

As the pressure ratio is lowered below the first critical point, a normal shock forms just downstream of the throat. The remainder of the *nozzle* is now acting as a diffuser since after the shock the flow is subsonic and the area is increasing. The shock will locate itself in a position such that the pressure changes that occur ahead of the shock, across the shock, and downstream of the shock will produce a pressure that exactly matches the outlet pressure. In other words, ***the operating pressure ratio determines the location and strength of the shock***. An example of this mode of operation is shown in Figure 11.1b.

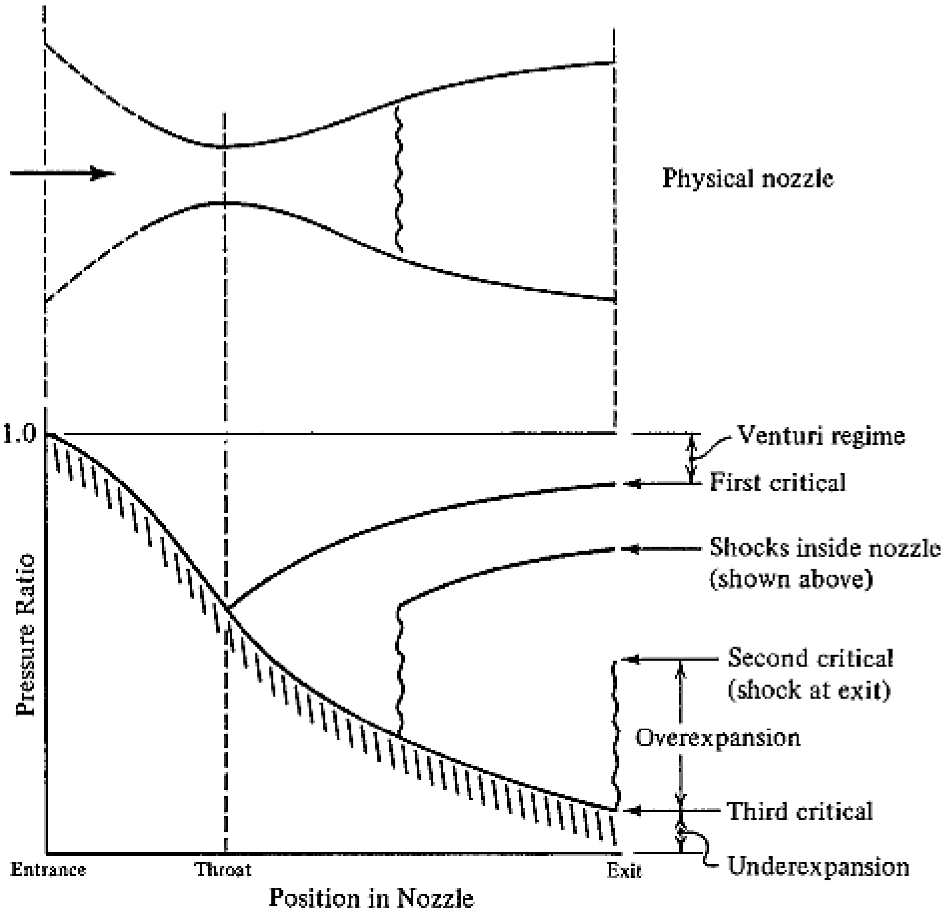


Figure 11.1b

As the pressure ratio is lowered further, the shock continues to move toward the exit. When the shock is located at the exit plane (curve d), this condition is referred to as the ***second critical point***.

When the operating pressure ratio is between the second and third critical points, a compression takes place *outside* the nozzle. This is called ***over-expansion*** (i.e., the flow has been expanded too far within the nozzle). As the back pressure is lowered below that of curve d, a shock wave inclined at an angle to the flow appears at the exit plane of the nozzle (Figure 11.2a). This shock wave, weaker than a normal shock, is called an ***oblique shock***. Further reductions in back pressure cause the angle between the shock and the flow to decrease, thus

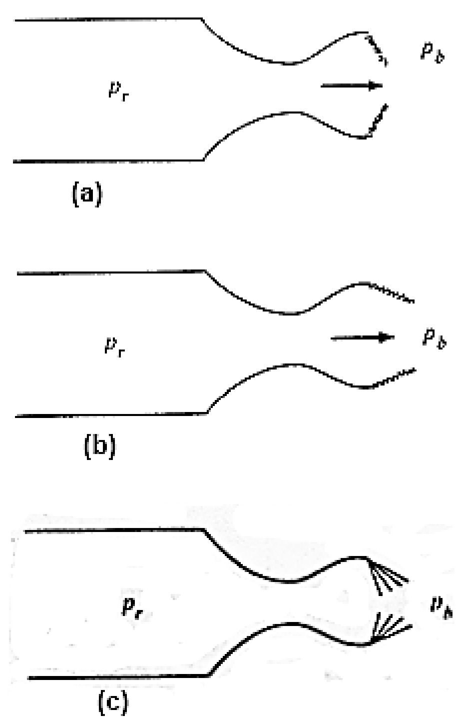


Figure 11.2c

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decreasing the shock strength (Figure 11.2b), until eventually the isentropic case, curve f, is reached

If the receiver pressure is below the third critical point, an expansion takes place *outside* the nozzle. This condition is called ***under-expansion***. A pressure decrease occurs outside the nozzle in the form of expansion waves (Figure 11.2c). Oblique shock waves and expansion waves represent flows that are not one dimensional flow and will be treated later.

Illustrative example:

For the present we proceed to investigate the operational regime between the first and second critical points. For the nozzle and inlet conditions illustrated in figure (11.3), the nozzle has *area ratio* to be $A_3/A_2 = 2.494$ and is fed by air at $6.0\ bar$ and 60 ℃ from a large tank.

Solution

The inlet conditions are essentially stagnation. For these fixed inlet conditions we find that a receiver pressure of $5.7642\ bar$ (for *operating pressure ratio* of 0.9607) identifies the first critical point and a receiver pressure of $0.3858\ bar$ (for *operating pressure ratio* of 0.06426) identifies the third critical point.

What receiver pressure do we need to operate at the second critical point? Figure 11.4 shows such a condition and you should recognize that the entire nozzle up to the shock is operating at its design or third critical condition.

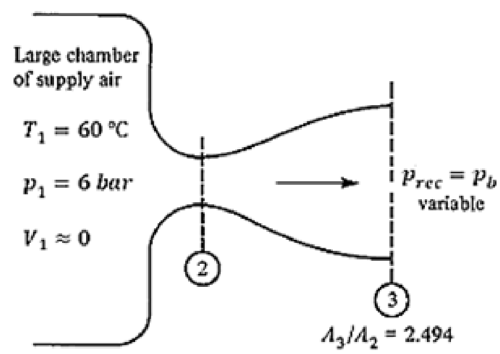

Figure 11.3 : coverging diverging nozzle

From the isentropic table at $A/A* = 2.494$,

$M_3 = 2.44 \quad and \quad p_3/p_{o3} = 0.06426$

From the normal-shock table for $M_3 = 2.44$,

$M_4 = 0.5189 \quad and \quad \dfrac{p_4}{p_3} = 6.7792$

and the operating pressure ratio will be

$$\frac{p_{rec}}{p_{o1}} = \frac{p_4}{p_{o3}} = \frac{p_4}{p_3} * \frac{p_3}{p_{o3}}$$

$$= 6.7792 * 0.06426 = 0.436$$

$p_1 = p_{reservoir=}p_{o1} = 6.0\ bar$

$p_4 = p_{receiver} = 6.0 * 0.436 = 2.616\ bar$

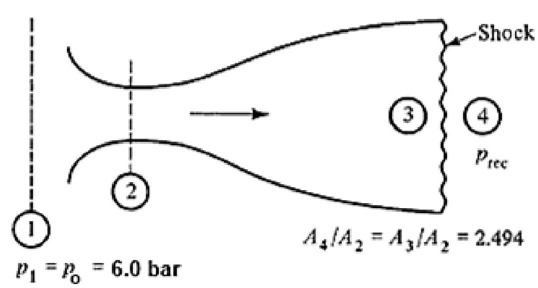

Figure 11.4: C.D.N operates at 2nd critical point

Thus for our converging–diverging nozzle with an area ratio of 2.494, any operating pressure ratio between 0.9607 and 0.436 will cause a normal shock to be located someplace in the diverging portion of the nozzle starting from the throat and ending at exit plane.

Suppose that we are given an operating pressure ratio of 0.60. The logical question to ask is: Where is the shock? This situation is shown in Figure 11.5. We must take advantage of the only two available pieces of information and from these construct a solution. We know that

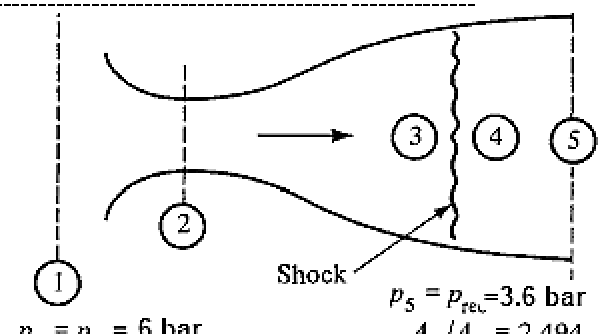

Figure 11.5

$$\frac{A_5}{A_2} = 2.494 \quad and \quad \frac{p_5}{p_{o1}} = 0.60$$

We assume that all losses occur across the shock and we know that $M_2 = 1.0$. Since there are no losses up to the shock, the flow is isentropic and we know that

$$A_2 = A_1^*$$

Thus

$$\frac{A_5}{A_2} * \frac{p_5}{p_{o1}} = \frac{A_5}{A_1^*} * \frac{p_5}{p_{o1}}$$

We know also across the normal shock $p_{o5}\, A_5^* = p_{o1}\, A_1^*$, i.e.

$$\frac{p_{o5}}{p_{o1}} = \frac{A_1^*}{A_5^*}$$

So

$$\frac{A_5}{A_1^*} * \frac{p_5}{p_{o1}} = \frac{A_5}{A_5^*} * \frac{p_5}{p_{o5}}$$

The following data is known, $A_5/A_2 = 2.494$, $p_5/p_{o1} = 0.60$ then;

$$\frac{A_5}{A_5^*}\frac{p_5}{p_{o5}} = 2.494 * 0.60 = 1.4964$$

And from isentropic table at $A_5\, p_5/A_5^*\, p_{o5} = 1.4964$

$M_5 \approx 0.38$ and $p_5/p_{o5} = 0.9052$

To locate shock position, we seek the ratio p_{o4}/p_{o3}.

We have $p_{o5} = p_{o4}$, isentropic after the shock, and $p_{o3} = p_{o1}$, isentropic before the shock. Then

$$\frac{p_{o4}}{p_{o3}} = \frac{p_{o5}}{p_{o1}} = \frac{p_{o5}}{p_5} * \frac{p_5}{p_{o1}} = \frac{1}{0.902} * 0.60 = 0.664$$

Then from normal shock table at $p_{o4}/p_{o3} = 0.664$

$M_3 = 2.12 \;\; and \;\; M_4 = 0.5583$

And then from the isentropic table that this Mach number, $M_3 = 2.12$, will occur at an area ratio of about $A_3/A^* = A_3/A_2 = 1.869$..

We see that if we are given a physical converging–diverging nozzle (area ratio is known) and an operating pressure ratio between the first and second critical points, it is a simple matter to determine the position and strength of the normal shock in the diverging section.

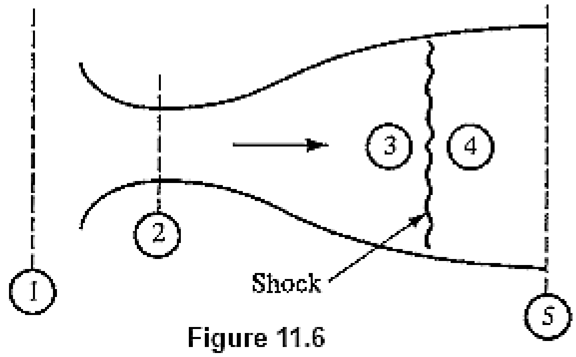

Figure 11.6

Example 11.1 A converging–diverging nozzle has an area ratio of 3.50. At off-design conditions, the exit Mach number is observed to be 0.3. What operating pressure ratio would cause this situation?

Solution

We have the nozzle area ratio $A_5/A_2 = 3.5$.

Using the section numbering system of Figure 10.6, for $M_5 = 0.3$, We have

$$\frac{A_5}{A_5^*}\frac{p_5}{p_{o5}} = 1.9119, \qquad \frac{A_5}{A_5^*} = 2.03507$$

$$p_{o5}A_5^* = p_{o1}A_1^*$$

$$\frac{p_5}{p_{o1}} = \left(\frac{p_5A_5}{p_{o5}A_5^*}\right) * \left(\frac{p_{o5}A_5^*}{p_{o1}A_1^*}\right) * \frac{A_1^*}{A_2} * \frac{A_2}{A_5} = 1.9119 * 1 * 1 * \frac{1}{3.50} = 0.546$$

Could you now find the shock location and Mach number?

$$\frac{p_{o5}}{p_{o1}} = \frac{A_1^*}{A_5^*} = \frac{A_1^*}{A_5} * \frac{A_5}{A_5^*} = \frac{1}{3.5} * 2.03507 = 0.58145 = \frac{p_{o4}}{p_{o3}}$$

From shock table at $p_{o4}/p_{o3} = 0.58145$ gives $M_3 =$

From isentropic table at $M_3 =$ gives $A_3/A_3^* = A_3/A_2 =$

Example 11.2 Air enters a converging–diverging nozzle that has an overall area ratio of 1.76. A normal shock occurs at a section where the area is 1.19 times that of the throat. Neglect all friction losses and find the operating pressure ratio. Again, we use the numbering system shown in Figure 11.6.

Solution

From the isentropic table at $A_3/A_2 = 1.19, \rightarrow M_3 = 1.52$.

From the shock table at $M_3 = 1.52, \rightarrow M_4 = 0.6941$ and $p_{o4}/p_{o3} = 0.9233$.

From isentropic table at $M_4 = 0.6941$ gives $A_4/A_4^* = 1.0988$. Then

$$\frac{A_5}{A_5^*} = \frac{A_5}{A_2} * \frac{A_2}{A_3} * \frac{A_4}{A_4^*} = 1.76 * \frac{1}{1.19} * 1.0988 = 1.625$$

Since $A_4 = A_3$ and $A_5^* = A_4^*$

Thus from isentropic Table at $A_5/A_5^* = 1.625 \rightarrow M_5 \approx 1.625$.

$$\frac{p_5}{p_{o1}} = \frac{p_5}{p_{o5}} * \frac{p_{o4}}{p_{o3}} = 0.9007 * 0.9233 = 0.8324$$

Where $p_{o5} = p_{o4}$ and $p_{o3} = p_{o1}$

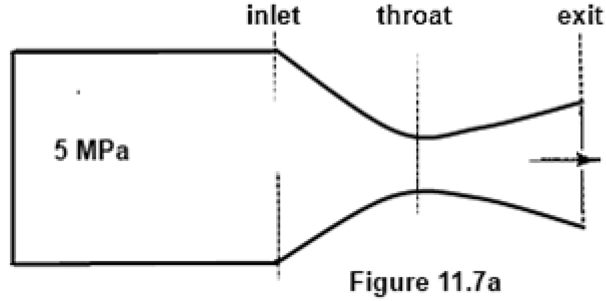

Figure 11.7a

Example 11.3 A converging-diverging nozzle is designed to operate with an exit Mach number of 1.75. The nozzle is supplied from an air reservoir at 5 MPa. Assuming one-dimensional flow, calculate the following:

a) Maximum back pressure to choke the nozzle.
b) Range of back pressures over which a normal shock will appear in the nozzle.
c) Back pressure for the nozzle to be perfectly expanded to the design Mach number.
d) Range of back pressures for supersonic flow at the nozzle exit plane.

Solution

The nozzle is designed for $M_{exit} = 1.75$. From Appendix A. at $M_{exit} = 1.75$, $A_{exit}/A^* = 1.386$ and $p_{exit}/p_o = 0.1878$

a) The nozzle is choked with $M = 1$ at the throat, followed by subsonic flow in the diverging portion of the nozzle. From Appendix A. at $A_{exit}/A^* = 1.386$. $M_{exit} = 0.477$ and $p_{exit}/p_o = 0.8558$.

$p_{exit} = p_{exit}/p_o * p_o = 0.8558 * 5 = 4.279\ MPa$

Therefore the nozzle is chocked for all back pressures bellow 4.279 MPa.

b) Or a normal shock at the nozzle exit plane (Figure 11.7b). $M_1 = 1.75$ and

$p_1 = 0.1878 * 5 = 0.939\ MPa$.

From normal shock, at $M_1 = 1.75$, $p_2/p_1 = 3.406$.

For a normal shock at the nozzle exit, the back pressure is

$p_b = 3.406(0.939) = 3.198\ MPa$.

For a shock just downstream of the nozzle throat, the back pressure is $p_b = 4.279\ MPa$, i.e. the flow downstream the throat in the divergent part is subsonic. So A normal shock will appear in the nozzle over the range of back pressures from 3.198 to 4.279 MPa.

c) From isentropic table , at $M_{exit} = 1.75$. $p_{exit}/p_o = 0.1878$. For a perfectly expanded, supersonic nozzle. the hack pressure is $0.939\ MPa$

d) Referring again to Figure 11.7a supersonic flow will exist at the nozzle exit plane for all back pressures less than $3.198\ MPa$.

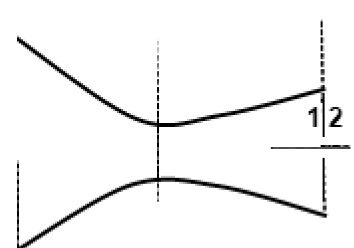

Figure 11.7b

Chapter Twelve/Converging–Diverging Supersonic Diffusers

12.1 Converging-Diverging Supersonic Diffuser

With the jet engine, the inlet (diffuser) takes the incoming air, traveling at high velocity with respect to the engine, and slows it down and then delivers it to the axial compressor of the turbojet or the combustion zone of the ramjet engine. The amount of static pressure rise achieved during deceleration of the flow in the diffuser is very important to the operation of the jet engine, since the pressure of the air entering the nozzle affects the nozzle exhaust velocity.

The maximum pressure that can be achieved in the diffuser is the isentropic stagnation pressure. Any loss in available energy (or stagnation pressure) in the diffuser, or for that matter in any other component of the engine, will have a harmful effect on the operation of the engine as a whole. For a supersonic diffuser, it would be highly desirable to provide shock free isentropic flow.

A first approach is to operate a converging-diverging nozzle in reverse (see Figure 12.1.) At the design Mach number, M_D, for such a diffuser, there is no loss in stagnation pressure (neglecting friction). However, off-design performance has to be considered, since the external flow must be accelerated to the design condition. For example, if a supersonic converging-diverging diffuser is to be designed for a flight $M_D = 2.0$, the ratio A_{inlet}/A_{throat} is 1.688 (see isentropic flow table).

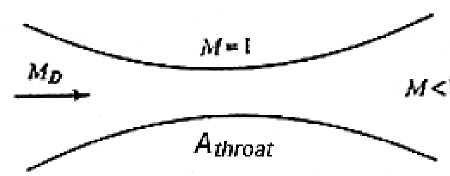

Figure 12.1

However, for a supersonic flight Mach number less than design Mach number, $M < M_D$, the area ratio A/A^* is less than 1.668, i.e. required throat area should be larger. This indicates that the actual throat area is not large enough to handle this flow. Under these conditions, flow must be bypassed around the diffuser. A normal shock stands in front of the diffuser with subsonic flow after the shock able to sense the presence of the inlet and an appropriate amount of the flow "spills over" or bypasses the inlet (see Figure 12.2).

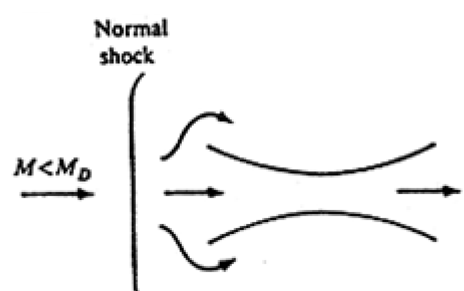

Figure 12.2

As the Might Mach number is increased, the normal shock moves toward the inlet lip. When the design Mach number is reached during start-up, however, with a normal shock in front of the diffuser, some of the flow must still be bypassed, since the throat area of less than A_2^* is still not able to handle the" entire subsonic flow after the shock.

As the flight Mach number is increased above M_D, the shock moves eventually to the inlet lip. A further increase in M causes the shock to reach a new equilibrium position in the diverging portion of the diffuser, in other words, the shock is "swallowed." Once the shock has been swallowed, a decrease in flight Mach number causes the shock to move back toward the throat, where it reaches an equilibrium position for M equal to M_D.

At this position, the shock is of vanishing strength, at $M_t = 1.0$, so no loss in stagnation pressure occurs at the design condition. In actual operation, it is desirable to operate with the shock slightly past the throat; since operation at the design condition is unstable in that a slight decrease in Mach number results in the shock's moving back out in front of the inlet. In this case, the operation of over speeding to swallow the shock would have to be repeated (see Figure 12.3).

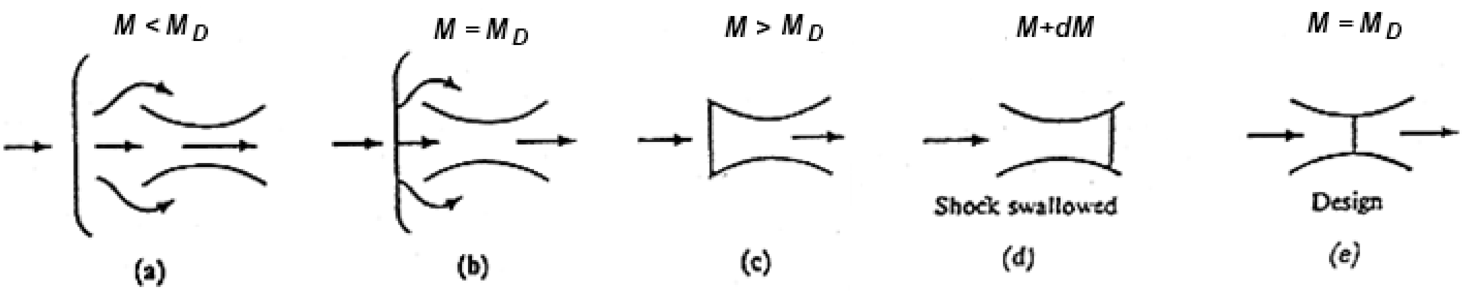

Figure 12.3

Another method for swallowing the shock is to use a variable throat area. With a shock in front of the diffuser, the throat area should be increased, which would allow more flow to pass through the inlet and consequently bring the shock closer to the inlet lip. To swallow the shock, the throat area would have to be slightly larger than that required to accept the flow with a shock at M_D at the inlet lip, that is, slightly larger than A_2^* with a normal shock at the design Mach number.

For $M_D = 2.0$, $A_1^*/A_2^* = 0.7209$, so an increase in area of greater than $(1 - 0.7209)/0.7209 = 39\,percent$ is required to swallow the shock. Once the shock is swallowed, the throat area must be decreased to reach the design condition.

Although the converging-diverging diffuser has favorable operating characteristics at the design condition, it involves severe losses at off-design operation. Operation with a normal shock in front of an inlet causes losses in the stagnation pressure.

To swallow this shock, the inlet must be accelerated beyond its design speed, or a variable throat area must be provided. Except for very low supersonic Mach numbers, the amount of over speeding required to swallow the shock during start-up becomes large enough to be totally impractical.

Furthermore; the incorporation of a variable throat area into a diffuser presents many mechanical difficulties. For these reasons, the converging-diverging diffuser is not commonly used; most engines utilize the oblique-shock type diffuser to be described later.

Example 12.1. A supersonic converging-diverging diffuser is designed to operate at a Mach number of 1.7 with design back pressure. To what Mach number would the inlet have to be accelerated in order to swallow the shock during stand-up?

Solution

From isentropic table at $M_{inlet} = 1.7, \Longrightarrow A/A^* = 1.338$

So the diffuser is designed with $A_{inlet}/A_{throat} = 1.338$

The inlet must be accelerated to a Mach number slightly greater than that required to position the shock at the inlet lip (see Figure 12.4).

Assume a normal shock stands at diffuser lips as shown. For $M = 1.0$ at the diffuser throat and subsonic flow after a shock at the inlet lip, we have:

From isentropic table at $A/A^* = 1.338 \Longrightarrow M_2 = 0.501$.

From normal shock table at $M_2 = 0.501 \Longrightarrow M_1 = 2.63$.

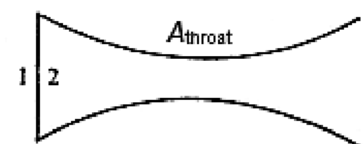

Figure 12.4

If the back pressure conditions imposed on the diffuser are such that a Mach number of 1.0 cannot be achieved at the throat, then M_2 will be less than 0.501, and a value of M_1 greater than 2.63 will be required. However, with $M = 1.0$ at the diffuser throat, the diffuser must be accelerated to a Mach number slightly greater than 2.63 to swallow the initial shock during start-up.

6.7 Supersonic Wind Tunnel

To provide a test section with supersonic flow requires a converging–diverging nozzle. To operate economically, the nozzle–test-section combination must be followed by a diffusing section which also must be converging–diverging.

Starting up such a wind tunnel is another example of nozzle operation at pressure ratios above the second critical point. Figure 12.5 shows a typical tunnel in its ***most unfavorable, off design,*** operating condition, which occurs at startup.

Figure 12.5, which shows the shock located in the test section. The variation of Mach number throughout the flow system is also shown for this case. This is called the most unfavorable condition because the shock occurs at the highest possible Mach number and thus the losses are greatest. We might also point out that the diffuser throat (section 5) must be sized (adjusting area) for this condition.

As the exhauster fan is started, this reduces the pressure $p_{out} = p_6$ and produces flow through the tunnel. At first the flow is subsonic throughout, but at increased power settings the exhauster fan reduces pressures still further and causes increased flow rates until the nozzle throat (section 2) becomes choked. At this point the nozzle is operating at its first critical condition. As power is increased further, i.e the ratio p_{out}/p_{in} is lowered further. a normal shock is formed just downstream of the throat, and if the tunnel pressure is decreased continuously, the shock will move down the diverging portion of the nozzle and pass rapidly through the test section and into the diffuser. If the ratio p_{out}/p_{in} is lowered further then the diffuser swallows the normal shock to the diverging part

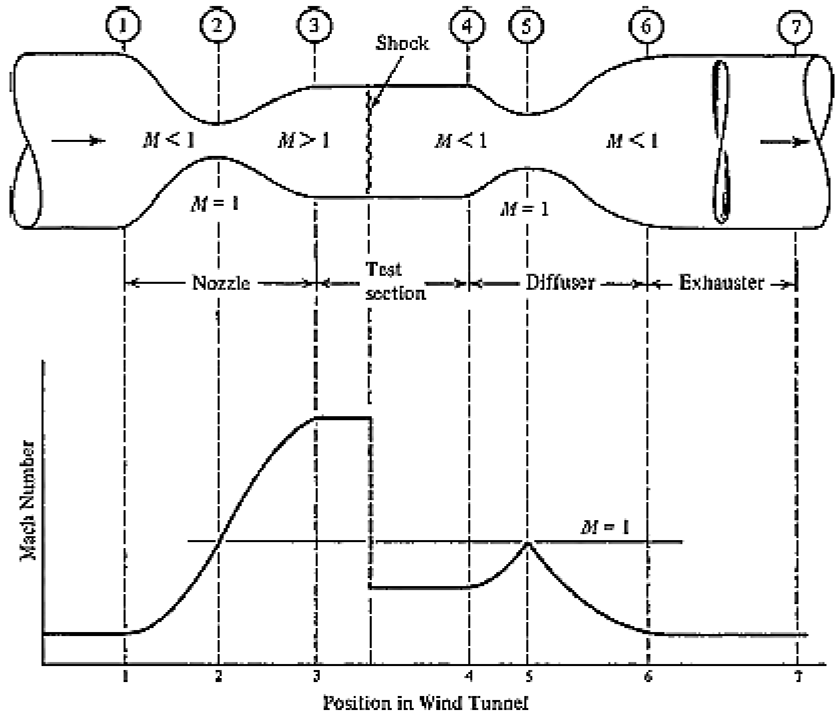

Figure 12.5 Supersonic tunnel at startup (with associated Mach number variation).

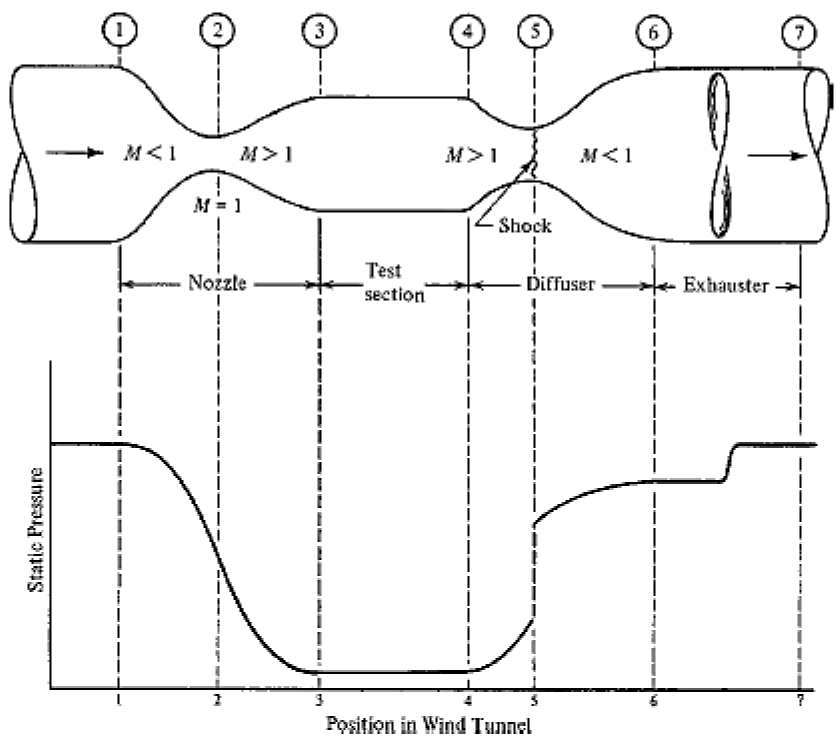

Figure 12.6 Supersonic tunnel in running condition (with associated pressure variation).

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of diffuser. Increasing this pressure ratio a little will move the normal shock upstream to the diffuser throat, the position at which the shock strength is a minimum .Figure 12.6 shows this general running condition, which is called the most favorable condition.

Across the shock of figure 12.5

$p_{o2}A_2^* = p_{o5}A_5^*$

At throat section 2 & 5 during start-up $M = 1$, then

$p_{o2}A_2 = p_{o5}A_5$

Due to the shock losses (and other friction losses) $p_{o5} < p_{o2}$ and then $A_5 > A_2$

For example if the test section Mach number is 2 then from normal shock table

$$\frac{p_{o5}}{p_{o2}} = 0.7209 = \frac{A_2}{A_5}$$

And $A_5 = 1/0.7209\ A_2 = 1.3872\ A_2$

Knowing the test-section-design Mach number fixes the shock strength in this unfavorable condition and A_5 is easily determined. Keep in mind that this represents a minimum area for the diffuser throat. If it is made any smaller than this, the tunnel could never be started (i.e., we could never get the shock into and through the test section). In fact, if A_5 is made too small, the flow will choke first in this throat and never get a chance to reach sonic conditions in section 2.

Once the shock has passed into the diffuser throat, knowing that $A_5 > A_2$ we realize that the tunnel can never run with sonic velocity at section 5. Thus, to operate as a diffuser, there must be a shock at this point, as shown in Figure 12.6.We have also shown the pressure variation through the tunnel for this running condition.

To keep the losses during running at a minimum, the shock in the diffuser should occur at the lowest possible Mach number, which means a small throat. However, we have seen that it is necessary to have a large diffuser throat in order to start the tunnel. A solution to this dilemma would be to construct a diffuser with a variable area throat. After startup, A_5 could be decreased, with a corresponding decrease in shock strength and operating power. However, the power required for any installation must always be computed on the basis of the unfavorable startup condition.

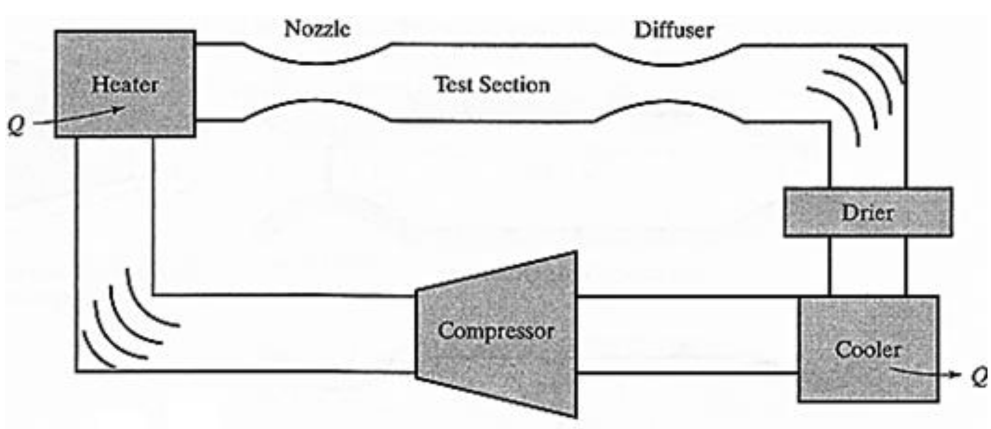


Figure 12.7 Continuous Closed-Circuit Supersonic Wind Tunnel

Example:

A continuous supersonic wind tunnel is designed to operate at a test section Mach number of 2.0, with static conditions duplicating those at an altitude of 20 km where $p = 5.5\ kPa$ and $T = 216.7\ K$. Take $\gamma = 1.4$ and $c_p = 1.004\ kJ/kg.K$. The test section is to be circular, 25 cm in diameter, with a fixed geometry and with a supersonic diffuser downstream of the test section. Neglecting friction and boundary-layer effects, determine the power requirements of the compressor during startup and during steady-state operation, [See Figure 12.8(a)]. Assume an isentropic compressor, with a cooler located between compressor and nozzle (after the compressor), so the compressor inlet static temperature can be assumed equal to the test section stagnation temperature.

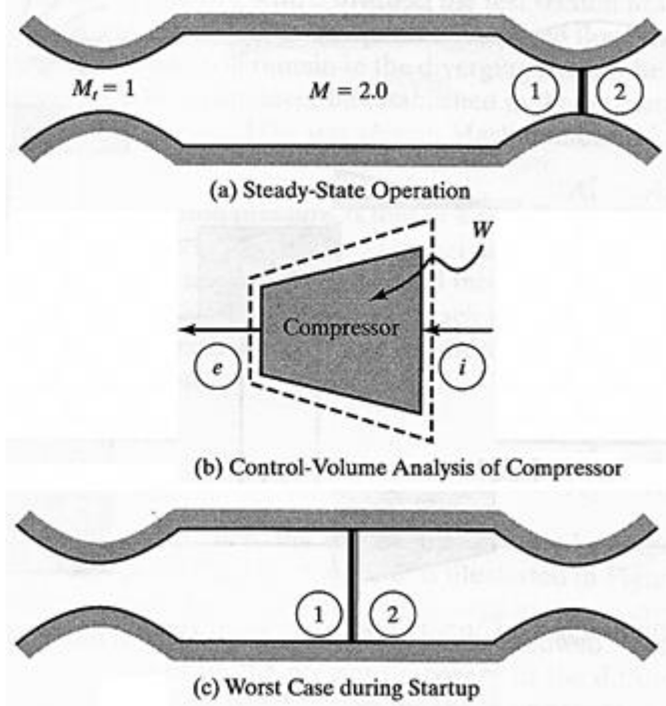


Figure 12.8 Continuous Supersonic Wind Tunnel

solution

During startup, the worst possible case [see Figure 12.8(c)] is that of a shock in the test section with $M_1 = 2.0$. For this situation, which fixes the ratio of the two throat areas, we have

$$\frac{p_{o2}}{p_{o1}} = 0.7209 = \frac{A_1^*}{A_2^*} = \frac{A_{t1}}{A_{t2}}$$

To fix the size of the diffuser throat area, we first use the design Mach number to find $(A/A^*)_{test}$. From isentropic table, $(A/A^*)_{test} = 1.6875$

$$A_{test} = \pi\frac{D^2}{4} = \pi\frac{0.25^2}{4} = 0.04909\ m^2$$

$$(T/T_o)_{test} = 0.5556$$

$$T_{o1} = \frac{T_1}{(T/T_o)_{test}} = \frac{216.7}{0.5556} = 390.03\ K$$

The throat area is then;

$$A_1^* = A_{t1} = \frac{A_{test}}{(A/A^*)_{test}} = \frac{0.04909}{1.6875} = 0.02909\ m^2$$

$$A_2^* = A_{t2} = \frac{A_1^*}{A_1^*/A_2^*} = \frac{0.02909}{0.7209} = 0.04035\ m^2$$

During steady-state operation [see Figure 12.8(a)], the mass flow through the test section is given by

$$\dot{m} = \rho AV = \frac{p_{test}}{RT_{test}} AM_{test}\sqrt{\gamma RT_{test}}$$

$$= \frac{5.5}{0.287 * 216.7}(0.04909)2\sqrt{1.4 * 287 * 216.7} = 2.5619\ kg/s$$

For this fixed geometry (i.e., $A_{t2}/A_{t1} = 1/0.7209 = 1.3872$), the optimum condition for steady- state operation is a normal shock at the diffuser throat. This means that the nozzle, test section and the converging part of the diffuser act as a duct of variable area with isentropic flow, where $M_{t1} = 1$ and $A_{t1} = A^* = 0.02909\ m^2$.

From isentropic table at $A_1/A^* = A_{t2}/A_{t1} = 1/0.7209 = 1.3872$

$$M_1 = 1.75 + 0.01\left(\frac{1.38720 - 1.38649}{1.39670 - 1.38649}\right) = 1.7507$$

From normal shock table at $M_1 = 1.7507$

$$\frac{p_{o2}}{p_{o1}} = 0.83457 + (0.83024 - 0.83457)\left(\frac{1.7507 - 1.7500}{1.7600 - 1.7500}\right) = 0.8343$$

The loss in stagnation pressure must be compensated for by the compressor. For isentropic compressor, [see Figure 12.7(b)], the energy balance is

$$w = h_{o,exit} - h_{o,inlet} = c_p\left(T_{o,exit} - T_{o,inlet}\right)$$

At design stage, i.e. steady state operation

$$\frac{T_{o,exit}}{T_{o,inlet}} = \left(\frac{p_{o1}}{p_{o2}}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{1}{0.8343}\right)^{\frac{0.4}{1.4}} = 1.0531$$

$$T_{o,exit} - T_{o,inlet} = T_{o,inlet}(1.0531 - 1) = 390.03 * 0.0531 = 20.72\ K$$

$$w = 1.004(20.72) = 20.8029\ kJ/kg$$

$$Power = \dot{m}w = 2.5619 * 20.8029 = 53.2949\ kW$$

At off-design stage, i.e. during startup

$$\frac{T_{o,exit}}{T_{o,inlet}} = \left(\frac{p_{o1}}{p_{o2}}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{1}{0.7209}\right)^{\frac{0.4}{1.4}} = 1.0980$$

$$T_{o,exit} - T_{o,inlet} = T_{o,inlet}(1.0980 - 1) = 390.03 * 0.0980 = 38.223\ K$$

$$w = 1.004(38.223) = 38.376\ kJ/kg$$

$$Power = \dot{m}w = 2.5619 * 38.376 = 98.3155\ kW$$

A more power is needed during startup by

$$\frac{98.3155 - 53.2949}{53.2949} = 84.47\ \%$$

Chapter Thirteen/Moving Normal Shock Waves

12.1 Moving Normal Shock Waves

Previous sections have dealt with the fixed normal shock wave. However, many physical situations arise in which a normal shock is moving. When an explosion occurs, a shock wave propagates through the atmosphere from the point of the explosion. As a blunt body reenters the atmosphere from space, a shock travels a short distance ahead of the body. When a valve in a gas line is suddenly closed, a shock propagates back through the gas. To treat these cases, it is necessary to extend the procedures already developed for the fixed normal shock wave.

Consider a normal shock moving at constant velocity into still air, $T_{oa} = T_a$, $and\ p_{oa} = p_a$, (Figure 13.1a). Let $V_s = absolute\ shock\ velocity$ and $V_g = velocity\ of\ gase$s behind the wave; both velocities are measured with respect to a fixed observer. For a fixed observer, the flow is not steady, since conditions at a point are dependent on whether or not the shock has passed over that point.

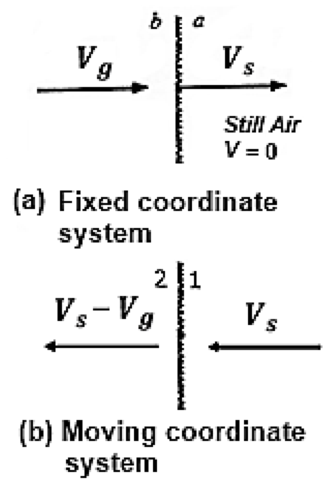


Figure 13.1

Now consider the same physical situation with an observer moving at the shock-wave velocity, a situation, for instance, with the observer "sitting on the shock wave."' The shock is now fixed with respect to the observer (Figure 13.1b). But this is the same case already covered in previously. Relations have been derived and results tabulated for the fixed normal shock-To apply these results to the moving shock, consideration must be given to the effect of observer velocity on static and stagnation properties.

Static properties are defined as those measured with an instrument moving at the absolute flow velocity. Thus static properties are independent of the observer velocity, so

$$\frac{p_2}{p_1} = \frac{p_b}{p_a}\ and\ \frac{T_2}{T_1} = \frac{T_b}{T_a}$$

Stagnation properties are measured by bringing the flow to rest. Comparing the situations shown in Figure 13.1, if $T_1 = T_a$ and $p_1 = p_a$, it is evident that $T_{o1} > T_a$ and $p_{o1} > p_a$ since the gas at state 1 has velocity V_s, and the gas at state a has zero velocity, $T_a = T_{oa}$ and $p_a = p_{oa}$. Thus stagnation properties are dependent on the observer velocity. To calculate the variation of stagnation properties across a moving shock wave, static conditions and velocities must first be determined.

Transformation of a stationary coordinate system to a coordinate system that moves with the shock makes analysis of the moving normal shock as of the steady-flow situation shown in Figure 13.1(b). The relations for stationary normal shock is now prevail.

$V_1 = V_s \qquad V_2 = V_s - V_g$

From continuity eq.:

$$\rho_2\left(V_s - V_g\right) = \rho_1 V_s \qquad \text{... (13.1a)}$$

$$\frac{\rho_1}{\rho_2} = 1 - \frac{V_g}{V_s} = \frac{V_s - V_g}{V_s} \qquad \text{... (13.1b)}$$

From momentum eq.:

$$p_2 + \rho_2\left(V_s - V_g\right)^2 = p_1 + \rho_1 V_s^2 \qquad \text{... (13.2)}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1} M_1^2 - \frac{\gamma-1}{\gamma+1} \qquad \text{... (10.1)}$$

From energy eq.:

$$h_2 + \frac{\left(V_s - V_g\right)^2}{2} = h_1 + \frac{V_s^2}{2} \qquad \text{... (13.4a)}$$

$$T_2 + \frac{\left(V_s - V_g\right)^2}{2c_p} = T_1 + \frac{V_s^2}{2c_p} \qquad \text{... (13.4b)}$$

$$\frac{T_2}{T_1} = \frac{\{1 + [(\gamma-1)/2]M_1^2\}\{[2\gamma/(\gamma-1)]M_1^2\}}{[(\gamma+1)^2/2(\gamma-1)]M_1^2} \qquad \text{... (10.2)}$$

And from eq.10.3 for velocity ratio:

$$\frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} = \frac{T_1}{T_2} * \frac{p_2}{p_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2} \qquad \text{... (10.3)}$$

$$\frac{V_s}{V_s - V_g} = \frac{(\gamma+1)\, V_s^2/\gamma R T_1}{(\gamma-1)\, V_s^2/\gamma R T_1 + 2} \qquad \text{... (13.5)}$$

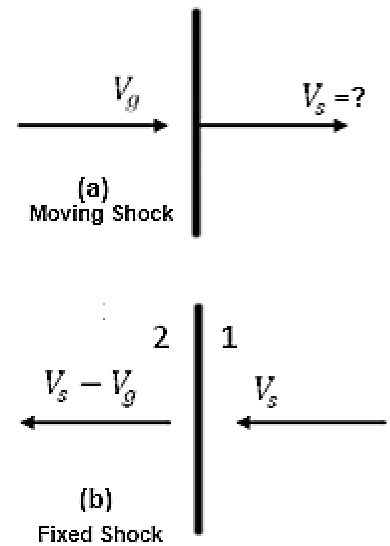


Figure 13.2

❖ First Case:

Either the shock velocity is known or the gas velocity behind the wave is known. When the shock velocity is known the gas velocity and other properties behind the moving wave are required. But when the velocity of the gas behind the shock is known, then shock velocity and other properties are required.

Example 13.1 A normal shock moves at a constant velocity of $500\, m/s$ into still air ($100\, kPa.\ 0°C$). Determine the static and stagnation conditions present in the air after passage of the wave, as well as the gas velocity behind the wave.

Solution

For a fixed observer, the physical situation is shown in Figure 13.3a. With respect to an observer moving with the wave, the situation transforms to that shown in Figure 13.3b.

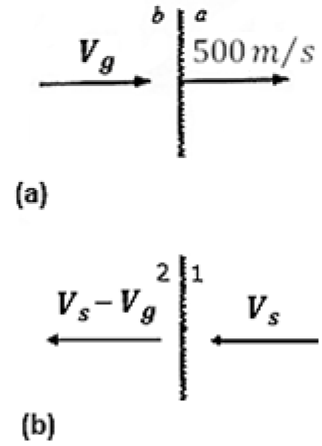


Figure 13.3

$$M_1 = \frac{V_s}{\sqrt{\gamma R T_1}} = \frac{500}{\sqrt{1.4 * 287 * 273}} = 1.510$$

From normal shock table

$$\frac{T_2}{T_1} = 1.327 \ \rightarrow \ T_2 = T_1 * \frac{T_2}{T_1} = 273 * 1.327 = 362.3\ K$$

$$\frac{p_2}{p_1} = 2.493 \ \rightarrow \ p_2 = p_1 * \frac{p_2}{p_1} = 100 * 2.493 = 249.3\ K$$

From continuity equation

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = 1.879$$

$$\frac{V_1}{V_2} = \frac{V_1}{500 - V_g} = 1.879$$

$$V_g = 233.9\ m/s$$

Since the velocity of the observer does not affect the static properties,

$$p_b = 249.3\ kPa$$

$$T_b = 362.3\ K$$

The Mach number of the gas flow behind the wave is given by

$$M_g = \frac{V_g}{\sqrt{\gamma R T_b}} = \frac{233.9}{\sqrt{1.4 * 287 * 362.3}} = 0.613$$

With the Mach number and static properties determined, the stagnation properties of the gas stream can be found from isentropic table at M = 0.613,

$T/T_o = 0.9301$ and $p/p_o = 0.7759$

After passage of the wave, the stagnation pressure is

$$T_{ob} = \frac{T_b}{T_b/T_{ob}} = \frac{362.3}{0.9301} = 389.5\ K$$

$$p_{ob} = \frac{p_b}{p_b/p_{ob}} = \frac{249.3}{0.7759} = 321.3\ kPa$$

Note that for a fixed observer the stagnation temperature after passage of the wave is greater than that before passage of the wave. For an observer "sitting on the wave," however, there is no change of stagnation temperature across the wave.

Example 13.2 An explosion occurs which produces a normal shockwave that propagates at a speed of $600\ m/s$ into still air. The pressure and temperature of the motionless air in front of the shock are $101.3\ kPa$ and $20\ °C$, respectively. Determine the velocity, static pressure, and static temperature of the air following the shock, i.e. ($V_2, p_2, and\ T_2$).

Solution

$$M_1 = \frac{V_s}{\sqrt{\gamma R T_1}} = \frac{600}{\sqrt{1.4 * 287 * 293}} = 1.749$$

From isentropic table at $M_1 = 1.749$ gives

$p_1/p_{o1} = 0.1882$, $T_1/T_{o1} = 0.6205$

And from normal table at $M_1 = 1.749$ gives

$p_2/p_1 = 3.4009$, $T_2/T_1 = 1.4936$, $p_{o2}/p_{o1} = 0.8351$ and $M_2 = 0.6284$. So;

$$T_{o1} = \frac{T_1}{(p_1/p_{o1})} = \frac{293}{0.6205} = 472.2\ K = T_{o2}$$

$$p_{o1} = \frac{p_1}{(T_1/T_{o1})} = \frac{101.3}{0.1882} = 538.2572\ kPa$$

$$p_{o2} = \left(\frac{p_{o2}}{p_{o1}}\right) p_{o1} = 0.8351 * 538.2572 = 449.4986$$

$$p_2 = \left(\frac{p_2}{p_1}\right) p_1 = 3.4009 * 101.3 = 344.5112\ kPa$$

$$T_2 = \left(\frac{T_2}{T_1}\right) T_1 = 1.4936 * 293 = 437.6248\ K = T_b$$

$$a_2 = a_b = \sqrt{\gamma R T_2} = \sqrt{1.4 * 287 * 437.6248} = 419.33\ m/s$$

$$(V_s - V_g) = a_2 M_2 = 419.33 * 0.6284 = 263.507\ m/s$$

$$V_g = V - (V_s - V_g) = 600 - 263.507 = 336.493\ m/s$$

$$M_b = \frac{V_g}{a_b} = \frac{336.493}{419.33} = 0.8025$$

From isentropic table at $M_b = 0.8025$, gives;

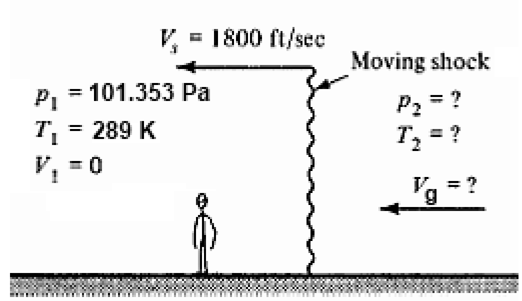
$p_b/p_{ob} = 0.6544$ and $T_b/T_{ob} = 0.8859$, then

$$p_{ob} = \frac{p_b}{p_b/p_{ob}} = \frac{344.5112}{0.6544} = 526.4535\ \ kPa$$

$$T_{ob} = \frac{T_b}{T_b/T_{ob}} = \frac{437.6248}{0.8859} = 493.9889\ K$$

Example 13.3 The shock was given as moving at $548.64\ m/s$ into air at $101.353\ Pa$ and 289 K. Solve the problem represented in Figure 13.4.

Solution



(a) Moving Normal Shock Fixed coordinate

(b) Standing Normal Shock Moving coordinate

Figure 13.4

➢ We solve for fixed normal shock, i.e. moving coordinate system, (figure 13.4b).

$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 * 287 * 289} =$

$= 340.76\ m/s$

$$M_1 = \frac{V_1}{a_1} = \frac{548.64}{340.76} = 1.61$$

From isentropic, at $M_1 = 1.61$,

$p_1/p_{o1} = 0.2318$, then

$$p_{o1} = \frac{p_1}{p_1/p_{o1}} = \frac{101.353}{0.2318} = 437.243\ kPa$$

From normal shock table, at $M_1 = 1.61$

$$M_2 = 0.6655, \qquad \frac{p_2}{p_1} = 2.8575, \qquad \frac{T_2}{T_1} = 1.3949$$

Thus

$$p_2 = p_1 * \frac{p_2}{p_1} = 101.353 * 2.8575 = 289.616\ kPa$$

$$T_2 = T_1 * \frac{T_2}{T_1} = 289 * 1.3949 = 403.13\ K$$

$$a_2 = \sqrt{\gamma R T_2} = \sqrt{1.4 * 287 * 403.76} = 402.78\ m/s$$

$$V_2 = a_2 M_2 = 402.78 * 0.6655 = 268.1\ m/s$$

And from isentropic table at $M_2 = 0.6655$, $p_2/p_{o2} = 0.7430$ and $T_2/T_{o2} = 0.9188$, then

$$p_{o2} = \frac{p_2}{p_2/p_{t2}} = \frac{289.616}{0.7430} = 389.8\ kPa$$

$$T_{o2} = \frac{T_2}{T_2/T_{o2}} = \frac{403.13}{0.9188} = 438.76\ K$$

$$V_g = V_s - V_2 = 548.64 - 268.1 = 280.54\ m/s$$

It is apparent that $p_{o2} < p_{o1}$ as expected.

➢ Now we solve for moving shock, i.e. fixed coordinate system (figure 13.4a). Remembering that pressure, temperature and sonic velocity values after the shock wave are not changed due to shock wave movement.

$p_2 = 289.616\ kPa$

$T_2 = 403.13\ K$

$a_2 = 402.78\ m/s$

$V_g = 280.54\ m/s$

$$M_g = \frac{V_g}{a_g} = \frac{280.54}{402.78} = 0.697$$

And from isentropic table, at $M_g = 0.697$, $p_2/p_{o2} = 0.7220$ and $T_2/T_{o2} = 0.9095$, then;

$$p_{o2} = \frac{p_2}{p_2/p_{o2}} = \frac{289.616}{0.7220} = 401.130\ kPa$$

$$T_{o2} = \frac{T_2}{T_2/T_{o2}} = \frac{403.13}{0.9095} = 443.2\ K$$

Therefore, after the shock passes (referring now to Figure 13.4a), the pressure and temperature will be $289.616\ kPa$ and $403.13\ K$, respectively, and the air will have acquired a velocity of $280.54\ m/s$ to the left. It will be interesting to compute and compare the stagnation pressures in each case. Notice that they are completely different because of the change in reference that has taken place.

❖ **Second case**

Developing an expressions for the case of a normal shock traveling at a constant speed V_s into a gas that is moving with a speed V. The shock induces a speed V_g of the gas it passes over, as shown in Figure 13.6. here simply replace each V_s & V_g in eqs. 13.1 to 13.5 by $V_s - V$ & $V_g - V$.

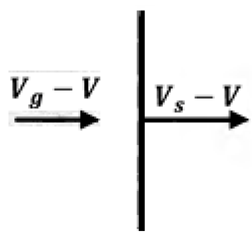


Figure 13.6

Example 13.4 A piston in a tube is suddenly accelerated to a velocity of $50\ m/s$, which causes a normal shock to move into the air at rest in the tube. Several seconds later, the piston is suddenly accelerated from $50\ to\ 100\ m/s$, which, causes a second shock to move down the tube. Calculate the velocities of the two shock waves for an initial air temperature of $300\ K$.

Solution

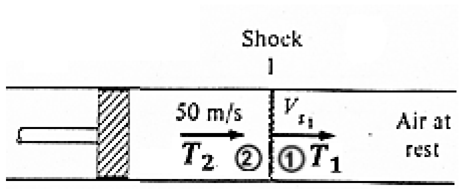


Figure 13.7

The air next to the piston must move at the same velocity as the piston, since it can neither move through the face of the piston nor move away from the piston and leave a vacuum behind. Therefore, for a fixed observer, the air velocities are as shown in Figure (13.7).

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 * 287 * 300} = 347.2\ m/s$$

From eq. 13.5

$$V_s = \frac{(\gamma+1)V_g}{4} \pm \sqrt{\left(\frac{(\gamma+1)V_g}{4}\right)^2 + a_1^2}$$

$$V_{s1} = \frac{(1.4+1)50}{4} \pm \sqrt{\left(\frac{(1.4+1)50}{4}\right)^2 + 347.2^2}$$

$$V_{s1} = 30 + 348.5 = 378.5 \ \ m/s$$

$$M_{s1} = \frac{V_{s1}}{a_1} = \frac{378.5}{347.2} = 1.090$$

From normal shock table, at $M_1 = 1.090 \ \rightarrow T_2/T_1 = 1.059$, so;

$$T_2 = 300 * 1.059 = 317.7\ K$$

For the second shock, the situation is shown in Figure (13.8a). Figure (13.8b) shows an observer "sitting on the second wave". Using eq. (10.5), we obtain

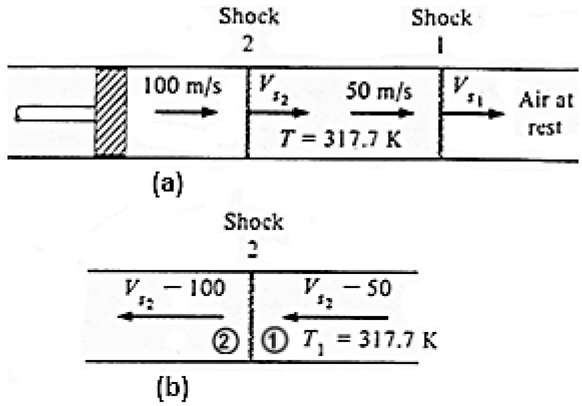


Figure 13.8

$$\frac{V_1}{V_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}$$

Where

$$V_1 = V_{s2} - 50, \qquad V_2 = V_{s2} - 100$$

$$M_1^2 = \frac{(V_{s2}-50)^2}{\gamma R T_1}$$

Substituting yields

$$\frac{V_{s2}-50}{V_{s2}-100} = \left[2.4 * \frac{(V_{s2}-50)^2}{1.4*287*317.7}\right] \Big/ \left[0.4 * \frac{(V_{s2}-50)^2}{1.4*287*317.7} + 2\right]$$

$$= \frac{2.4(V_{s2}-50)^2}{0.4(V_{s2}-50)^2 + 2*127651.86}$$

To solving this quadratic equation, Let $x = (V_{s2} - 50)$

$$\frac{x}{x-50} = \frac{2.4x^2}{0.4x^2 + 255303.72}$$

$$0.4x^3 + 255303.72x = 2.4x^3 - 120x^2$$

$$2x^2 - 120x - 255303.72 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{120 \pm \sqrt{120^2 + 4*2*255303.72}}{2*2}$$

$$x = \frac{120 \pm \sqrt{120^2 + 4*2*255303.72}}{2*2} = \frac{120 + 1434.165}{4}$$

$$V_{s2} - 50 = 388.543$$

$V_{s2} = 438.54\ m/s$

Thus, the second wave travels at a greater velocity than the first and eventually overtakes it. This result is a demonstration of the principles formation of normal shock. Compression waves are able to overtake and reinforce one another. In this example problem, the second wave travels at a greater velocity because it is both moving into the compressed, higher-temperature gas behind the first wave and also moving into a gas stream already traveling in the same direction with a velocity of $50\ m/s$. A new set of gas properties now can be computed before and after the second shock.

12.2 Reflected Waves.

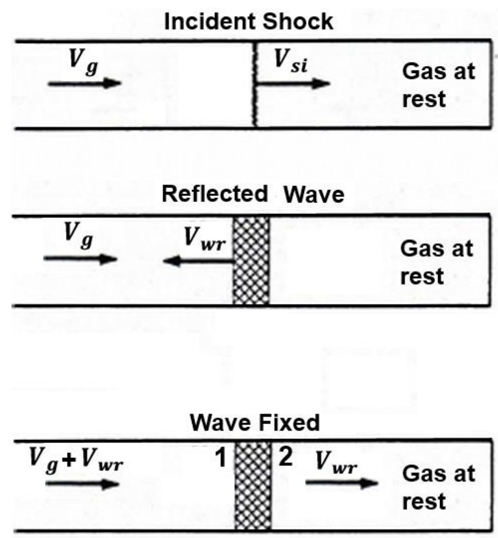


Figure 13.9: Incident and reflected wave in closed tube

When a wave impinging on the end of a tube, two cases should be studied, a closed tube and a tube open to the atmosphere. The reflected wave in closed end tube is treated as a reflected normal shock while for open end tube is treated as reflected expansion waves.

To complete this study of moving normal shock waves, consider the result of a wave impinging on the end of a tube. Two cases will be studied; a closed tube and a tube open to the atmosphere. In both cases it is desired to determine whether the reflected wave is a compression shock wave or a series of weak expansion waves. For reflected wave in closed tube, (see Figure 13.9), the gas next to the fixed end of the tube must be at rest, with the gas behind the incident shock moving to the right with velocity Vg. For an observer moving with the reflected wave, the physical indicates that a decrease in velocity and a corresponding increase in static pressure across the reflected wave, which is physically the situation for a normal shock. Therefore, a normal shock reflects from a closed tube as a normal shock.

For reflected in open tube to atmosphere, the boundary condition imposed on the system is the static pressure at the end of the tube. Because the flow in front of the moving shock is subsonic, the back pressure and the exit pressure must be the same, see figure 13.10. there will be a decrease in pressure across the reflected wave and a normal shock reflects from an open end of a tube as a series of expansion waves.

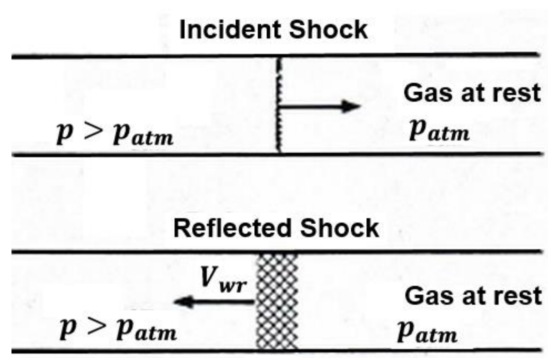


Figure 13.10: Incident and reflected wave in open tube to atmospher

Example **13.4** A normal shock wave with pressure ratio of 4.5 impinges on a plane wall (see Figure 13.11a). Determine the static pressure ratio for the reflected normal shock wave. The air temperature in front of the incident wave is 20°C.

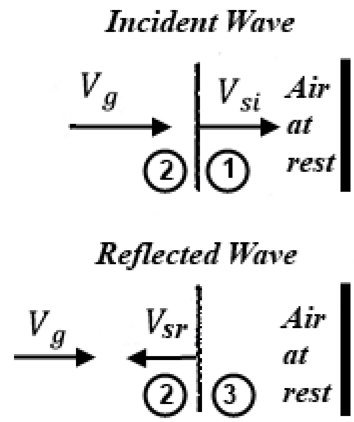


Figure 13.11a
Fixed coordinate

Solution

❖ Solution for incident wave:

To determine the velocity Vg of the gas behind the incident wave, utilize a reference system moving with the wave, as shown in Figure 13.11b.

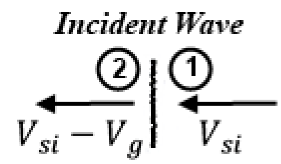


Figure 13.11b
Moving coordinate

From normal shock table $p_2/p_1 = 4.5$, gives:

$M_1 = 2.0$, $\rho_2/\rho_1 = 2.667$ and $T_2/T_1 = 1.688$

$$V_{si} = M_1 * \sqrt{\gamma R T_1} = 2.0 * \sqrt{1.4 * 287 * 293} = 686.2\ m/s$$

$$\frac{V_{si}}{V_{si} - V_g} = \frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} = 2.667$$

$$\left(686.2 - V_g\right) = 686.2 \div 2.667$$

$$\therefore V_g = 428.9\ m/s$$

$$T_2 = T_1 * \frac{T_2}{T_1} = 293 * 1.688 = 494.6\ K$$

❖ Solution for reflected wave:

To find the reflected shock velocity, fix the reflected shock by using (see Figure 13.11c)

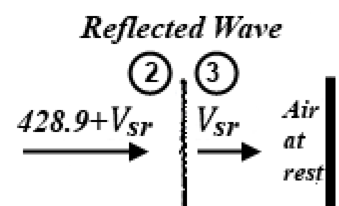


Figure 13.11c
Moving coordinate

$$\frac{V_2}{V_3} = \frac{(\gamma + 1)M_2^2}{(\gamma - 1)M_3^2 + 2} \qquad (8.16)$$

For this case

$$V_2 = 428.9 + V_{sr}$$

$$V_3 = V_{sr} = V_2 - 428.9$$

$$T_2 = 494.6\ K$$

$$M_2^2 = \frac{V_2^2}{\gamma R T_2} = \frac{V_2^2}{1.4 * 287 * 494.6} = \frac{V_2^2}{198730.28}$$

$$\frac{V_2}{V_3} = \frac{2.4 \frac{V_2^2}{198730.28}}{0.4 \frac{V_2^2}{198730.28} + 2} = \frac{2.4 V_2^2}{0.4 V_2^2 + 397460.56}$$

$$\frac{V_2}{V_2 - 428.9} = \frac{2.4\,V_2^2}{0.4\,V_2^2 + 397460.56}$$

$$0.4\,V_2^3 + 397460.56\,V_2 = 2.4\,V_2^3 - 1029.36V_2^2$$

$$2\,V_2^2 - 1029.36\,V_2 - 397460.56 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1029.36 \pm \sqrt{1029.36^2 + 4 * 2 * 397460.56}}{2 * 2}$$

$$V_2 = \frac{1029 - 2058.948}{4} = -257.487\ m/s\ \ ignored$$

$$V_2 = \frac{1029 + 2058.948}{4} = 771.987\ m/s$$

$$V_{sr} = 771.987 - 428.9 = 343.1\ m/s$$

For the fixed shock, back to fig. 13.10a

$$\frac{V_2}{V_3} = \frac{428.9 + V_{sr}}{V_{sr}} = \frac{771.987}{343.1} = 2.250 = \frac{\rho_3}{\rho_2}$$

From normal shock table, at $\rho_3/\rho_2 = 2.250$, gives

$p_3/p_2 = 3.333$ static pressure ratio for reflected normal shock.

$$\frac{p_3}{p_1} = \frac{p_3}{p_2} * \frac{p_2}{p_1} = 3.333 * 4.5 \approx 15$$

That means the in zone 3 after reflection becomes fifteen times the pressure in zone 1 before incident.

Another type of moving shock is occurred when air is flowing through a duct under known conditions and a valve is suddenly closed, as shown in fig. 13.12.. The fluid is compressed as it is quickly brought to rest. This results in a shock wave propagating back through the duct. In this case the problem is not only to determine the conditions that exist after passage of the shock but also to predict the speed of the shock wave.   This can also be viewed as the reflection of a shock wave, similar to what happens at the end of a shock tube. We transfer the fixed coordinate into a moving coordinate system by riding the shock wave and superimpose the reflected wave velocity V_{sr} on the entire flow field. With this new frame of reference we have the standing normal-shock problem shown in Figure 13.12.

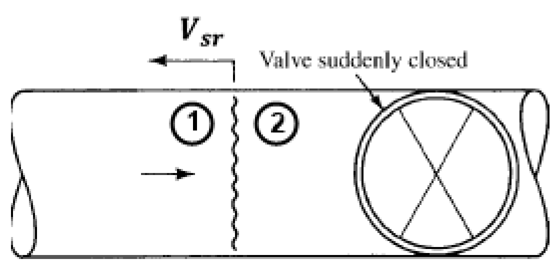


Figure 13.12

***Example* 13.5** Air of speed of 240 m/s is flowing through a duct where its pressure and temperature are 2 bar and 300 k respectivilly. Then a valve exists in the duct is suddenly closed. Find fluid properties nest to the valve after it closed and shock velocity, as show in figure 13.13.

Answer

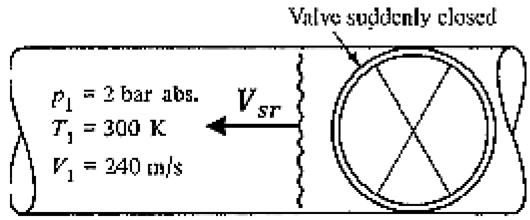


Figure 13.12a

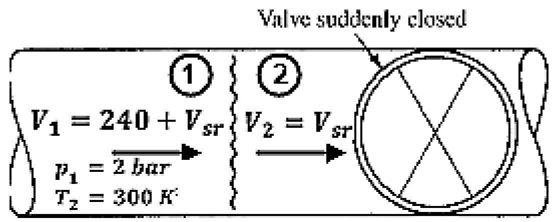


Figure 13.12b

$V_2 = V_1 - 240$

$\frac{V_1}{V_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}$

$\frac{V_1}{V_1 - 240} = \frac{2.4\,V_1^2/120540}{0.4\,V_1^2/120540+2}$

$0.4V_1^3 + 2*120540V_1 = 2.4V_1^3 - 576V_1^2$

$2V_1^2 - 576V_1 - 241080 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$V_1 = \frac{576 + \sqrt{576^2 - 4*2*241080}}{2*2} = 519.867\ m/s$

$V_2 = 519.867 - 240 = 279.867\ m/s$

$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4*287*300} = 347.189\ m/s$

$M_1 = V_1/a_1 = 519.867/347.2 = 1.497$

From normal shock table at $M_1 = 1.5$ gives

$M_2 = 0.7011$, $p_2/p_1 = 2.458$ and $T_2/T_1 = 1.320$

$p_2 = 2.458*2 = 4.916\ bar$

$T_2 = 1.320*300 = 396\ K$

12.3 Shock Tube

The shock tube is a device in which normal shockwaves are generated by the rupture of a diaphragm separating a high-pressure gas from a gas at low pressure. As such, the shock tube is a useful research tool for investigating not only shock phenomena, but also the behavior of materials and objects when subjected to the extreme conditions of pressure and temperature prevalent in the gas flow behind the wave. Thus, the kinetics of a chemical reaction taking place at high temperature can be studied, as well as the performance, for example, of a body during reentry from space back into the earth's atmosphere.

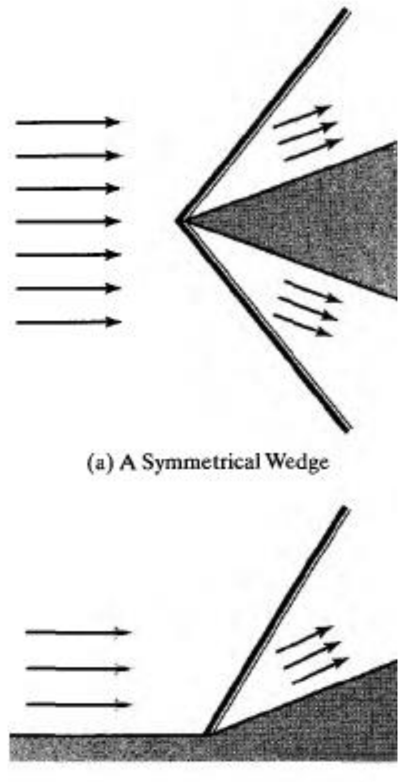
Chapter Fourteen/Oblique Shock Waves

14.1 Introduction.

An oblique shock wave, a compression shock wave that is inclined at an angle to the flow, either straight or curved, can occur in such varied examples as supersonic flow over a thin airfoil or in supersonic flow through an over-expanded nozzle.

The oblique shock wave is a two-dimension problem. The method of handling the oblique shock is alike that of handling the normal shock. Even though inclined to the flow direction, the oblique shock still represents a sudden, almost discontinuous change in fluid properties, with the shock process itself being adiabatic. Attention will be focused on the two-dimensional straight oblique shock wave, a type that might occur during the presence of a wedge in a supersonic stream (Figure 14.1a) or during a supersonic compression in a corner (Figure 14.1b). As with the normal shock wave, the equations of continuity, momentum, and energy will first be derived. An additional variable is introduced because of the change in flow direction across the wave. However, momentum is a vector quantity, so two momentum equations are derivable for this two-dimensional flow.

With the additional variable and equation, the analysis of two-dimensional shock flow is somewhat more complex than that for normal shock flow. However, as with the normal shock wave, solutions to the equations of motion will be presented in a form suitable for the working of practical engineering problems.



(a) A Symmetrical Wedge

(b) A Concave Corner

Figure14.1 Oblique Shocks

14.2 Equations of Motion for a Straight Oblique Shock Wave

When a uniform supersonic stream is forced to undergo a finite change in direction due to the presence of a body in the flow, the stream cannot adjust gradually to the presence of the body; rather, a shock wave or sudden change in flow properties must occur. A simple case is that of supersonic flow about a two-dimensional wedge with axis aligned parallel to the flow direction.

For small wedge angles, the flow adjusts by means of an oblique shock wave, attached to the apex of the wedge. Flow after the shock is uniform, parallel to the wedge surface (as shown in Figure 14.2), with the entire flow having been turned through the wedge half-angle δ.

The equations of continuity, momentum, and energy will now be written for uniform, supersonic flow over a fixed wedge. If one selects the control volume indicated in Figure 14.2. The continuity equation for steady flow is

$$\iint_{cs} \rho\,(\boldsymbol{V}.\hat{n})\,dA = 0$$

For the case under steady, it simplifies to

$$\rho_1 V_{1n}\,A = \rho_2 V_{2n}\,A$$

$$\rho_1 V_{1n} = \rho_2 V_{2n} \qquad (14.1)$$

Where $V_{1n}$and V_{2n} are the velocity components normal to the wave. A is the control volume surface and it is the same for both sides. The momentum equation for steady flow is;

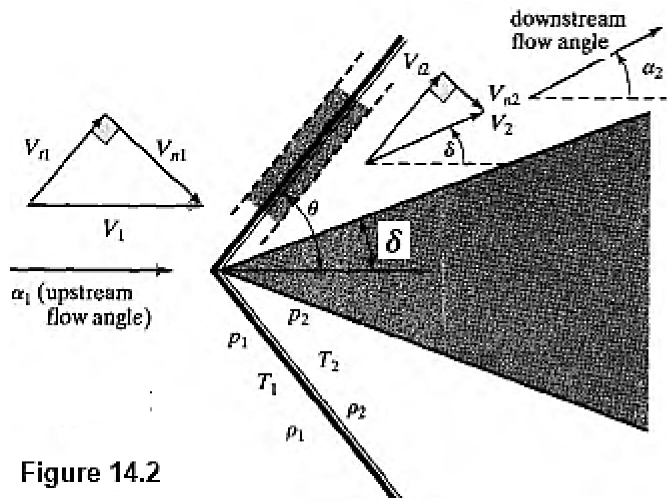


Figure 14.2
Notation and Control Volume for an Oblique Shock

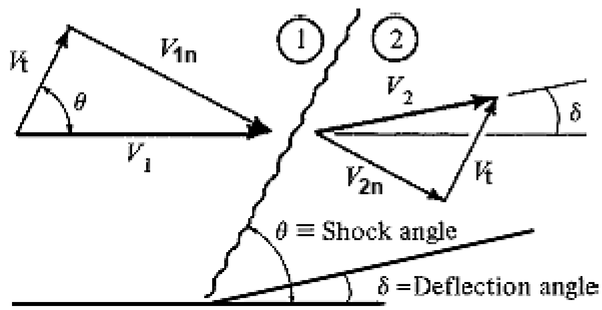


Figure 14.3 Oblique shock with angle definitions.

$$\sum \boldsymbol{F} = \iint_{cs} \boldsymbol{V}\rho\,(\boldsymbol{V}.\hat{n})\,dA = 0$$

Momentum is a vector quantity, so momentum balance equations can be written both in the direction normal to the wave and in the direction tangential to the wave. The normal momentum equation yields;

$$p_1 A_1 - p_2 A_2 = \rho_2 A_2 V_{2n}^2 - \rho_1 A_1 V_{1n}^2$$

The shock is very thin so as we assume that $A_2 = A_1$. Thus;

$$p_1 - p_2 = \rho_2 V_{2n}^2 - \rho_1 V_{1n}^2 \qquad (14.2)$$

In the tangential direction there is no change in pressure so;

$$0 = \iint_{cs} V_t \rho\,(\vec{\boldsymbol{V}}.\hat{n})\,dA = 0$$

$$(\rho_1 V_{1n} A_1) V_{1t} = (\rho_2 V_{2n} A_2) V_{2t}$$

Cancelling, we obtain;

$$V_{1t} = V_{2t} \qquad (14.3)$$

where V_{1t} & V_{2t} are the velocity components tangential to the wave. The energy equation for adiabatic, no work steady flow simplifies to;

$$\left(h_1 + \frac{\vec{\boldsymbol{V}}_1^2}{2} + g z_1\right) = \left(h_2 + \frac{\vec{\boldsymbol{V}}_2^2}{2} + g z_2\right)$$

Expanding this equation and ignoring potation term for gas and remembering that a velocity is a vector ($\vec{\boldsymbol{V}} = V_n + V_t$), we get;

$$\left(h_1 + \frac{V_{1n}^2}{2} + \frac{V_{1t}^2}{2}\right) = \left(h_2 + \frac{V_{2n}^2}{2} + \frac{V_{2t}^2}{2}\right)$$

Since $V_{1t} = V_{2t}$ then;

$$\left(h_1 + \frac{V_{1n}^2}{2}\right) = \left(h_2 + \frac{V_{2n}^2}{2}\right) \qquad \dots (14.4a)$$

$$T_{o1} = T_{o2} \qquad \dots (14.4b)$$

$$M_{1n} = M_1 \sin\theta \qquad \dots (14.5a)$$

$$M_{1t} = M_1 \cos\theta \qquad \dots (14.5b)$$

$$M_{2n} = M_2 \sin\,(\theta - \delta) \qquad \dots (14.6a)$$

$$M_{2t} = M_2 \cos\,(\theta - \delta) \qquad \dots (14.6b)$$

From the geometry of the oblique wave;

It can be seen that eqs. (14.1), (14.2).and (14.4) contain only the normal velocity components, and as such are the same as eqs. (9.1), (9.2) and (9.4) for the normal shock wave. In other words, an oblique shock acts as a normal shock for the component normal to the wave, while the tangential velocity component remains unchanged. The pressure ratio, temperature ratio, and so on, across an oblique shock can be determined by first calculating the component of M_n, normal to the wave and then referring this value to the normal shock tables.

Note that the Mach number after an oblique shock wave can be greater than 1 without violating the second law of thermodynamics. The normal component of M_2 however, must still be less than 1. In most cases, the shock wave angle θ is not known, but rather incoming Mach number M_1 and deflection angle δ appear as the independent variables. Therefore, it is more convenient to express the wave angle θ and M_2 in terms of $M_1\ and\ \delta$, From eq. 14.1

$$\rho_1 V_{1n} = \rho_2 V_{2n}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_{1n}}{V_{2n}} = \frac{V_{1t}\tan\theta}{V_{2t}\tan(\theta - \delta)} = \frac{\tan\theta}{\tan(\theta - \delta)} \qquad \dots (14.7)$$

A cross the normal shock

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_{1n}^2}{(\gamma - 1)M_{1n}^2 + 2} \qquad \dots (10.3)$$

$$\frac{\tan\theta}{\tan(\theta - \delta)} = \frac{(\gamma + 1)M_{1n}^2}{(\gamma - 1)M_{1n}^2 + 2} \qquad \dots (14.8a)$$

$$\frac{\tan\theta}{\tan(\theta - \delta)} = \frac{(\gamma + 1)M_1^2\sin^2\theta}{(\gamma - 1)M_1^2\sin^2\theta + 2} \qquad \dots (14.8b)$$

Eq. 14.8 relates deflection angle δ incoming Mach number M_1 and shock wave angle θ.

Now θ can be plotted versus δ for a given value of M_1. Also M_2 can be plotted versus δ for given M_1. For $M_1 = 2.0$, the results appear as shown in Figures 14.4a and 14.4b.

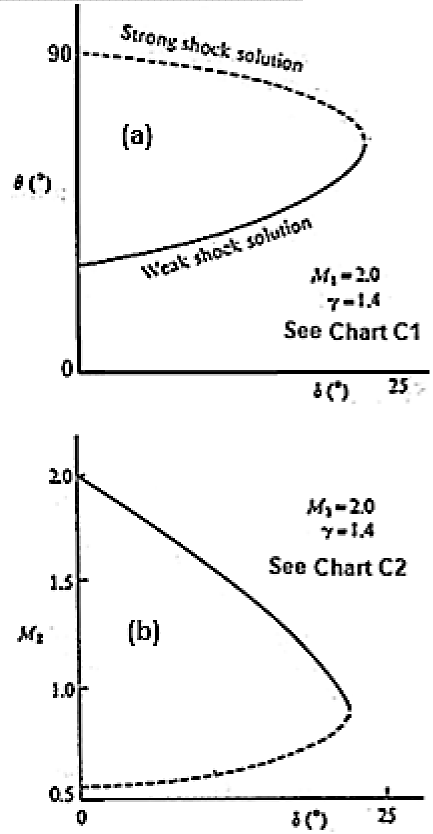


Figure 14.4

Detailed oblique shock charts are provided in charts C1 and C2 for $\gamma = 1.4$. But chart C2 is not accurate and it will not recommended. Several characteristics of the solution to the oblique shock equations can be seen from these charts. For a given M_1 and δ, either two solutions are possible or none at all. For supersonic flow in varying area channels, it is the pressure boundary conditions imposed on the channel that determines the type of solution.

If a solution exists, there may be

1. A weak oblique shock, with M_2 either supersonic or slightly less than 1.
2. A strong oblique shock, with M_2 subsonic.

Both oblique shocks have different characteristics, see figure 14.5, such as;

a. For the strong oblique shock:
- The wave makes a large angle θ (close to 90°) with the approach flow.
- It accompanied by a relatively large pressure ratio

b. For the weak oblique shock,
- The wave makes a much less angle θ with the approach flow.
- It accompanied by a relatively small pressure ratio

c. The supersonic flow is turned through the same angle in both cases.

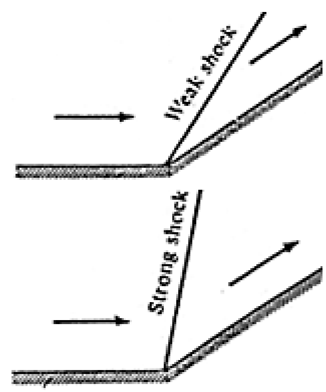


Figure 14.5

A strong oblique shock with ($\delta = 0$), gives a normal shock. A weak oblique shock with ($\delta = 0$**)** gives an isentropic flow (no shock). Therefore, the normal wave can be generalized to the oblique shock. The strong oblique shock occurs when a large back pressure is imposed on a supersonic flow**,** as might possibly take place during flow through a duct or intake.

When a wedge or airfoil travels through the atmosphere at supersonic velocities with an oblique shock attached to the body only a weak shock solution is found to occur, since**,** with a uniform pressure after the shock, large pressure differences cannot be exist. This is identical to determine whether isentropic flow or a normal shock will occur in a supersonic flow for flow through converging-diverging nozzles, we know that for low enough back pressures, isentropic flow occurs in the nozzle; for higher back pressures, a normal shock takes place in the diverging section of the nozzle.

14.3 Detached shock Wave

Another characteristic of the oblique shock equations is that, for a great enough turning angle $\delta > \delta_{max}$, no solution is possible. Under these conditions it is observed that the shock is no longer attached to the wedge, but stands detached, in front of the body (see Figure 14.6).

The detached shock is curved, as shown, with the shock strength decreasing progressively from that of a normal shock at the apex of the wedge to that of a Mach wave far from the body. Thus, with a detached shock, the entire range of oblique shock solutions is obtained for the given Mach number M_1.

The shape of the wave and the shock-detachment distance are dependent on the Mach number and the body shape. Flow over the body is subsonic in the vicinity of the wedge apex, where the strong oblique shocks occur, and it is supersonic farther back along the wedge, where the weak oblique shocks are present.

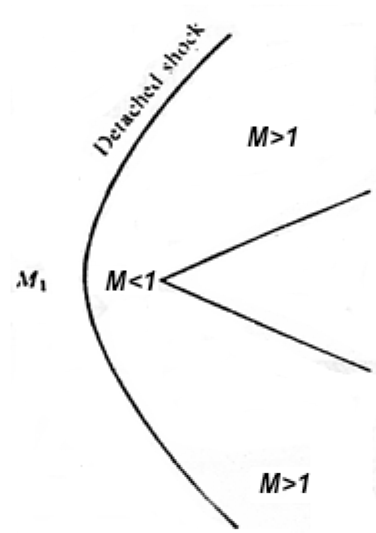


Figure 14.6

A detached oblique shock can also occur with supersonic flow in a concave corner. Again, if the turning angle is too great, a solution cannot be found in Charts Cl and C2, so a detached shock forms ahead of the corner (see Figure 14.7**).** The characteristics of this shock are exactly the same as those of the upper half of the detached shock shown in Figure 14.6. Thus flow after the shock is subsonic near the wall and supersonic farther out in the flow and it is treated as a stationary normal shock near the wall.

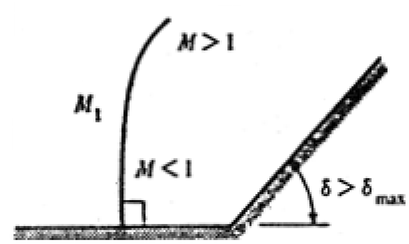


Figure 14.7

Example 14.1 A uniform supersonic airflow traveling at $Mach\ 2.0$ passes over a wedge (Figure 14.4**).** An oblique shock, making an angle of 40° with the flow direction, is attached to the wedge under these flow conditions. If the static pressure and temperature in the uniform flow are, respectively, $20\ kPa$ and $-10\ ℃$, determine the static pressure and temperature behind the wave, the Mach number of the flow passing over the wedge, and the wedge half-angle.

Solution

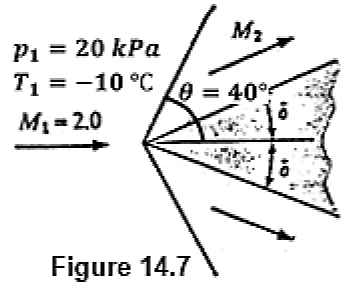


Figure 14.7

From Figure 14.4,

$M_{1n} = M_1\ sin\ 40° = 2.0 \sin 40° = 1.286.$

$M_{1t} = M_1 \cos 40° = 2.0 \cos 40° = 1.532$

Therefore, from normal shock table at $M_{1n} = 1.286$

$$M_{2n} = 0.794, \quad \frac{p_2}{p_1} = 1.763, \quad \frac{T_2}{T_1} = 1.182$$

$$p_2 = p_1 * \frac{p_2}{p_1} = 20 * 1.763 = 35.26\ kPa$$

$$T_2 = T_1 * \frac{T_2}{T_1} = 263 * 1.1814 = 310.7\ K$$

For the adiabatic shock process, $T_{o1} = T_{o2}$. From isentropic table at $M_1 = 2.0$,

$T_1/T_{o1} = 0.5556$, Then

$$T_{o1} = T_{o2} = \frac{T_1}{T_1/T_{o1}} = \frac{263}{0.5556} = 473.4\ K$$

Now

$T_2/T_{o2} = 310.7/473.4 = 0.6563$

From isentropic table A at $T_2/T_{o2} = 0.6563$; → $M_2 = 1.617$

$$si\,n(\theta - \delta) = \frac{V_{2n}}{V_2} = \frac{M_{2n}a_{2n}}{M_2 a_2} = \frac{0.794}{1.617} = 0.491$$

$a_{2n} = a_2\ scaler$

$\theta - \delta = 29.4^{\circ}$

$\delta = 40 - 29.4 = 10.6^{\circ}$ end of the solution.

Solving graphically;

From Chart C1 at $M_1 = 2.0$ & $\theta = 40°$ gives $\delta = 10.6°$

From Chart C2 at $M_1 = 2.0$ & $\delta = 10.6°$ gives $M_2 = 1.62$

Solving by the exact equations;

$$\tan\delta = (\cot\theta)\left(\frac{M_1^2 \sin^2\theta - 1}{\frac{\gamma+1}{2}M_1^2 - (M_1^2 \sin^2\theta - 1)}\right)$$

$$\tan\delta = (\cot 40)\left(\frac{2.0^2 \sin^2 40 - 1}{\frac{\gamma+1}{2}M_1^2 - (2.0^2 \sin^2 40 - 1)}\right)$$

$$= (1.19175)\left(\frac{0.6527}{4.8 - 0.6527}\right) = 0.1756$$

$\delta = \tan^{-1} 0.1756 = 10.6°$

$$M_2 = \sqrt{\frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 \sin^2\theta - \frac{\gamma-1}{2}} + \frac{M_1^2 \cos^2\theta}{1+\frac{\gamma-1}{2}M_1^2 \sin^2\theta}}$$

$$M_2 = \sqrt{\frac{1+\frac{1.4-1}{2}2^2}{1.4*2^2 \sin^2 40 - \frac{1.4-1}{2}} + \frac{2^2 \cos^2 40}{1+\frac{1.4-1}{2}2^2 \sin^2 40}}$$

$$M_2 = \sqrt{\frac{1.8}{2.1138} + \frac{2.3473}{1.3305}} = 1.617$$

Example 14.2 Uniform flow at $M = 2.0$ passes over a wedge of 10° half-angle., find M_2, p_2/p_1, T_2/T_1 *and* p_{o2}/p_{o1}, and also the half-angle above which the shock will become detached.

Solution

From Chart Cl at $M = 2.0$ and $= 10°$, the weak solution yields $\theta = 39.3°$

$M_{1n} = M_1 \sin\theta = 2.0 \sin 39.3 = 1.267$

$M_{1t} = M_1 \cos\theta = 2.0 \cos 39.3 = 1.548$

From the normal shock tables at $M_{1n} \approx 1.27$

$p_2/p_1 = 1.71505$; $T_2/T_1 = 1.17195$; $p_{o2}/p_{o1} = 0.98422$ and $M_{2n} = 0.80164$

From Chart Cl it can be seen that δ_{max}, for $M = 2.0$ *is* 23°.

Example 14.3 A supersonic two-dimensional inlet is to be designed to operate at $M = 3.0$. Two possibilities will be considered, as shown in Figure 14.8. In one, the compression and slowing down of the flow take place through one normal shock; in the other, a wedge-shaped diffuser, the deceleration occurs through two weak oblique shocks, followed by a normal shock. The wedge turning angles are each 8°. Compare the loss in stagnation pressure for the two cases shown.

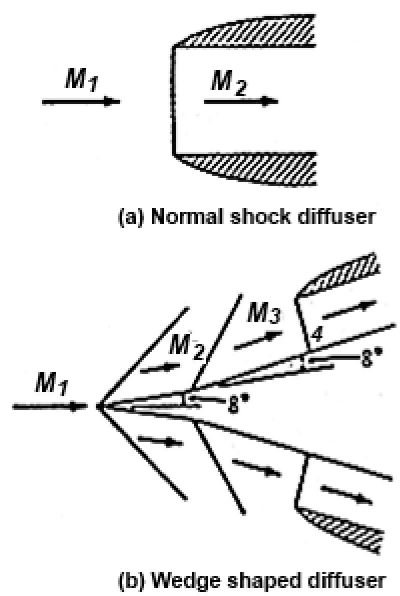


Figure 14.8

Solution

For the normal shock diffuser, the ratio p_{2o}/p_{1o} can be found from normal shock table at $M_1 = 3.0$: so

$p_{o2}/p_{o1} = 0.328$.

For the wedge-shaped diffuser, M_2 and M_3, as well as the wave angles, can be found from Charts Cl and C2.Thus

$M_2 = 2.60$ and $M_3 = 2.255$.

The wave angles are, respectively, 25.6° and 29.0°.

$M_{1n} = M_1 \sin\theta_1 = 3.0 \sin 25.6 = 1.3$

From normal shock table at $M_{1n} = 1.30,\ p_{o2}/p_{o1} = 0.979$

$M_{2n} = M_2 \sin\theta_2 = 2.60 \sin 29.0 = 1.26$

From normal shock table at $M_{2n} = 1.26,\ \mathrm{p_{o3}/p_{o2}} = 0.986$.

From normal shock table at $M_3 = 2.255,\ p_{o4}/p_{o3} = 0.603$, so that;

$$\frac{p_{o4}}{p_{o1}} = \frac{p_{o4}}{p_{o3}} * \frac{p_{o3}}{p_{o2}} * \frac{p_{o2}}{p_{o1}} = 0.603 * 0.986 * 0.979 = 0.582$$

Note; Solve the same example without using chart C2.

Therefore, the overall stagnation pressure ratio is 0.582. The advantage of diffusing through several oblique shocks rather than one normal shock can be seen. The greater the number of oblique shocks, the less the overall loss in stagnation pressure. Theoretically, if the flow is allowed to pass through an extremely large number of oblique shocks, each turning the flow through a very small angle, the inlet flow should approach that of an isentropic compression. The oblique shock diffuser will be discussed in detail in later.

14.4 Oblique Shock Reflections

When a weak, two-dimensional oblique shock impinges on a plane wall, the presence of a reflected wave is required to straighten the flow, since there can be no flow across the wall surface (see Figure 14.11**)**.

Flow after the incident wave is deflected toward the wall. Hence, a reflected oblique shock wave must be present to deflect the flow back through the same angle and restore the flow direction parallel to the wall. The reflected shock is weaker than the incident shock, since $M_2 < M_1$**.**

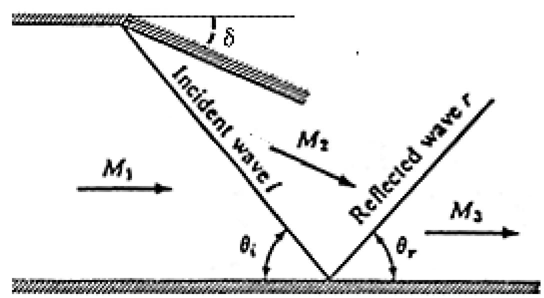


Figure 14.11

Example 14.4 For $M_1 = 2.0,$ and $\theta_i = 40°$, determine $\theta_r, M_2\ and\ M_3$. Refer to Figure 14.11.

Solution

From Chart C1, for $M_1 = 2.0$ and $\theta_i = 40°$, the deflection angle δ is equal to 10.6°. This corresponds to the angle through which the flow is turned after the incident wave and also the angle through which the flow is turned back after the reflected wave.

From Chart C2, for $M_1 = 2.0$ and $\delta = 10.6°$, M_2 is equal to 1.62.
From the same chart, for $M_2 = 1.62$ and $\delta = 10.6°$, M_3 is equal to 1.24.
From Chart C1, for $M_2 = 1.6$ and $\delta = 10.6°$, the shock wave angle θis $51.2°$, which is the angle between the flow direction in region 2 and the reflected wave. From geometrical consideration, $\theta_r = 51.2° - 10.6° = 40.6°$.

If M_2 is low enough, a simple shock reflection may be impossible. That is, for a given M_2, the required turning angle may be great enough so that no solution exists from Charts C1 and C2.

In a real fluid, the problem of oblique shock reflections is complicated by the presence of a boundary layer on the wall. the analysis presented here of oblique shock reflections is an approximate one, which neglects real fluid effects.

14.5 Conical Shock Waves

Supersonic flow about a right circular cone is considerably more complex than that about a wedge. But it has many similarities to wedge flow. For a cone at zero angle of attack with the oncoming stream, a conical shock is attached to the apex of the cone for small cone angles. (see Figure 14.12.)

It is interesting to compare the resultant wedge and cone flows (see Figure 14.13.) For a wedge, straight parallel flow exists before the oblique shock and after the shock.

For the three-dimensional semi-infinite cone, this is no longer possible. Streamlines after the conical shock must be curved in order that the three-dimensional continuity equation be satisfied. For axisymmetric flow about a semi-infinite cone, with no characteristic length along the cone surface, conditions after the shock are dependent only on the conical coordinate ω. That is, along each line of constant ω, the flow pressure, velocity, and so on, are constant. This indicates that the pressure on the surface of the cone after the shock is constant, independent of distance from the cone apex.

At each point on the conical wave, the oblique shock equations already presented are valid. Conical flow behind the wave is isentropic, with the static pressure increasing to the cone surface pressure. A solution for the conical shock thus requires fitting the isentropic compression behind the shock to the shock equations already derived. Results are shown

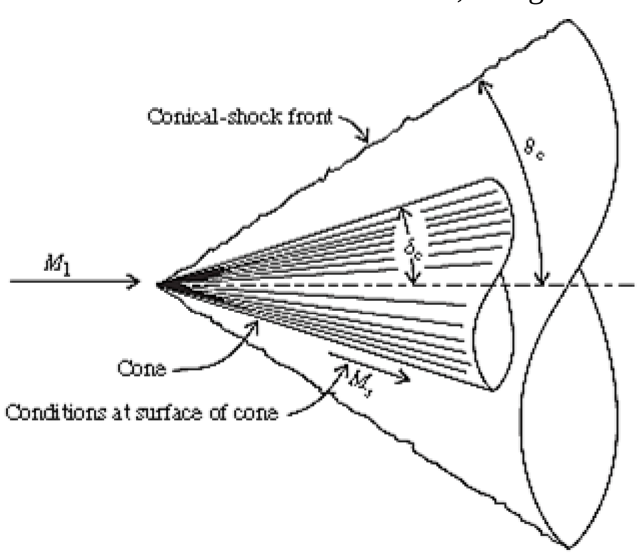


Figure 14.12 Conical shock with angle definitions.

in Charts C3, C4, and C5, which show the variation of shock wave angle, surface pressure coefficient, and surface Mach number with cone semi-vertex angle and Mach number.

Whereas the conical flow equations yield two shock solutions, the only one observed on an isolated conical body is the weak shock. As with wedge flow, for large enough cone angles there is no solution; the shock stands detached from the cone.

If we compare again the wedge and cone solutions, it can be seen from Charts C3, C4. and C5 that, for a given body half-angle and M_1 the shock on the wedge is inclined at a greater angle to the flow direction than the shock on the cone; this indicates that a stronger compression takes place across the wedge oblique shock. In other words, the wedge presents a greater flow disturbance than the cone. Again, this results from three-dimensional effects.

From a physical standpoint, the flow is unable to pass around the side of the two-dimensional wedge since it extends to infinity in the third dimension. Flow can pass around the sides of the three-dimensional cone, however, so the cone presents less overall disruption to the supersonic flow.

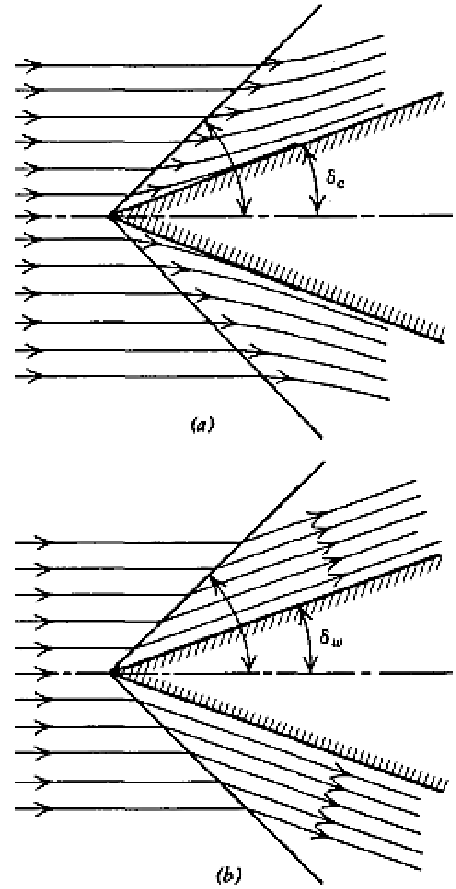


Figure 14-13 (a) cone, (b) wedge

Example 14.5 Uniform supersonic flow at Mach 2.0 and $p = 20\ kPa$ passes over a cone of semi-vertex angle of 10° aligned parallel to the flow direction. Determine the shock wave angle, Mach number of the flow along the cone surface, and the surface pressure coefficient.

Solution

From Chart C3, the shock wave angle is 31.2°.

From Chart C4, the Mach number along the cone surface is 1.85.

From Chart C5, the surface pressure ratio is 1.29

$$p_c = 20 * 1.29 = 25.8\ kPa$$

$$= \frac{(p_c - p_1)}{0.5\rho_1 V_1^2} = \frac{(p_c - p_1)}{0.5\rho_1 \gamma R T_1 M_1^2} = \frac{(p_c - p_1)}{0.5\gamma p_1 M_1^2}$$

$$C_p = \frac{25.8 - 20}{0.5 * 1.4 * 20 * 2^2} = 0.104$$

14.6 Supersonic oblique Shock Diffuser.

For a turbojet or ramjet traveling at high velocity, it is necessary to provide an inlet, or diffuser, that will perform the function of slowing down the incoming air with a loss of stagnation pressure. The use of a converging-diverging passage as an inlet for supersonic flow was studied in Chapter 4. Because such an internal deceleration device can operate isentropically only at the design speed, this type of diffuser has been found to be impractical during startup and when operating in an off-design condition. In fact without provisions for either varying the throat area or over speeding, the design condition could not be attained.

To eliminate the starting problem involved with the converging-diverging passage, the internal throat must be removed. Thus, a possible design is the normal-shock diffuser, where the deceleration takes place through a normal shock followed by subsonic diffusion in a diverging passage. (See Figure 14.14.) The disadvantage of this setup is the large loss in stagnation pressure incurred by the normal shock. Only at Mach numbers close to unity would this design be practicable.

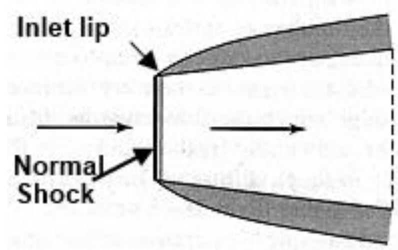


Figure 14.14 A Normal-Shock Diffuser

The advantage of decelerating through several oblique shocks rather than one normal shock was shown. The oblique-shock spike-type diffuser takes advantage of this condition and hence represents a practical device for decelerating a supersonic flow. The operation of a single oblique-shock inlet at design speed is depicted in Figure 14.15. External deceleration is accomplished through an oblique shock attached to the spike. Further deceleration takes place through a normal shock at the engine cowl inlet, with subsonic deceleration occurring internally. Even though a normal shock occurs in this system, the flight Mach number M has been reduced by the oblique shock, thus reducing the normal-shock strength and resultant stagnation pressure loss.

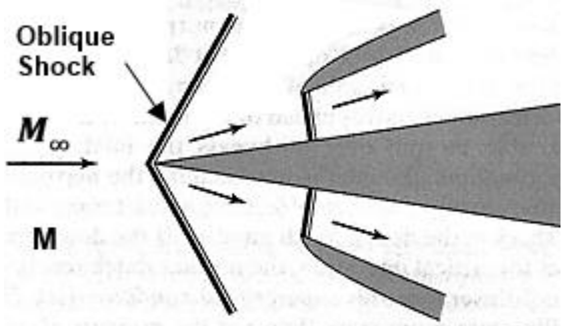


Figure 14.15 A Single Oblique-Shock Spike-Type Inlet at Design Speed

Theoretically, the greater the number of oblique shocks, the less the resultant total loss in stagnation pressure becomes. For example, a two-shock inlet is shown in Figure 14.16. Note, however, that along the surface of the spike, the boundary layer increases in thickness. The adverse pressure gradient created by the second shock may be sufficient to cause flow separation, with resultant loss of available energy. The greater the number of shocks, then, the

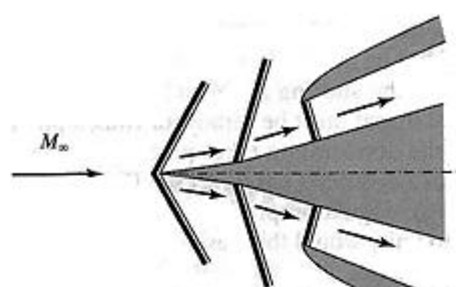


Figure 14.16 A Two-Oblique-Shock Spike-Type Inlet at Design Speed

greater the tendency toward flow separation is.

It is necessary to affect a compromise in supersonic diffuser design between the increased total-pressure recovery achieved by increasing the number of oblique shocks through which the flow must be diffused and the increased tendency toward separation brought about by the shocks. For this reason, with flight Mach numbers up to 2.0, a single-shock diffuser is generally employed, whereas multiple-shock inlets are required for higher flight Mach numbers.

Several different modes of operation of the spike diffuser may occur, depending on the downstream engine conditions such as nozzle opening, turbine speed, and fuel flow rate. This situation is in contrast to the converging-diverging inlet, where operation was dependent on the inlet's geometry. The spike diffuser's modes of operation are termed *subcritical, critical,* and *supercritical*, depending on the location of the normal shock.

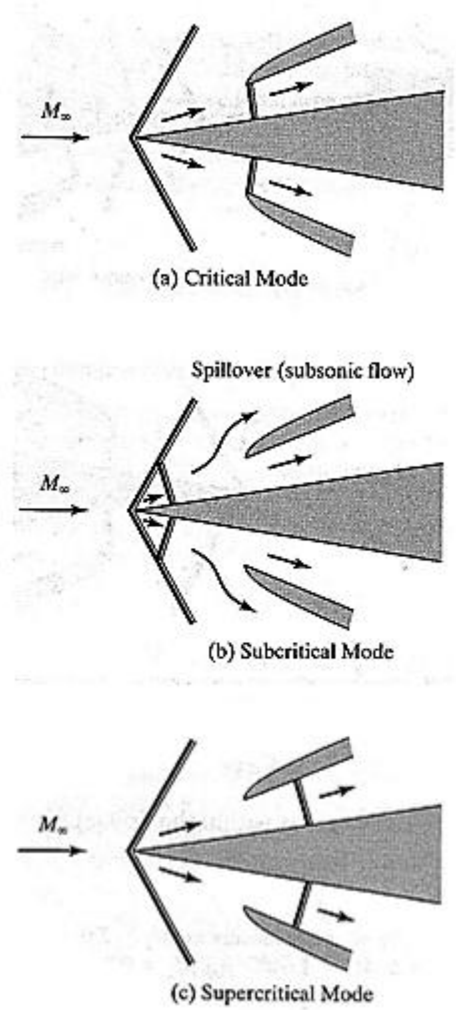


Figure 14.17 Modes of Operation of the Spike Diffuser (continue)

Critical operation occurs with the normal shock at the cowl inlet, as shown in Figure 14.17(a), with the engine operating at design speed. If the flow resistance downstream of the inlet is increased, with the engine still at the design flight Mach number, the normal shock moves ahead of the inlet, with some of the subsonic flow after the shock able to spill over or bypass the inlet. [See Figure 14.17(b).] For this *subcritical condition,* the inlet is not handling the maximum flow rate; furthermore, the pressure recovery is unfavorable, since at least some of the inlet air passes through a normal shock at the design Mach number.

If the downstream resistance is reduced below that for critical operation, the normal shock reaches an equilibrium position inside the diffuser. For this *supercritical condition* [see Figure 8.4(c)], the inlet is still handling maximum mass flow, yet the pressure recovery is less than that for critical operation, since the normal shock occurs at a higher Mach number in the diverging passage.

A turbojet engine must be able to operate efficiently both at other-than-design speeds and at different angles of attack. An engine operating at the critical mode may be pushed over into the undesirable subcritical mode by a small change of speed or angle of attack. For this reason, in actual operation, it is more practical to operate in the supercritical mode. While not providing quite as good a pressure recovery as critical operation, the supercritical mode still yields maximum engine-mass flow and makes a safety margin so that a small decrease in engine speed will not cause a transition to the subcritical mode. Thus, the supercritical mode provides a more stable engine operation.

Example 14.6. Compute the pressure recovery in one- and two-shock spike inlets. Compare the loss in total pressure for a one-shock spike diffuser (two dimensional) with that for two-shock diffuser operating at $\boldsymbol{Mach\ 2.0}$. Also repeat for inlet $\boldsymbol{Mach\ 4.0}$. (See Figure 14.18.). Assume that each oblique shock turns the flow through an angle of $\boldsymbol{\delta = 10^o}$. Take $\boldsymbol{\gamma = 1.4}$.

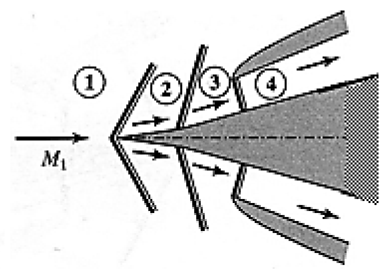


Figure 14.18 Flow Regions within the Spike Diffuser
(b) Two-shock inlet

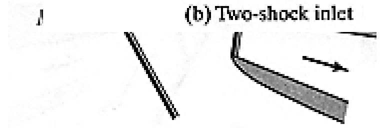


Figure 14.18 Flow Regions within the Spike Diffuser
(a) One-shock inlet

Solution

From the charts C1 & C2 at $M_1 = 2.0$ and $\delta = 10°$, the weak solution yields

$\theta_1 = 39.3°$. and $M_2 = 1.65$.

$M_{1n} = M_1 \sin\theta_1 = 2.0\ \sin 39.3 = 1.2668$

❖ For one oblique shock spike diffuser

From normal shock wave table at $M_{1n} = 1.2668$

$$M_{2n} = 0.80709 + (0.80164 - 0.80709) * \frac{1.2668 - 1.2600}{1.2700 - 1.2600}$$
$$= 0.80344$$

$\theta_2 = \sin^{-1}(M_{2n}/M_2) = \sin^{-1}(0.80344/1.65) = 29.14^o$

$$p_{o2}/p_{01} = 0.98568 + (0.98422 - 0.98568) * \frac{1.2668 - 1.2600}{1.2700 - 1.2600} = 0.9847$$

From normal shock wave table at $M_2 = 1.65$

$M_3 = 0.65396$ and $p_{o3}/p_{02} = 0.87599$

$\frac{p_{o3}}{p_{o1}} = \frac{p_{o3}}{p_{o2}} * \frac{p_{o2}}{p_{o1}} = 0.9847 * 0.87599 = 0.8626$

❖ For two oblique shock spike diffuser

From the charts C1 & C2 at $M_2 = 1.65$ and $\delta = 10°$, the weak solution yields

$\theta_2 = 49.4°$. and $M_3 = 1.28$.

$M_{2n} = M_2 \sin\theta_2 = 1.65\ \sin 49.4 = 1.2524$

From normal shock wave table at $M_{2n} = 1.2524$

$$M_{3n} = 0.81264 + (0.80709 - 0.81264) * \frac{1.2524 - 1.2500}{1.2600 - 1.2500} = 0.8113$$

$$\theta_3 = \sin^{-1}(M_{3n}/\mathrm{M}_3) = \sin^{-1}(0.8113/1.28) = 39.33^o$$

$$p_{o3}/p_{02} = 0.98706 + (0.98568 - 0.98706) * \frac{1.2668 - 1.2600}{1.2700 - 1.2600} = 0.9867$$

From normal shock wave table at $M_3 = 1.28$

$M_4 = 0.79631$ and $p_{o4}/p_{03} = 0.98268$

$$\frac{p_{04}}{p_{o1}} = \frac{p_{04}}{p_{o3}} * \frac{p_{03}}{p_{o2}} * \frac{p_{02}}{p_{o1}} = 0.9847 * 0.98679 * 0.98268 = 0.9548$$

$$improvement = \frac{0.9548 - 0.8626}{0.9548} * 100 = 9.66\ \%$$

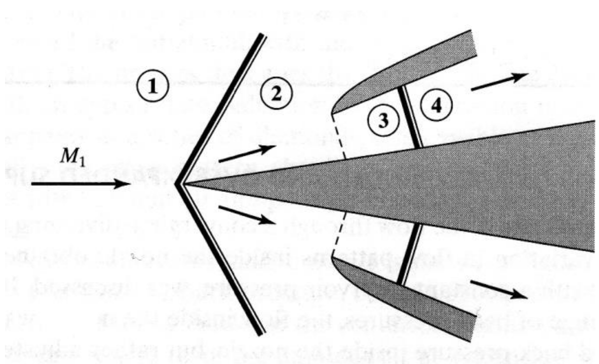


Figure14.19 Flow Regions within a Spike Diffuser Operating in the Supercritical Mode

When $M_1 = 4.0$:

$$\frac{p_{03}}{p_{o1}} = 0.2372 \quad and \quad \frac{p_{04}}{p_{o1}} = 0.3629$$

$$improvement = \frac{0.3629 - 0.2372}{0.3629} * 100 = 34.6\ \%$$

The improvement in total-pressure ratio gained by using a two- shock inlet over a one-shock inlet is (9.66%) when $M_1 = 2.0$ and (34.6%) when $M_1 = 4.0$. Thus, at flight Mach numbers of 2.0 and below, the use of an inlet with one oblique shock is satisfactory; at flight Mach numbers of 4.0, an inlet with two oblique shocks (or more) is necessary.

Example 14.7 A two-dimensional, spike-type inlet is operating in the supercritical mode at a flight Mach number of $\mathbf{3.0}$. The local static pressure and temperature are $\mathbf{50\ kPa}$ and $\mathbf{260\ K}$, respectively. The flow cross-sectional area at the cowl inlet $\mathbf{A_2 = 0.1\ m^2}$; the cross-sectional area at the location where the normal shock occurs in the diverging passage $\mathbf{A_3 = A_4 = 0.12\ m^2}$. (See Figure 14.19.) Calculate the mass-flow rate and total-pressure ratio $\boldsymbol{p_{o4}/p_{o3}}$. Neglect friction. The spike half-angle is $\mathbf{10°}$, and the ratio of specific heats is $\boldsymbol{\gamma = 1.4}$.

Solution

From the oblique shock wave charts C1 and C2 $\mathrm{M}_1 = 3.0$ and $\delta = 10°$, the weak solution yields $\theta_1 = 27.4°$ and $M_2 = 2.5$

$M_{1n} = M_1 \sin\theta_1 = 3.0\ \sin 27.4 = 1.3806$

From normal shock wave table at $M_{1n} = 1.3806$

$$M_{2n} = 0.74829 + (0.74396 - 0.748299) * \frac{1.3806 - 1.3800}{1.3900 - 1.3800} = 0.748$$

$$\theta_2 = \sin^{-1}(M_{2n}/\mathrm{M}_2) = \sin^{-1}(0.748/2.5) = 17.41^o$$

$$p_{o2}/p_{01} = 0.96304 + (0.96065 - 0.96304) * \frac{1.3806 - 1.3800}{1.3900 - 1.3800} = 0.9630$$

The flow from region 2 to region 3 is assumed to be isentropic. Thus, from isentropic flow table at $M_2 = 2.5$ gives $A_2/A_2^* = 2.63672$, then:

$$\frac{A_3}{A_3^*} = \frac{A_3}{A_2} * \frac{A_2}{A_2^*} = \frac{0.12}{0.1} * 2.63672 = 3.178 \qquad (A_3^* = A_2^* \, for \, isentropic \, flow)$$

From isentropic at this value gives

$$M_3 = 2.69 + (2.70 - 2.69)\frac{3.178 - 3.15299}{3.18301 - 3.15299} = 2.6983$$

$$p_{o3}/p_{02} = 1 \; (isentropic \; flow)$$

From normal shock table at $M_3 = 2.6983$

$$p_{o4}/p_{03} = 0.42714 + (0.42359 - 0.42714) * \frac{2.6983 - 2.690}{2.7000 - 2.690} = 0.4242$$

So the total pressure ratio is:

$$\frac{p_{o4}}{p_{o1}} = \frac{p_{o4}}{p_{o3}} * \frac{p_{o3}}{p_{o2}} * \frac{p_{o2}}{p_{o1}} = 0.4242 * 1.0 * 0.9630 = 0.4085$$

To calculate mass flow rate

$$p_{o1} = p_1 * \frac{p_{o1}}{p_1} = 50 * \left(1 + \frac{1.4 - 1}{2} 3^2\right)^{\frac{1.4}{1.4-1}} = 50 * 36733 = 1836.636 \; kN/m^2$$

$$p_{o2} = p_{o1} * \frac{p_{o2}}{p_{01}} = 1836.636 * 0.9630 = 1768.68 \; kN/m^2$$

$$p_2 = p_{o2}/\left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{\frac{1.4}{1.4-1}} = 1836.636/\left(1 + \frac{1.4 - 1}{2} 2.5^2\right)^{\frac{1.4}{1.4-1}} = 110.207 \; kN/m^2$$

$$\frac{T_{01}}{T_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right) = \left(1 + \frac{\gamma - 1}{2} 3^2\right) = 2.8$$

$$\frac{T_{02}}{T_2} = \left(1 + \frac{\gamma - 1}{2} M_2^2\right) = \left(1 + \frac{\gamma - 1}{2} 2.5^2\right) = 2.25$$

$$T_2 = \frac{T_2}{T_{o2}} * \frac{T_{01}}{T_1} * T_1 = \frac{1}{2.25} * 2.8 * 260 = 323.6 \, K \qquad stagnation \; temp \; is \; constant$$

$$\dot{m} = \rho_2 A_2 V_2 = \left(\frac{p_2}{RT_2}\right) A_2 M_2 \sqrt{\gamma R T_2}$$

$$\dot{m} = \left(\frac{110.2071}{0.287 * 323.6}\right) * 0.1 * 2.5 * \sqrt{1.4 * 287 * 323.6} = 106.971 \; kg/s$$

Lecture Fifteen / Prandtl Meyer Flow

15.1 Introduction

When a supersonic compression takes place at a concave corner, an oblique shock has been shown to occur at the corner. When supersonic flow passes over a convex corner, it is evident that some sort of supersonic expansion must take place. Previous results indicate that an expansion shock is impossible. However, a means must be available for the supersonic flow of Figure (15.1) to negotiate the corner. Here will present an analysis of the mechanism of two-dimensional, supersonic expansion flow, as might occur, for example during supersonic flow over a convex corner or at the exit of an under-expanded supersonic nozzle.

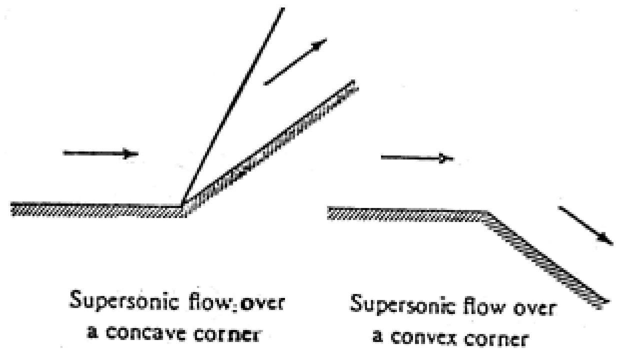


Figure 15.1

15.2 Thermodynamic Considerations

Two-dimensional, supersonic flow is to be turned through a finite angle at a convex corner. The mechanism of the resultant flow is of interest. Consider first the possibility of an oblique adiabatic shock occurring at the corner. Figure 15.2 shows the velocity vectors normal and tangential to such a wave. For this two-dimensional flow, uniform conditions prevail upstream and downstream of the wave. The equations of motion are exactly the same as those presented for oblique shock compression shock. Again, with no pressure gradient in the direction tangential to the wave, the tangential momentum equation yields

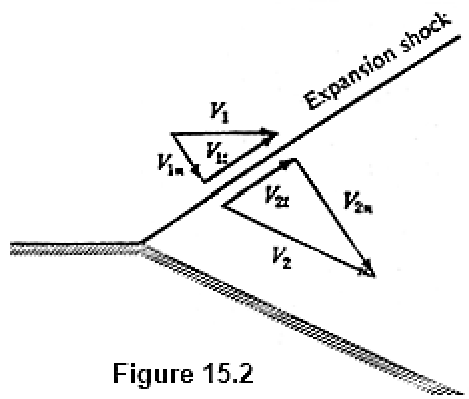


Figure 15.2

$$V_{1t} = V_{2t} \qquad (15.1)$$

From geometrical considerations, as $V_2 > V_1$, it follows that V_{2n} must be greater than V_{1n}. The normal momentum equation, eq. (14.2), yields

$$p_1 + \rho_1 V_{1n}^2 = p_2 + \rho_2 V_{2n}^2$$

Combining this with the continuity equation, eq. (14.1), where $A = constant$;

$$\rho_1 V_{1n}\, A = \rho_2 V_{2n}\, A$$

We obtain,

$$p_2 - p_1 = \rho_1 V_{1n}(V_{1n} - V_{2n}) \qquad (15.2)$$

Since $V_{2n} > V_{1n}$, see figure (15.2), it follows that $p_2 < p_1$, indicating that the resultant flow must be an expansion.

It has been shown that an oblique shock reduces to a normal shock for the velocity component normal to the wave, with the tangential component remaining unchanged. The ratios of pressure, temperature, and density across an oblique shock are functions of M_1 alone. The entropy change across an oblique shock can be written, then, in terms of M_{1n}, the resultant variation of Δs with M_{1n} being exactly the same as that for the normal shock. Hence, an oblique expansion shock $V_{2n} > V_{1n}$, just as a normal expansion shock, would involve a decrease in entropy during an adiabatic process. This violates the second law of thermodynamics and is impossible since $\Delta s \geq 0$. Therefore, the expansion shock, with sudden changes in flow properties, cannot occur at the convex corner. Instead, a more gradual type of supersonic expansion must take place.

15.3 Gradual Compressions and Expansions

When a supersonic stream undergoes a compression due to a finite, sudden change of direction at a concave corner, an oblique shock occurs at the corner. However, if the flow is allowed to change direction in a more gradual fashion, the compression can approach an isentropic process. Allowing supersonic flow to pass through several weak oblique shocks rather than one strong shock has been shown to reduce the resultant loss in stagnation pressure (or entropy rise) for a given change in flow direction (see Figure 15.3). In the limit, as the number of oblique shocks gets larger and larger, with each shock turning the flow through a smaller and smaller angle, the oblique shocks approach the Mach waves. The Mach wave, brought about by the presence of an infinitesimal disturbance in a supersonic flow, here corresponds to an oblique shock of vanishing strength, with infinitesimally small changes of velocity, flow direction, entropy, and so on, taking place across the wave (see Figure 15.4).

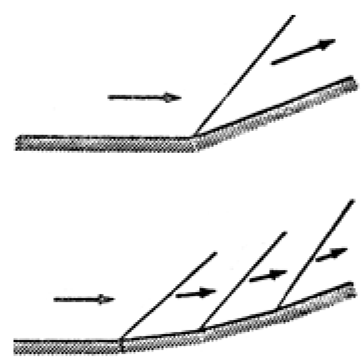


Figure 15.3

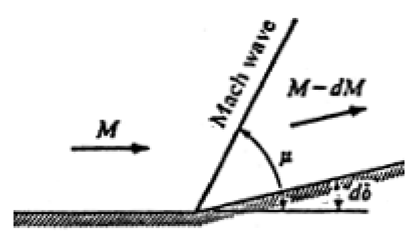


Figure 15.4

The wave angle is given by Equation $\mu = \sin^{-1}(1/M)$. Note that, from the oblique shock charts, Tables C, for an oblique shock of vanishing strength ($\delta = 0$), μ is evaluated from Mach number; for example, at $M_1 = 2.0$, $\delta = 0$ and $\mu = \theta = 30°$.

So, by employing a smooth turn, with the resultant oblique shocks approaching Mach waves, a continuous compression is achieved in the vicinity of the wall with vanishingly small entropy rise (see Figure 15.5).

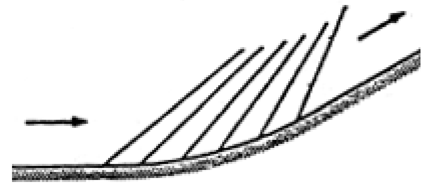


Figure 15.5

Away from the wall, however, the compression waves converge (Figure 15.6), coalescing to form a finite oblique shock wave. The characteristics of this shock are the same as those already discussed previously for an oblique shock wave of given M_1 and turning angle δ. In fact, far enough away from the wall, flow about the smooth turn cannot be distinguished from the flow about a sharp, concave corner of angle δ. It is important to note that here, again, the weak compression waves, each involving only an infinitesimal entropy rise, are able to reinforce one another to form a compression shock wave, with the resultant shock process involving a finite increase of entropy.

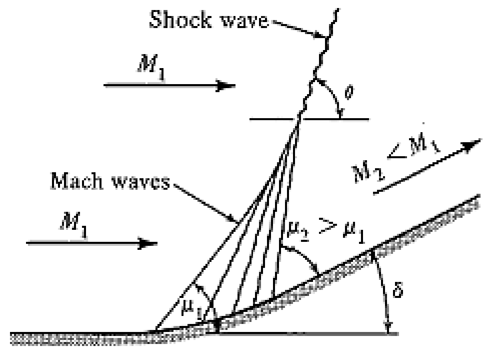


Figure 15.6 Smooth turn

Now consider a supersonic expansion through a series of infinitesimally small convex turns (see Figure 15.7). Mach waves are generated at each corner, with each wave inclined at an angle to the flow direction. For this expansion flow, unlike the compressive flow discussed previously, waves do not coalesce but rather spread out. The divergent waves cannot reinforce one another; the oblique expansion shock is physically impossible.

Flow between each of the waves in Figure (15.7) is uniform, so the length of the wall between waves has no effect on the variation of flow properties. Thus the lengths of the wall segments can be made vanishingly small, without affecting the overall variation of flow properties across the expansion. By thus reducing the wall segments, the series of convex turns becomes a sharp corner (see Figure 15.8.) The resultant series of expansion waves, centered at the corner, is called a **Prandtl Meyer expansion fan.**

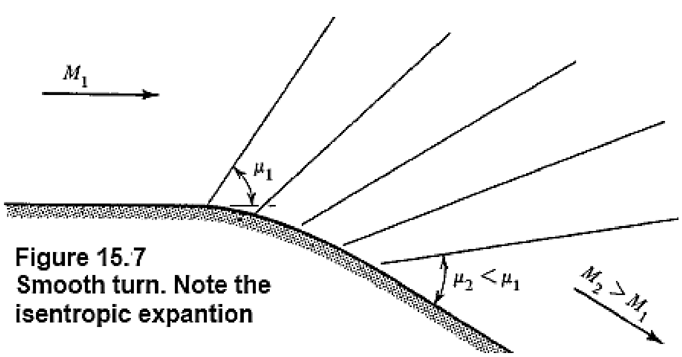


Figure 15.7
Smooth turn. Note the isentropic expantion

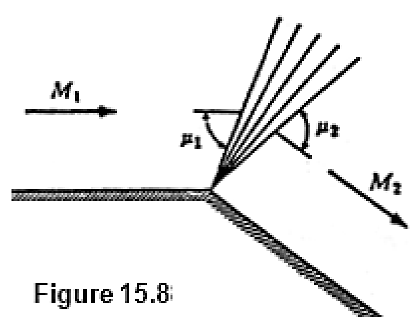


Figure 15.8

15.4 Flow Equations for a Prandtl Meyer Expansion Fan

It has been shown that supersonic expansion flow around a convex corner involves a smooth, gradual change in flow properties. The Prandtl Meyer fan consists of a series of Mach waves, centered at the convex corner. The initial wave is inclined to the approach flow at an angle $\mu_1 = \sin^{-1}(1/M_1)$ the final wave is inclined to the downstream flow at an angle $\mu_2 = \sin^{-1}(1/M_2)$. Flow conditions along each Mach wave are uniform; the variation of pressure, velocity and so on, through the expansion is only a function of angular position.

The equations for two-dimensional Prandtl Meyer flow will now be presented so that the variation of flow properties can be determined for a given flow turning angle. A perfect gas with constant specific heats will be assumed in the following analysis.

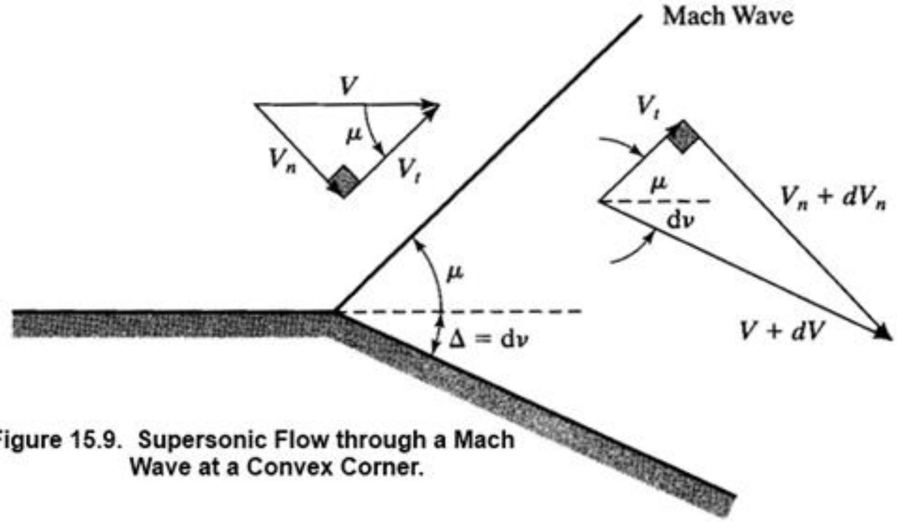


Figure 15.9. Supersonic Flow through a Mach Wave at a Convex Corner.

Consider first a single Mach wave, expanding the supersonic flow through an angle of magnitude dv. With no pressure gradient in the tangential direction, there is no change of the tangential velocity component across the wave. Equating the expressions for V_t upstream and downstream of the Mach wave (see figure 15.9);

$V\cos\mu = (V + dV)\cos(\mu + dv)$

$\quad = (V + dV)(\cos\mu\cos dv - \sin\mu\sin dv)$

Since dv is very small, then

$\cos dv = 1 \quad and \quad \sin dv = dv$, therefor;

$V\cos\mu = (V + dV)(\cos\mu - dv\sin\mu)$

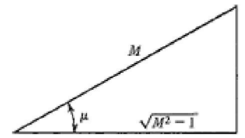
$V\cos\mu = V\cos\mu + dV\cos\mu - Vdv\sin\mu - dVdv\sin\mu$ (15.3)

The last term, containing the product of two differentials, can be dropped in comparison with the other terms of the equation. Simplifying, we obtain

$0 = dV\cos\mu - Vdv\sin\mu$

$$\frac{dV}{V} = d\nu \tan\mu$$

Since $\mu = \sin^{-1}(1/M)$, i.e. $\sin\mu = 1/M$, it follows that



$$\tan\mu = \frac{1}{\sqrt{M^2-1}}$$

$$\frac{dV}{V} = \frac{1}{\sqrt{M^2-1}} d\nu \qquad (15.4)$$

To solve for M as a function of ν, velocity V must be expressed in terms of M. For a perfect gas with constant specific heats, we can write,

$$V = M\sqrt{\gamma R T}$$

Taking log and differentiatng, we obtain

$$\log V = \log M + \log\sqrt{\gamma R} + \frac{1}{2}\log T$$

$$\frac{dV}{V} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T} \qquad (15.5)$$

But, for this adiabatic flow, there is no change in stagnation temperature.

$$T_o = constant = T\left(1 + \frac{(\gamma-1)}{2}M^2\right)$$

Taking logs and differentiating, we obtain

$$0 = \frac{dT}{T} + \frac{(\gamma-1)MdM}{1 + \frac{(\gamma-1)}{2}M^2} \qquad (15.6)$$

Cobining eqs. 5 & 6 gives

$$\frac{dV}{V} = \frac{dM}{M} - \frac{(\gamma-1)MdM}{2\left(1 + \frac{(\gamma-1)}{2}M^2\right)} \qquad (15.7)$$

$$\frac{dV}{V} = \frac{dM}{M}\left[1 - \frac{\frac{(\gamma-1)}{2}M^2}{\left(1 + \frac{(\gamma-1)}{2}M^2\right)}\right]$$

$$\frac{dV}{V} = \frac{dM}{M}\left[\frac{1}{\left(1 + \frac{(\gamma-1)}{2}M^2\right)}\right] \qquad (15.8)$$

Substitute eq. 8 into eq.4 gives

$$d\nu = \frac{dM}{M}\left[\frac{\sqrt{M^2-1}}{\left(1 + \frac{(\gamma-1)}{2}M^2\right)}\right] \qquad (15.9)$$

To determine the change of Mach number associated with a finite turning angle, the above eq. (15.9) can be integrated

$$\Delta v = (v_2 - v_1) = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{M\left(1+\frac{(\gamma-1)}{2}M^2\right)} dM$$

$$\Delta v = (v_2 - v_1)$$
$$= \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}\right]_{M_1}^{M_2} \qquad (15.10)$$

For the purpose of tabulating this result, it is convenient to define a reference state 1, so that

$$\Delta v = (v_2 - v_{ref}) = \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}\right]_{M_{ref}}^{M_2}$$

Let the reference state be $v = 0 \;\; at \;\; M = 1$. Now

$$v = \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}\right] \qquad (15.11)$$

The symbol v represents the angle through which a stream, initially at $M = 1$, must be expanded to reach a supersonic Mach number $M > 1$. Values of v have been tabulated in isentropic table, for Mach numbers from $1.0 \; to \; 5.0$ for $\gamma = 1.4$. Also presented are values of the wave angle μ, with both v and μ expressed in degrees.

To determine the angle through which a flow would have to be turned to expand from M_1 to M_2 with M_1 not equal to 1, it is necessary only to subtract the value of v_1 at M_1 from the value of v_2 at M_2, where v_1 and v_2 are found in isentropic table (see Figure 15.10).

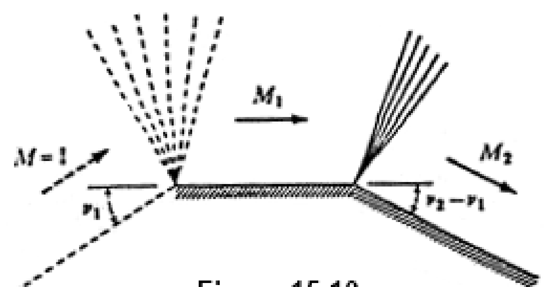


Figure 15.10

The variation of pressure, temperature, and other thermodynamic properties through the expansion can be found from the usual thermodynamic relations for isentropic flow, presented in Chapter 3. For this isentropic process, with no change in stagnation pressure;

$$\frac{p_2}{p_1} = \left[\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right]^{\gamma/(\gamma-1)} \qquad (15.12)$$

$$\frac{T_2}{T_1} = \frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2} \qquad (15.13)$$

Example 15.1 A uniform supersonic flow at Mach 2.0, with static pressure of 75 kPa and a temperature of 250 K, expands around a 10° convex corner. Determine the downstream Mach number M_2, pressure p_2, temperature T_2, and the fan angle. See Figure (15.11).

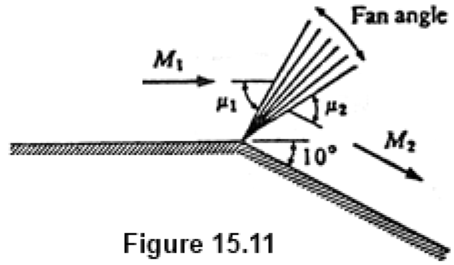


Figure 15.11

Solution

From isentropic table , at $M_1 = 2.0 \rightarrow$

$v_1 = 26.380° \;\; and \;\; \mu_1 = 30.00°$

But $v_2 = v_1 + 10° = 36.38°$

Again from isentropic table at $v_2 = 36.38 \rightarrow M_2 = 2.385 \; and \; \mu_2 = 24.79°$

From isentropic table at $M_2 = 2.385 \rightarrow p_2/p_{2o} = 0.07003, \; T_2/T_{2o} = 0.4678$

From Table A at $M_1 = 2.000 \rightarrow p_1/p_{1o} = 0.12780 \;\; and \;\; T_1/T_{1o} = 0.5556.$

With no change in stagnation pressure $p_{1t} = p_{2t}$ and constant stagnation temperature

$$\frac{p_2}{p_1} = \frac{p_2}{p_{2o}} * \frac{p_{1o}}{p_1} = \frac{0.07003}{0.1278} = 0.548$$

$$p_2 = 75 * 0.548 = 41.10 \; kPa$$

$$\frac{T_2}{T_1} = \frac{T_2}{T_{2o}} * \frac{T_{1o}}{T_1} = \frac{0.4678}{0.5556} = 0.842$$

$$T_2 = 250 * 0.842 = 210 \; K$$

$$fan \; angle = (\mu_1 + v_2 - v_1) - \mu_2$$
$$= 30.0 + 36.38 - 26.38 - 24.79 = 15.21°$$

EXAMPLE 15.2 FLOW in Example 15.1 is expanded through a second convex turn of angle 10° (sec Figure 15.12). Determine the downstream Mach number M_3 and the angle of the second fan.

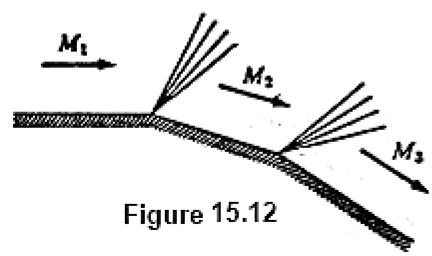


Figure 15.12

Solution

The initial wave of the second fan must he parallel to the final wave of the first fan. Again, the distance between waves can have no effect on the resultant flow, since the flow between the waves is uniform. Therefore, the variation of properties is the same whether the flow is expanded through two 10° turns or one 20° turn.

$v_3 = v_2 + 10° = 36.38° + 10° = 46.38°$

From isentropic table at $v_3 = 46.38 \rightarrow M_3 = 2.831 \rightarrow \mu_3 = 20.68°$

$$fan \; angle_{2nd} = v_3 - v_2 + \mu_2 - \mu_3$$
$$= 46.38 - 36.38 + 24.79 - 20.68 = 14.11°$$

EXAMPLE 15.3 An under-expanded, two-dimensional, supersonic nozzle exhausts into a region where $p_2 = 100\ kPa$ (Figure 15.13). Flow at the nozzle exit plane is uniform, with $p_1 = 200\ kPa$ and $M_1 = 2.0$. Determine the flow direction and Mach number after the initial expansion.

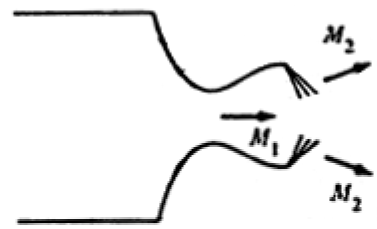


Figure 15.13

Solution

From isentropic table at $M_1 = 2.0 \ \rightarrow \ p_1/p_{1o} = 0.1278$

Since $p_{1o} = p_{2o}$ for an isentropic expansion, then

$$\frac{p_2}{p_{2t}} = \frac{p_2}{p_1} * \frac{p_1}{p_{1o}} = \frac{100}{200} * 0.1278 = 0.0639$$

From isentropic table at $p_2/p_{2o} = 0.0639 \ \rightarrow \ M_2 = 2.444$

From isentropic table, at $M_1 = 2.000 \ \rightarrow \ v_1 = 36.830°$

$M_2 = 2.444 \ \rightarrow \ v_2 = 37.803°$

So the flow is turned through

$v_2 - v_1 = 37.803° - 26.830° = 11.42°$

15.5 Prandtl Meyer Row in a Smooth Compression

It was shown in Section 15.3 that, at a smooth compressive turn in supersonic flow, Mach waves emanate from the wall, coalescing farther out in the stream to form an oblique shock wave. In the region from the wall out to the point of coalescence of the waves (see Figure 15.6), the flow is isentropic and possesses the same characteristics as Prandtl Meyer flow. Therefore, the equations derived for Prandtl Meyer flow can be applied to the isentropic flow region at a concave corner, even though a compression takes place at the corner. Naturally, the turning angle, Δv will here be negative, corresponding toa decrease inMach number. The extent of the isentropic flow region at a concave corner depends on the curvature of the wall. For a sharp turn, the region that can be treated as Prandtl Meyer flow is negligible; for a gradual turn, with a large radius of curvature, a much greater region has the characteristics of Prandtl Meyer now.

15.6 Maximum Turning Angle for Prandtl Meyer Flow

From Eq. (15.11), it can be seen that, as $M \rightarrow \infty$, or as the static pressure $p_2 \rightarrow 0$ (see Figure 15.14), the turning angle approaches a finite value of 130.4°. This result has significance, for example, in a determination of the shape of the exhaust plume of an under-expanded nozzle discharging

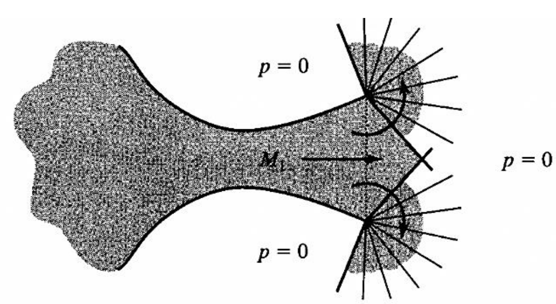


Figure 15.14 Maximum Turning Angle for a Supersonic Flow Exiting a Nozzle into a Vacuum

into the vacuum of Space. To prevent the impingement of rocket exhaust gases on a part of a Spacecraft, the designer must have knowledge of the shape of the rocket-nozzle exhaust plume; modification of a spacecraft geometrical design may be (required to prevent possible damage from the hot exhaust gases. Furthermore, the axial thrust of a rocket depends on the direction of the exhaust velocity vectors.

The actual magnitude of the maximum turning angle presented here has only academic interest, in that effects such as liquefaction of air gases and other departures from perfect gas flow would occur long before the ultimate pressure could be attained. However, the result does indicate the presence of a maximum turning angle for a supersonic expansion.

15.7 Reflections

When a Prandtl Meyer expansion flow impinges on a plane wall, as shown in Figure (15.15), sufficient waves must be generated to maintain the wall boundary condition; that is, at the wall surface, the flow must be parallel to the wall. Each Mach wave of the initial Prandtl Meyer fan, then, must reflect as an expansion Mach wave. The resultant wave interactions present complexities that render an exact analysis of the flow extremely difficult; however, the general nature of the flow can be recognized. An application is the expansion that takes place at the exit of an under-expanded, two-dimensional nozzle. Since, from symmetry, there can be no flow across the center streamline; this streamline can be replaced by a plane wall. The resultant flow situation is shown in Figure (15.16)

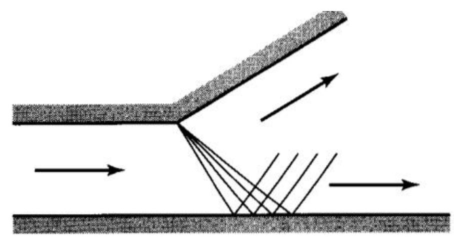


Figure 15.15 Reflection of a Prandtl–Meyer from a Plane Wall Expansion Fan

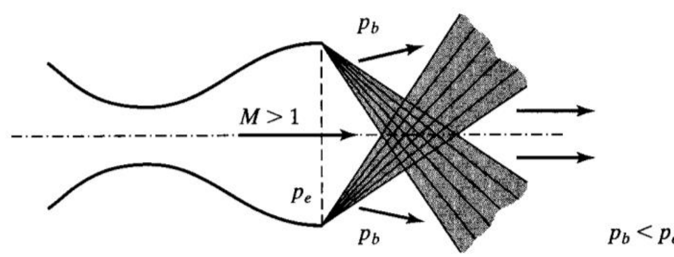


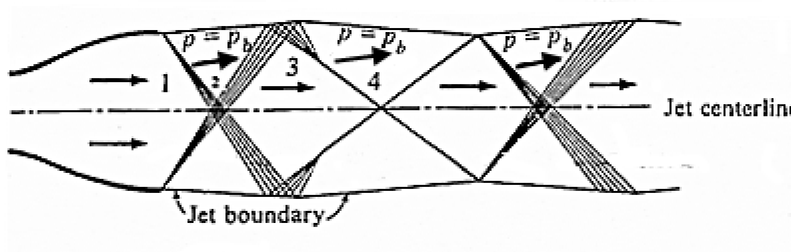
Figure 15.16 Supersonic Flow from an Underexpanded Nozzle

Chapter Sixteen / Plug, Underexpanded and Overexpanded Supersonic Nozzles

16.1 Exit Flow for Underexpanded and Overexpanded Supersonic Nozzles

The variation in flow patterns inside the nozzle obtained by changing the back pressure, with a constant reservoir pressure, was discussed early. It was shown that, over a certain range of back pressures, the flow was unable to adjust to the prescribed back pressure inside the nozzle, but rather adjusted externally in the form of compression waves or expansion waves. We can now discuss in detail the wave pattern occurring at the exit of an underexpanded or overexpanded nozzle.

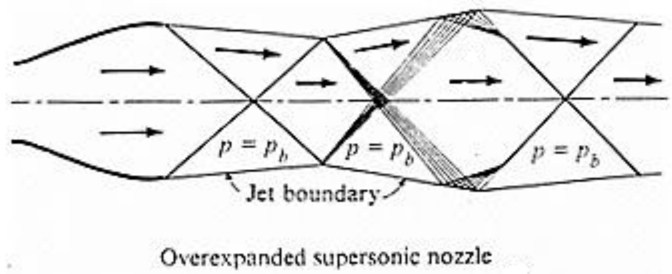
Consider first, flow at the exit plane of an underexpanded, two-dimensional nozzle (see Figure 16.1). Since the expansion inside the nozzle was insufficient to reach the back pressure, expansion fans form at the nozzle exit plane. As is shown in Figure (16.1), flow at the exit plane is assumed to be uniform and parallel, with $p_1 > p_b$. For this case, from symmetry, there can be no flow across the centerline of the jet. Thus the boundary conditions along the centerline are the same as those at a plane wall in nonviscous flow, and the normal velocity component must be equal to zero. The pressure is reduced to the prescribed value of back pressure in region 2 by the expansion fans. However, the flow in region 2 is turned away from the exhaust-jet centerline. To maintain the zero normal-velocity components along the centerline, the flow must be turned back toward the horizontal. Thus the intersection of the expansion fans centered at the nozzle exit yields another set of expansion waxes, just as did the reflection of the expansion fan from a plane wall (reflected Pradtl-Myer waves. The second expansion, however, produces a pressure in region 3 less than the back pressure, so the expansion waves reflect from the external air as oblique shocks. These compression waves produce a static pressure in region 4 equal to the back pressure, but again turn the flow away from the centerline. The intersection of the oblique shocks from either side of the jet then requires another set of oblique shocks to turn the flow back toward the horizontal, with the shocks reflecting from the external air as expansion waves.



Underexpanded supersonic nozzle
Figure 16.1

The process thus goes through a complete cycle and continues to repeat itself. The flow pattern discussed appears as a series of diamonds, often visible at the exit of high-speed rocket nozzles. Theoretically, the wave pattern should extend to infinity. Actually, however, mixing of the jet with ambient air along the jet boundaries eventually causes the wave pattern to die out.

Flow at the exit of an overexpanded nozzle is shown in Figure (16.2). Since the exit-plane pressure is less than the back pressure, oblique shock waves form at the nozzle exit. The intersection of these shocks at the centerline yields a second set of oblique shocks, which in turn reflect from the ambient air as expansion waxes. Thus, except for being out of phase with the wave pattern from the underexpanded nozzle, the jet flow of the overexpanded nozzle exhibits the same characteristics as the underexpanded nozzle.



Overexpanded supersonic nozzle
Figure 16.2

Example 16.1 A supersonic nozzle is designed to operate at Mach 2.0. Under a certain operating condition, however, an oblique shock making a 45° angle with the flow direction is observed at the nozzle exit plane, as in figure (16.3). What percent of increase in stagnation pressure would be necessary to eliminate this shock and maintain supersonic flow at the nozzle exit?

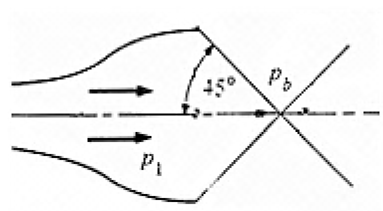


Figure 16.3

Solution

From isentropic table, for $M = 2.0$ gives $p_1/p_{o1} = 0.128$.

The component of M_1 normal to the oblique wave is $M_1 \, sin\, 45° = 1.41$.

From normal shock table, $p_b/p_1 = 2.15$. Therefore, with the oblique shock, the ratio

$$\frac{p_b}{p_{o1}} = \frac{p_b}{p_1} * \frac{p_1}{p_{o1}} = 2.15\, x\, 0.12S \; = \; 0.276$$

With the shock, p_{o1} is equal to

$$p_{o1} = \frac{1}{p_b/p_{o1}} p_b = (1/0.276)\, p_b = 3.62\, p_b$$

For supersonic exit flow with no shocks (perfectly expanded case),

$p_{1o} = (1/0.128)\, p_b = 7.81\, p_b$

$(7.81 \; - \; 3.62)/3.62 = 116\, percent$

Thus, an increase of 16% in stagnation pressure is required.

16.2 Plug Nozzle

The thrust developed by a nozzle is dependent on the nozzle exhaust velocity and the pressure at the nozzle exit plane. In a jet propulsion device, when an exit-plane pressure greater than ambient gives a positive contribution to the thrust of the device, whereas when an exit-plane pressure less than ambient gives a negative thrust component.

$$F = \dot{m}V_e + (p_e - p_a)A_e \qquad (16.01)$$

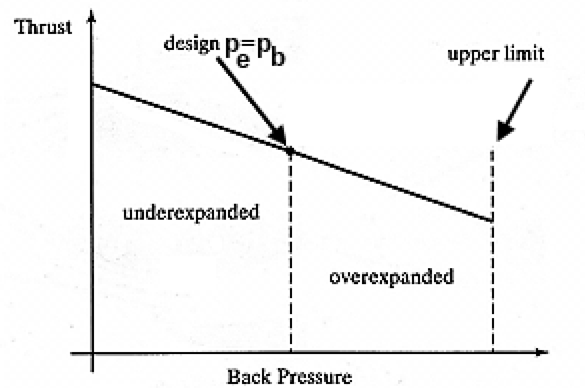
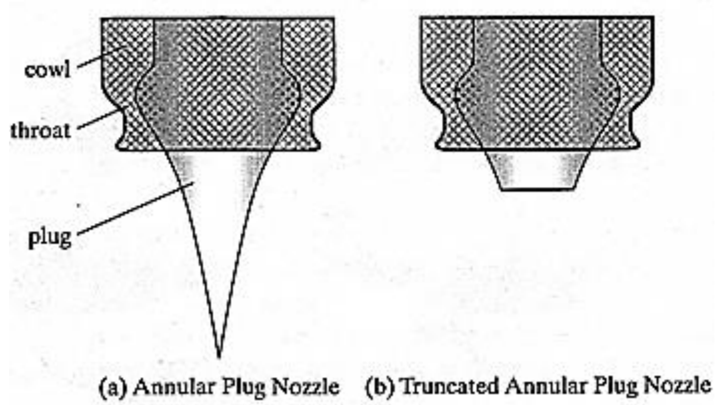


Figure 16.4: Illusterative diagram for thrust vs p_b for c-d nozzle

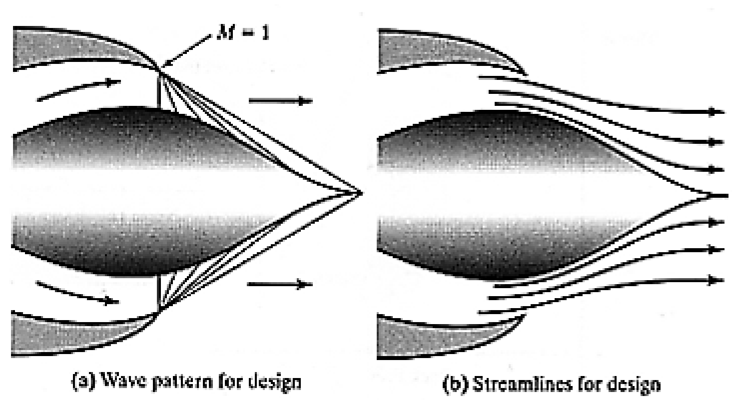
When a supersonic nozzle is operating in the under- or overexpanded regimes, with flow in the nozzle independent of back pressure, the exit velocity is unaffected by back pressure ($V_e = c$). Thus, over this range of back pressures, Eq. (16.01) shows that the greater thrusts are developed in the underexpanded case ($p_e > p_a$), and the lesser in the overexpanded case ($p_e < p_a$). A plot of thrust versus back pressure for a converging-diverging nozzle is shown in Figure 16.4. For back pressures greater than the upper limit indicated, a normal shock starts to appear in the diverging portion of the nozzle, the exit velocity becoming subsonic, and this analysis no longer applies.



(a) Annular Plug Nozzle (b) Truncated Annular Plug Nozzle

Figure 16.5 Plug Nozzles

The plug nozzle (figure 16.5) is a device that is intended to allow the flow to be directed or controlled by the ambient pressure rather than by the nozzle walls. In this nozzle, the supersonic flow is not confined within solid walls, but is exposed to the ambient pressure. Plug nozzle operation at the design pressure ratio is depicted in Figure 16.6. Figure 16.6a shows the expansion wave pattern and part b shows the streamlines at the nozzle exit. The annular flow first expands internally up to $M = 1$ at the throat. The remainder of the expansion to the back pressure occurs with the flow exposed to ambient pressure. Since the throat pressure is considerably higher than the



(a) Wave pattern for design (b) Streamlines for design

Figure 16.6 Wave Pattern and Streamlines within a Plug Nozzle at Design Mode back pressure, a Prandtl Meyer expansion fan is attached to the throat cowling as shown. The plug is designed so that, at the design pressure ratio, the final expansion wave intersects the plug apex. Thus, under this operating condition, the pressure at the plug wall decreases continuously from throat pressure to ambient pressure, just as with the converging-diverging perfectly expanded nozzle.

To produce a maximum axial thrust, it is necessary for the exit flow to have an axial direction. Therefore, the flow at the throat cowling must be directed toward the axis so that the turning produced by the expansion fan will yield axial flow at the plug apex.

For the underexpanded case, the operation of the plug nozzle (Figure 16.7) is similar to that of the converging-diverging nozzle. The pressure along the plug is the same as for the design case, just as the static pressure along the converging-diverging nozzle wall is the same as for the perfectly expanded case. With a lower back pressure than that for the design case depicted in Figure 16.6, the flow continues to expand after the apex pressure, yielding a non-axial jet velocity component, just as with the underexpanded supersonic converging-diverging nozzle.

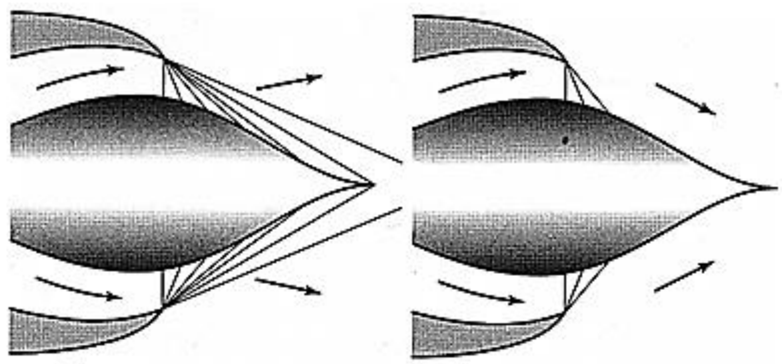


Figure 16.7 Wave Patterns of a Plug Nozzle Operating in Under- and Overexpanded Modes

The major improvement to be derived from the plug nozzle occurs with the overexpanded mode of operation. This is significant, in that a rocket nozzle, for example, accelerating from sea level up to design speed and altitude, must pass through the overexpanded regime. With the ambient pressure greater than the design back pressure, the flow expands along the plug only up to the design back pressure. The final wave of the expansion fan centered at the cowling intersects the plug at a point upstream of the apex. As shown in Figure 16.7, the outer boundaries of the exhaust jet are directed inward. Further weak compression and expansion waves occur downstream of the point of impingement of the final wave from the fan; the strength and location of these waves are dependent on the plug contour. Thus the expansion along the plug is controlled by the back pressure, whereas the converging-diverging nozzle expansion is controlled by nozzle geometry.

A plot of pressure along the plug surface versus x is given in Figure 16.8. The pressure along the plug surface does not decrease below ambient, so there is not a negative thrust term due to pressure difference. As a result, the plug nozzle provides improved thrust over the converging-diverging nozzle for the overexpanded case (see Figure 16.9).

It would appear desirable to design the plug so as to provide for isentropic expansion flow along its curved pointed surface. However, this design leads to a rather long plug and heavy design. it has been shown that replacement of the curved shape with a simple cone results in only a small loss of thrust for a cone half angles up to 30°. Thus the plug nozzle has the further advantage over the converging-diverging nozzle of being short and compact. One major problem with the plug nozzle, however, is that of designing a plug to withstand the high temperatures that exist, for example, in the exhaust gases of a rocket engine. This requires cooling of the plug or allowance for its ablation is necessary.

Studies have shown that one half of the plug length provides almost no thrust and only added weight. a truncated plug have been considered. The flow pattern of these shortened plugs is complicated.

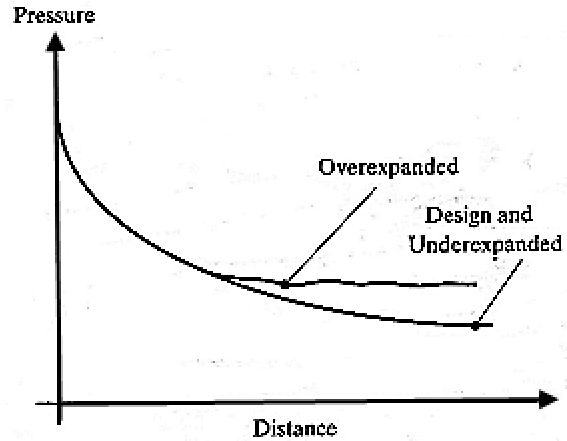


Figure 16.8 Pressure Distribution within a Plug Nozzle

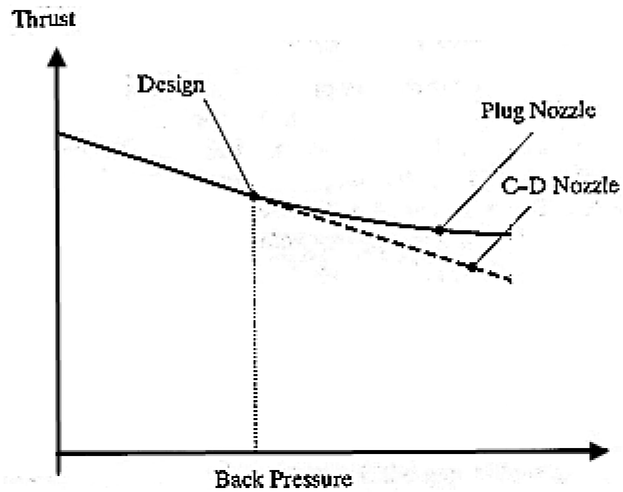


Figure 16.9 Comparison of Thrust and Back Pressure for Plug and C–D Nozzles

Example 15.2

A rocket nozzle is designed to operate with a ratio of chamber pressure to ambient pressure (p_o/p_a) of 50. Compare the performance of a plug nozzle with that of a converging-diverging nozzle for two cases where the nozzle is operating overexpanded; ($p_o/p_b = 40$) and ($p_o/p_b = 20$). Make the Comparison on the basis of thrust coefficient; $CT = thrust/(p_o * A_{throat})$. Assume $\gamma = 1.4$ and in both cases neglect the effect of non-axial exit velocity components.

Solution

❖ For the design case,

From ($p_b/p_o = 1/50 = 0.02$) and since the flow in the design case the flow is isentropic, then:

$M_e = 3.208$ and $(T_e/T_o) = 0.3270$, $A_e/A_{th} = 5.1584$

$F = C_F A_{thr}\, p_c$

$$C_F = \frac{\dot{m}_{th} V_e}{p_o A_{th}} = \frac{(\rho_{th} A_{th} V_{th}) V_e}{p_o A_{th}}$$

$$C_F = \left(\frac{p_{th}}{RT_{th}}\right)\frac{V_{th} V_e}{p_o} = \left(\frac{p_{th}}{p_o}\right)\left(\frac{p_o}{RT_o}\right)\left(\frac{T_o}{T_{th}}\right)\frac{(M_{th} a_{th})(M_e a_e)}{p_o}$$

$$C_F = \left(\frac{p_{th}}{RT_{th}}\right)\frac{(M_{th} a_{th})(M_e a_e)}{p_o}$$

For design condition the nozzle is choked and Mach number at throat is unity, then

$p_{th}/p_o = 0.5283$ and $T_{th}/T_o = 0.8333$

$$C_F = \left(\frac{0.5283\, p_o}{R * 0.8333 T_o}\right)\frac{\sqrt{1.4 * R * 0.8333 T_o} * 3.2077\sqrt{1.4 * R * 0.3270 T_o}}{p_o}$$

$C_F = 1.4862$

❖ For the converging-diverging nozzle operating off design:

$$C_F = \frac{\dot{m}_{th} V_e}{p_o A_{th}} + \frac{A_e (p_e - p_a)}{p_o A_{th}} = \frac{\dot{m}_{th} V_e}{p_o A_{th}} + \frac{A_e}{A_{th}}\left(\frac{p_e}{p_o} - \frac{p_a}{p_o}\right)$$

For $p_o/p_a = 40$

$$C_F = 1.4862 + 5.1584\left(\frac{1}{50} - \frac{1}{40}\right) = 1.4604$$

For $p_o/p_a = 20$

$$C_F = 1.4862 + 5.1584\left(\frac{1}{50} - \frac{1}{20}\right) = 1.3314$$

❖ For the plug nozzle operating off design:

Flow in the plug nozzle does not continue to expand below ambient pressure, so there is no pressure term in the expression for thrust.

Now from isentropic table at $p_c/p_a = 40 \rightarrow$ gives

$$M_e = 3.04 + (3.06 - 3.04)\frac{0.0250 - 0.0256}{0.0249 - 0.0256} = 3.0486$$

$$\frac{T_e}{T_o} = 0.3511 + (0.3481 - 0.3511)\frac{0.0250 - 0.0256}{0.0249 - 0.0256} = 0.3485$$

$$C_F = \left(\frac{0.5283\, p_o}{R * 0.8333 T_o}\right)\frac{\sqrt{1.4 * R * 0.8333 T_o} * 3.0486\sqrt{1.4 * R * 0.3485 T_o}}{p_o}$$

$C_F = 0.63399 * 1.08010 * 2.12944 = 1.4582$

Now from isentropic table at $p_o/p_a = 20 \rightarrow$ gives

$$M_e = 2.60 + (2.62 - 2.60)\frac{0.0500 - 0.0501}{0.0486 - 0.0501} = 2.6013$$

$$\frac{T_e}{T_o} = 0.4252 + (0.4214 - 0.4252)\frac{0.0500 - 0.0501}{0.0486 - 0.0501} = 0.4249$$

$$C_F = \left(\frac{0.5283\,p_o}{R * 0.8333T_o}\right)\frac{\sqrt{1.4 * R * 0.8333T_o} * 2.6013\sqrt{1.4 * R * 0..4249T_o}}{p_o}$$

$$C_F = 0.63399 * 1.08010 * 2.0063 = 1.3739$$

Chapter Seventeen / Supersonic Airfoils

17.1. Supersonic lift and drag coefficients

The shape of a wing section to be used in low-speed, incompressible flow is the teardrop, or streamlined, profile. This shape is predicated on incompressible aerodynamics, where, for example, drag is composed of skin friction on the airfoil surface and pressure or profile drag, due to the effects of flow separation at the rear of the airfoil.

In supersonic flow, however, the design must be completely modified, owing to the occurrence of shocks. For example, if a streamlined profile with a rounded blunt nose were used in supersonic flow, either an attached shock of relatively high strength would occur at the nose or, if θ were great enough, a detached shock (Figure 17.1) would take occur in front of the airfoil. In both cases, the high pressures after the shockwave produce excessive drag forces on the airfoil. To minimize the drag due to the presence of shocks, the supersonic airfoil must have a pointed nose and be as thin as possible. The ideal case is a flat-plate airfoil possessing zero thickness.

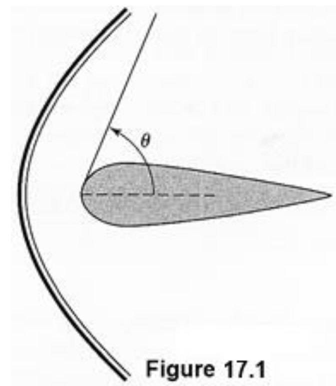


Figure 17.1
Detached Shock in Front of a Streamlined Airfoil

Consider a two-dimensional flat plate at an *angle of attack* (AoA) to the approach flow as shown in Figure 17.2. (It should be noted that the flat plate is an idealization; structurally, such an airfoil is not exist). Flow over the upper surface is turned through an expansion fan centered at the nose; flow over the lower surface is compressed through an oblique shock attached to the nose. The difference in pressure between the upper and lower surfaces causes a net upward force, directed normal to the flow direction, the *lift*, on the airfoil. A force opposing the motion of the airfoil, the *drag*, on the airfoil, accompanies this

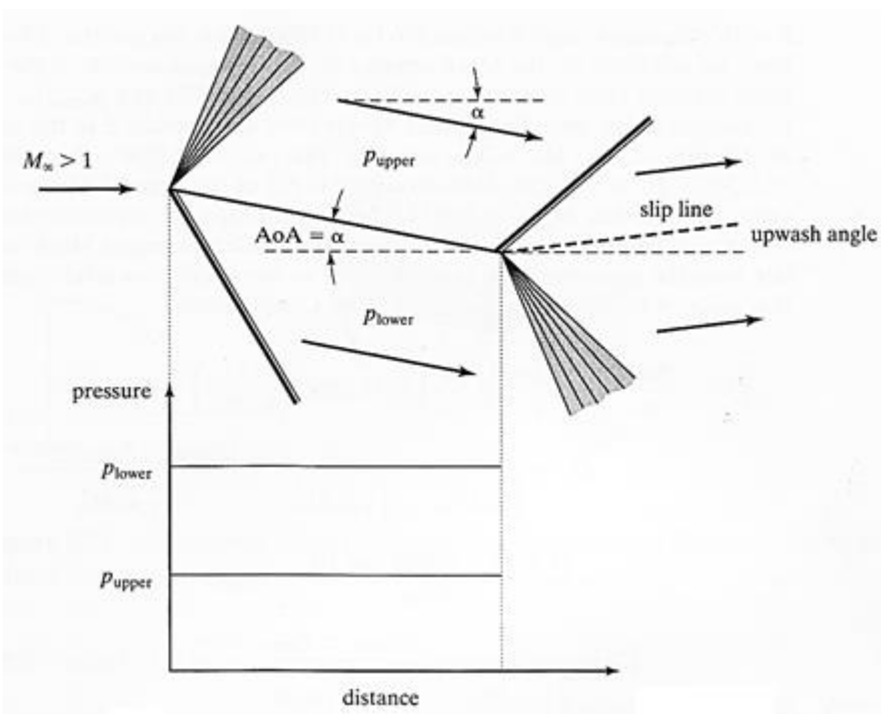


Figure 17.2 Supersonic Flow Past a Flat Plate at an Angle of Attack

lift. The latter force is called *wave drag*, since it exists only because of the supersonic wave pattern involved with this flow.

For the lift and drag for supersonic flow past a flat-plate airfoil operating at an angle of attack a to the flow direction are given by

$$L = -(p_{upper} * Area_{upper\,surface}) \cos\alpha + (p_{lower} * Area_{lower\,surface}) \cos\alpha$$

$$L = -(p_{upper} * c) \cos\alpha + (p_{lower} * c) \cos\alpha$$

$$= c(p_{lower} - p_{upper}) \cos\alpha \qquad (17.1)$$

$$D = -(p_{upper} * c) \sin\alpha + (p_{lower} * c) \sin\alpha$$

$$= c(p_{lower} - p_{upper}) \sin\alpha \qquad (17.2)$$

17.2. Existence of an Oblique Shock and an Expansion Fan.

When a thin body, for example a flat plate of zero thickness, is placed at an angle of attack within a supersonic stream, both oblique shocks and expansion fans will appear at various locations on the body, (See Figure 17.3.). Oblique shocks will appear at locations where the flow must be turned because the plate forms a concave corner with the stream (on the bottom of the plate at the leading edge and on the top of the plate at the trailing edge).

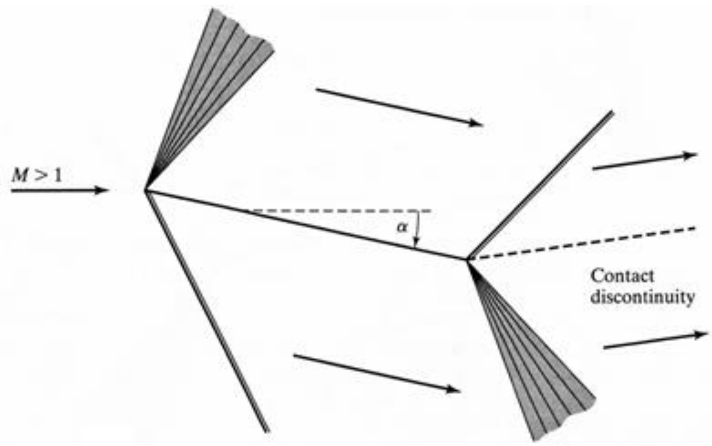


Figure 17.3 Supersonic Flow past a Flat Plate at an Angle of Attack to the Flow

Expansion fans will appear at locations where the flow must be turned because the plate forms a convex corner with the stream (on the top of the plate at the leading edge and the bottom of the plate at the trailing edge). Here, we are interested only in the flow at the trailing edge of the plate. At this location, there is a confluence of an oblique shock and an expansion fan, as shown in Figure 17.3.

Moreover, because the streams that pass over the top and bottom surfaces of the plate will not have the same value of entropy as after they have passed through the shock and expansions on each side of the plate, a **contact discontinuity**, originating at the trailing edge, will separate the two streams. The flow direction of the contact discontinuity is determined by requiring that the flow on either side of the discontinuity have the same flow angle and that the pressure across the discontinuity remain constant. And the following is valid (see figure 17.4):

$$\alpha_1 = \alpha_2 = AoA$$

$$\alpha_3 = \alpha_4$$

$\alpha_3 = \alpha_2 + v_3 - v_1$

$\frac{p_3}{p_\infty} = \frac{p_4}{p_\infty}$

$M_3 \neq M_4$

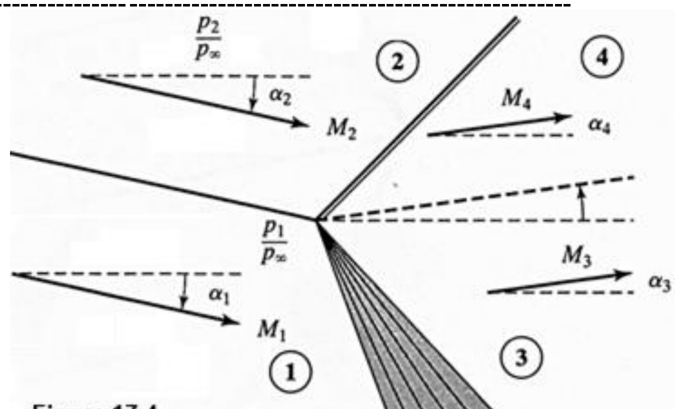


Figure 17.4
The flow direction at the rear of a flat plate in supersonic flow

At rear of trailing edge there are many unknowns (α_3, v_3, p_3 and M_3) and the solution procedure is iterative and it is left for the interest student.

For a supersonic airfoil, a thin airfoil with a pointed nose is required. The curved, symmetrical airfoil represents one possibility. For small angles of attack, oblique shocks are attached to the nose, with the stronger shock occurring on the lower surface, since the flow turning angle must be greater on this surface. (See Figure 17.5.) Due to the continuous curvature of the airfoil, flow over the airfoil continually changes direction, and a gradual expansion occurs over the upper and lower surfaces. Expansion waves are produced as shown in Figure 17.5. If the angle of attack becomes too great, or if the nose half-angle A is too large, the oblique shocks may detach from the nose, yielding excessive drag.

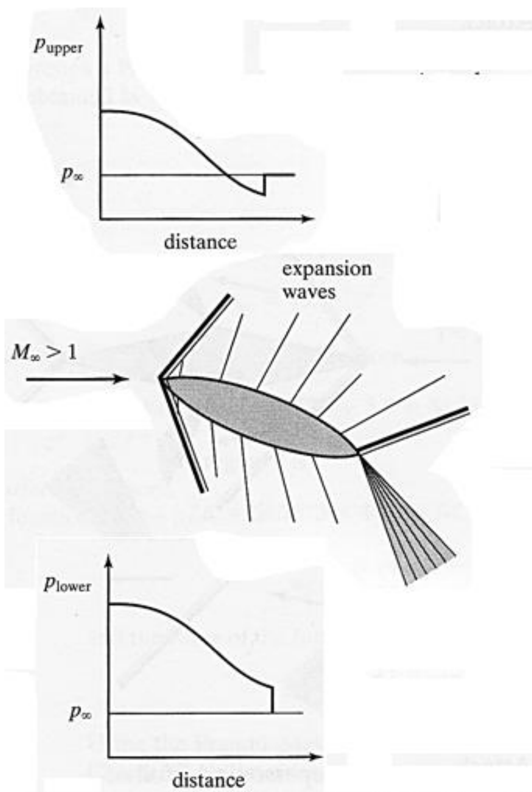


Figure 17.5 Supersonic Flow Past a Curved, Symmetrical Airfoil

Another airfoil shape for supersonic flow is the diamond profile, shown in Figure 17.6. Flow over the upper surface is first expanded through a fan centered at A and then is turned through another expansion fan at B. If the angle of attack is small enough, or if the airfoil is thick enough, flow over the upper surface may first be compressed through an oblique shock attached at A. (See Figure 17.7.) Flow over the lower surface is turned through an oblique shock at A and then through an expansion fan at C. As shown by the pressure distribution, higher pressures over the lower surfaces yield a lift force; higher pressures at the front surfaces caused a drag force.

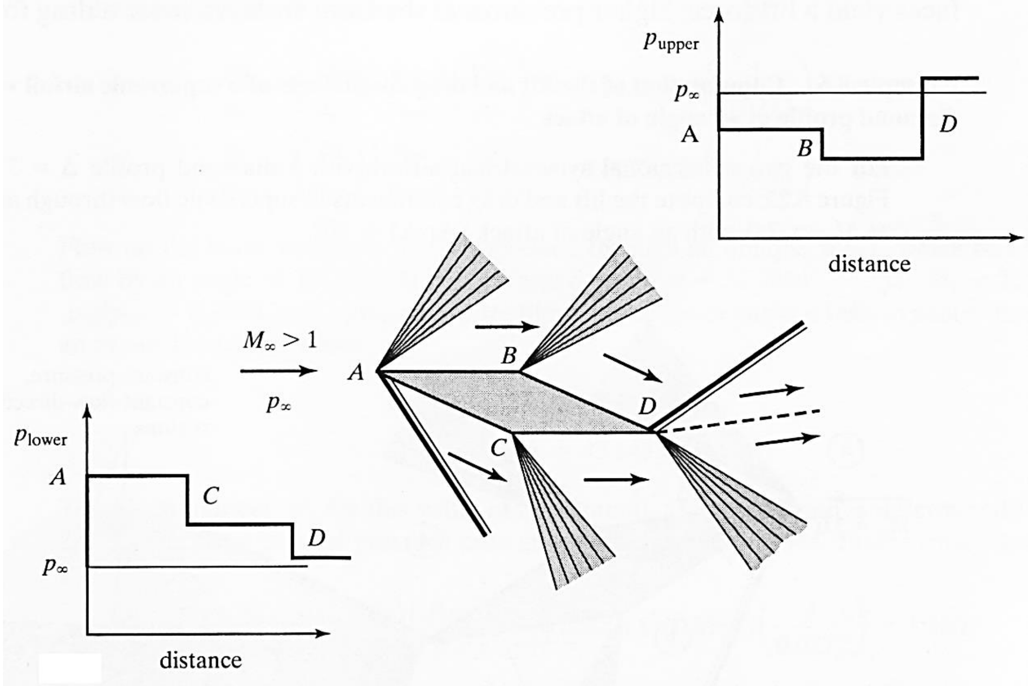


Figure 25.6 Wave Pattern on a Supersonic Airfoil of Diamond Profile at an Angle of Attack

Example 17.1. Compute of the lift and drag coefficients of a flat-plate airfoil at an angle of attack in a supersonic stream. The flat-plate airfoil is of chord length $c = 1$ m in supersonic flow through air at $M = 2.5$ and $\alpha = 10^o$.

Solution

From figure 17.2

For lower surface: find the static pressure on the lower surface behind the oblique shock.

From oblique shock tables at $M_\infty = 2.5$ and $\delta = 10^o$ gives

The shock angle $\theta = 31.85^o$ and $M_{lower} = 2.1$

$M_{\infty,n} = M_\infty \sin\theta = 2.5 \sin 31.85 = 1.3192$

From normal shock table at $M_{\infty,n} = 1.3192$ gives

$$\frac{p_{lower}}{p_\infty} = 1.83545 + (1.86613 - 1.83545)\frac{1.3192 - 1.31}{1.3200 - 1.31} = 1.8637$$

For upper surface: find the static pressure on the upper surface behind the Prandtl-Meyer fan.

From Prandtl-Meyer table at $M_\infty = 2.5$ gives $v_\infty = 39.1236^o$

And the final shock wave angle is

$v_{upper} = v_\infty + AoA = 39.1236 + 10 = 49.1236^o$

From Prandtl-Meyer table at $v_{upper} = 49.1236^o$ gives

$$M_{upper} = 2.96 + (2.97 - 2.96)\frac{49.1236 - 48.78333}{49.17520 - 48.78333} = 2.9687$$

The flow through the expansion fan is isentropic; that is stagnation pressure is constant, so $p_{o,\infty} = p_{o,upper}$, and from isentropic flow table at $M_{upper} = 2.9687$

$$\frac{p_{upper}}{p_o} = 0.02891 + (0.02848 - 0.02891)\frac{2.9687 - 2.96}{2.9700 - 2.96} = 0.028536$$

And from isentropic flow table at $M_\infty = 2.5$ gives $p_\infty/p_o = 0.05853$

Then

$$\frac{p_{upper}}{p_\infty} = \frac{p_{upper}}{p_o} * \frac{p_o}{p_\infty} = \frac{0.028536}{0.05853} = 0.48755$$

$$C_l = \frac{L}{0.5\rho_\infty V_\infty^2 S_w} = \frac{L}{0.5\gamma p_\infty M_\infty^2 c} = \frac{c\left(p_{lower} - p_{upper}\right)\cos\alpha}{0.5\gamma p_\infty M_\infty^2 c}$$

$$= \frac{(1.8637 - 0.48755)\cos 10}{0.5 * 1.4 * 2.5^2} = \frac{1.3552}{4.375} = 0.3098$$

$$C_d = \frac{D}{0.5\rho_\infty V_\infty^2 S_w} = \frac{\left(p_{lower} - p_{upper}\right)\sin\alpha}{0.5\gamma p_\infty M_\infty^2} = C_l \tan\alpha$$

$$= 0.3098 \tan 10 = 0.0546$$

Example 17.2. For the two-dimensional symmetrical airfoil with a diamond profile $\Delta = 5^o$, shown in Figure 17.7, compute the lift and drag coefficients in supersonic flow through air $M_\infty = 3.0$, with an angle of attack $(AoA) = 10°$.

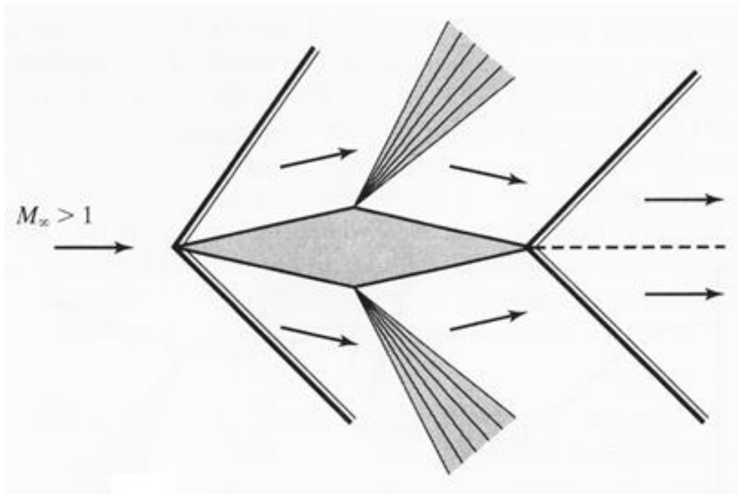


Figure 17.7 Wave Pattern on a Supersonic Airfoil of Diamond Profile at Zero Angle of Attack

Solution

On the *upper surface*, supersonic flow is first expanded through a Prandtl-Meyer fan. The Prandtl-Meyer function for the free stream conditions is obtained as

From Prandtl Meyer tables at $M_\infty = 3.0$, $v_\infty = 49.7573°$

The Prandtl-Meyer function in region 2 is therefore

$v_2 = v_\infty + \Delta = 49.7573 + 5.0 = 54.7573°$

And the value of the Prandtl Meyer function in region 4 is

$v_4 = v_2 + 2\Delta = 54.7573 + 10.0 = 64.7573°$

Using the Prandtl-Meyer tables, we determine the respective Mach numbers for these functions to be

$$M_2 = 3.27 + (3.28 - 3.27)\frac{54.7573 - 54.7035}{54.8770 - 54.7035} = 3.2731$$

$$M_4 = 3.92 + (3.93 - 3.92)\frac{64.7573 - 64.7125}{64.8483 - 64.7125} = 3.9233$$

The static-to-total-pressure ratios at these two Mach numbers, as well as the freestream ratio, can be readily determined,

$$\frac{p_{o\infty}}{p_\infty} = \left(1 + \frac{\gamma - 1}{2}M_\infty^2\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{0.4}{2}3.0^2\right)^{1.4/0.4} = 36.7327$$

$$\frac{p_{o2}}{p_2} = \left(1 + \frac{\gamma - 1}{2}M_2^2\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{0.4}{2}3.2731^2\right)^{1.4/0.4} = 55.0211$$

$$\frac{p_{o4}}{p_4} = \left(1 + \frac{\gamma - 1}{2}M_4^2\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{0.4}{2}3.9233^2\right)^{1.4/0.4} = 137.0047$$

And since the flow between the freestream and regions 2 and 4 is isentropic

$$p_{o\infty} = p_{o2} = p_{o4}$$

Then

$$\frac{p_2}{p_\infty} = \frac{p_2}{p_{o2}} * \frac{p_{o\infty}}{p_\infty} = \frac{36.7327}{55.0211} = 0.6676$$

$$\frac{p_4}{p_\infty} = \frac{p_4}{p_{o4}} * \frac{p_{o\infty}}{p_\infty} = \frac{36.7327}{137.0047} = 0.2681$$

Flow on the *lower surface* is first compressed through an oblique shock, and from oblique shock charts at $M_\infty = 3.0$ and $\delta = (\Delta + AoA) = 5 + 10 = 15^o$, give

$\theta = 4$ and $M_1 = 2.255$

$$M_{n\infty} = M_\infty \sin\theta = 3.0 \sin 32.24 = 1.6004$$

From normal shock tables at $M_{n\infty} = 1.6004$ gives

$$\frac{p_{o1}}{p_{o\infty}} = 0.89520 + (0.89145 - 0.89520)\frac{1.6004 - 1.6000}{1.6100 - 1.6000} = 0.8951$$

$$\frac{p_1}{p_\infty} = 2.8200 + (2.85745 - 2.8200)\frac{1.6004 - 1.6000}{1.6100 - 1.6000} = 2.8215$$

Now from Prandtl Meyer tables at $M_1 = 2.255$ gives

$$v_1 = 33.01841 + (33.27301 - 33.01841)\frac{2.255 - 2.250}{2.260 - 2.250} = 33.1457^o$$

And

$$\frac{p_{o1}}{p_1} = \left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{0.4}{2}2.255^2\right)^{1.4/0.4} = 11.6540$$

$v_3 = v_1 + 2\Delta = 33.1457 + 2 * 5 = 43.1457^o$

And from Prandtl Meyer tables at $v_3 = 43.1457^o$

$$M_3 = 2.67 + (2.68 - 2.67)\frac{43.1457 - 42.96819}{43.18678 - 42.96819} = 2.6781$$

$$\frac{p_{o3}}{p_3} = \left(1 + \frac{\gamma - 1}{2} M_3^2\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{0.4}{2} 2.6781^2\right)^{1.4/0.4} = 22.5112$$

As $p_{o3} = p_{o1}$ for isentropic flow throw Prandtl-Meyer fan, then

$$\frac{p_3}{p_\infty} = \frac{p_3}{p_{o3}} * \frac{p_{o1}}{p_1} * \frac{p_1}{p_\infty} = \frac{1}{22.5112} * 11.654 * 2.8215 = 1.4607$$

The lift force is calculated, (we have 4 equal quarters for the diamond airfoil), as

The straight segment line length for each quarter, ℓ, is

$$\ell = \frac{c/2}{\cos\Delta} = \frac{c}{2\cos 5} = 0.502\ c$$

The depth of the airfoil is unity and the surface area is $0.502\ c$. Now

$$L = +(p_1 * 0.502\ c)\cos(\alpha + \Delta) + (p_3 * 0.502\ c)\cos(\alpha - \Delta) \\ -(p_2 * 0.502\ c)\cos(\alpha - \Delta) - (p_4 * 0.502\ c)\cos(\alpha + \Delta)$$

$$L = +(p_1 * 0.502\ c)\cos 15^o + (p_3 * 0.502\ c)\cos 5^o \\ -(p_2 * 0.502\ c)\cos 5^o - (p_4 * 0.502\ c)\cos 15^o$$

$$L = +2.8215 p_\infty * 0.502\ c * 0.9659 + 1.4607 p_\infty * 0.502\ c * 0.9962 \\ -0.6676 p_\infty * 0.502\ c * 0.9962 - 0.2681 p_\infty * 0.502\ c * 0.9659$$

$$L = +1.3681 p_\infty c + 0.7305 p_\infty c - 0.3339 p_\infty c - 0.13 p_\infty c$$

$$L = +1.6347 p_\infty c$$

$$C_l = \frac{L}{0.5\rho_\infty V_\infty^2 S_w} = \frac{L}{0.5\gamma p_\infty M_\infty^2 c} = \frac{1.6347 p_\infty c}{0.5\gamma p_\infty M_\infty^2 c} \\ = \frac{1.6347}{0.5 * 1.4 * 3.0^2} = 0.2595$$

$$D = +(p_1 * 0.502\ c)\sin 15^o + (p_3 * 0.502\ c)\sin 5^o \\ -(p_2 * 0.502\ c)\sin 5^o - (p_4 * 0.502\ c)\sin 15^o$$

$$D = +2.8215 p_\infty * 0.502\ c * 0.2588 + 1.4607 p_\infty * 0.502\ c * 0.0872 \\ -0.6676 p_\infty * 0.502\ c * 0.0872 - 0.2681 p_\infty * 0.502\ c * 0.2588$$

$$D = +0.3666 p_\infty c + 0.0639 p_\infty c - 0.0292 p_\infty c - 0.0348 p_\infty c$$

$$D = +0.3665 p_\infty c$$

$$C_d = \frac{D}{0.5\rho_\infty V_\infty^2 S_w} = \frac{(p_{lower} - p_{upper})\sin\alpha}{0.5\gamma p_\infty M_\infty^2} = \frac{0.3665 p_\infty c}{0.5\gamma p_\infty M_\infty^2} = C_l \tan\alpha \\ = \frac{0.3665}{0.5 * 1.4 * 3.0^2} = 0.0582$$

The computation of the angle of the slip line, and therefore the angle of the flow downstream of the airfoil at regions 5 and 6 is left for the interested student.

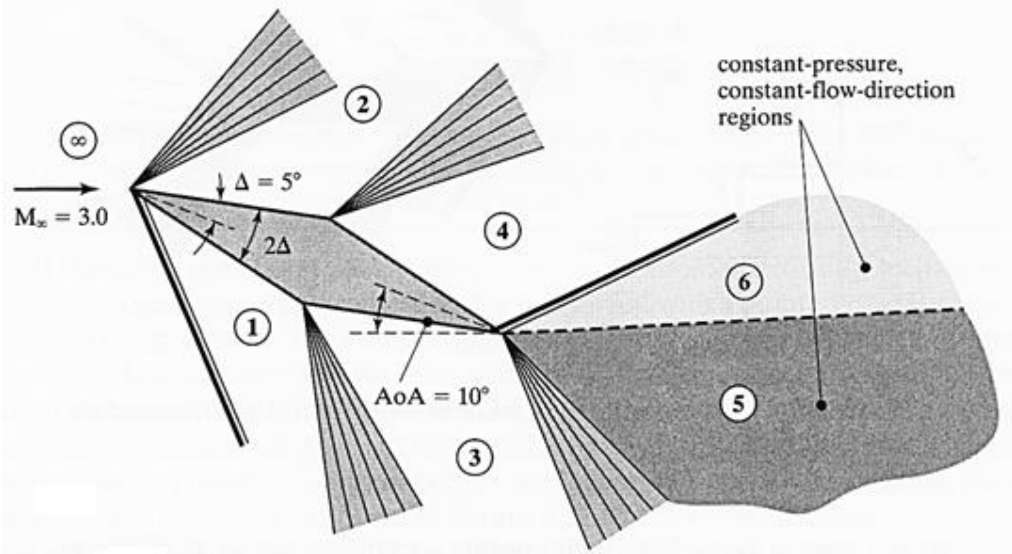


Figure 17.8 Supersonic Flow Past an Airfoil with a Diamond Profile
illustrative drawing for example 17.2.

Chapter Eighteen/ Fanno flow-Part 1

18.1. Introduction

We have mentioned that area changes, friction, and heat transfer are the most important factors affecting the properties in a flow system. Up to this Chapter we have considered only one of these factors, that of variations in area. We now wish to take a look at the subject of friction losses. To study only the effects of friction, we analyze flow in a constant-area duct without heat transfer. We consider first the flow of an arbitrary fluid and discover that its behavior follows a definite pattern which is dependent on whether the flow is in the subsonic or supersonic regime.

Working equations are developed for the case of a perfect gas, and the introduction of a reference point allows a table to be constructed. As before, the table permits rapid solutions to many problems of this type, which are called ***Fanno flow***.

18.2. Working Relations for Fanno Flow

Consider one-dimensional steady flow of perfect gas with constant specific heats through constant area duct. In case of adiabatic, no work exchange, the flow is Fanno flow where friction effect is considered. The basic equations of continuity, energy, and momentum under the following assumptions, are derived:

Adiabatic $ds_{ext} = 0, \delta q = 0$

Friction exist $ds_{int} \neq 0$

No shaft work $\delta w_s = 0$

Neglect potential $dz = 0$

Constant area $dA = 0$

Constant specific heat $c_p = const$

The stagnation temperature will be proved to be constant along the duct while the stagnation pressure will suffer from losses due to friction. The entropy is expected to increase.

- **State**

$p = \rho RT$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad (18.1)$$

- **Continuity**

$\dot{m} = \rho AV = const.$

$$\rho V = G = const \quad (18.2)$$

The flow area is constant. G is a constant, which is referred to as the ***mass velocity***.

- **Energy**

We start with s.f.e.e.

$$h_{o1} + q = h_{o2} + w_s$$

For adiabatic and no work, this becomes

$$h_{o1} = h_{o2} \tag{18.3}$$

If we neglect the potential term, this means that

$$h_o = h + \frac{V^2}{2} = const$$

$$c_p T_o = c_p T_1 + \frac{V_1^2}{2}$$

$$T_o = T_1 + \frac{V_1^2}{2c_p} = T_2 + \frac{V_2^2}{2c_p}$$

$$V = Ma = M\sqrt{\gamma RT}$$

$$T_1 + \frac{\gamma R T_1 M_1^2}{2c_p} = T_2 + \frac{\gamma R T_2 M_2^2}{2c_p}$$

$$T_1\left(1 + \frac{\gamma R M_1^2}{2c_p}\right) = T_2\left(1 + \frac{\gamma R M_2^2}{2c_p}\right)$$

$$\boxed{\frac{T_2}{T_1} = \frac{1 + [(\gamma-1)/2]M_1^2}{1 + [(\gamma-1)/2]M_2^2}} \tag{18.4}$$

From continuity equation

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{M_1 a_1}{M_2 a_2} = \frac{M_1}{M_2} * \left(\frac{T_1}{T_2}\right)^{1/2}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2} * \left(\frac{1 + [(\gamma-1)/2]M_2^2}{1 + [(\gamma-1)/2]M_1^2}\right)^{1/2}} \tag{18.5}$$

$$\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} * \frac{T_2}{T_1} = \frac{M_1}{M_2}\left(\frac{T_1}{T_2}\right)^{1/2} * \frac{T_2}{T_1}$$

$$\frac{p_2}{p_1} = \frac{M_1}{M_2} * \left(\frac{T_2}{T_1}\right)^{1/2})$$

$$\boxed{\frac{p_2}{p_1} = \frac{M_1}{M_2} * \left(\frac{1 + [(\gamma-1)/2]M_1^2}{1 + [(\gamma-1)/2]M_2^2}\right)^{1/2}} \tag{18.6}$$

- **Entropy**

$$Tds = c_p dT - vdp = c_p dT - RT\frac{dp}{p}$$

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

$$\Delta s = c_p \ln T - R \ln p$$

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma - 1} \ln \frac{T_2}{T_1} - \ln \frac{p_2}{p_1}$$

Substitute for Temperature and pressure ratio, from eqs. (18.4) and (18.6) gives

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma - 1} \ln \left(\frac{1 + [(\gamma - 1)/2] M_1^2}{1 + [(\gamma - 1)/2] M_2^2} \right)^{\gamma/(\gamma - 1)} - \ln \left[\frac{M_1}{M_2} \left(\frac{1 + [(\gamma - 1)/2] M_1^2}{1 + [(\gamma - 1)/2] M_2^2} \right)^{1/2} \right]$$

$$\boxed{\frac{s_2 - s_1}{R} = \ln \frac{M_2}{M_1} * \sqrt{\left(\frac{1 + [(\gamma - 1)/2] M_1^2}{1 + [(\gamma - 1)/2] M_2^2} \right)^{(\gamma + 1)/(\gamma - 1)}} \qquad (18.7)}$$

To derive an expression for stagnation pressure ratio for adiabatic, no-work flow of a perfect gas, we start from the following thermodynamic relation for stagnation (total) properties

$$T_o ds_o = dh_o - \frac{dp_o}{\rho_o} \qquad (18.8)$$

$$ds_o = ds_{external} + ds_{internal} \qquad (18.9)$$

Since $\delta q = T ds_{ext} = 0$ for adiabatic flow and $dh_o = 0$ from energy equation, then

$$\frac{dp_o}{\rho_o} = -T_o ds_{int} \qquad (18.10)$$

$$p_o = \rho_o R T_o$$

$$\frac{dp_o}{p_o} = -\frac{ds_{int}}{R}$$

$$\frac{\Delta s_{int}}{R} = -\ln \frac{p_{o2}}{p_{o1}} \qquad (18.11)$$

Substitute from eq. (18.7) into eq. (18.11) gives

$$\boxed{\frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2} * \sqrt{\left(\frac{1 + [(\gamma - 1)/2] M_2^2}{1 + [(\gamma - 1)/2] M_1^2} \right)^{(\gamma + 1)/(\gamma - 1)}} \qquad (18.12)}$$

- **Momentum**

$$\sum \mathbf{F} = \iint_{cs} V_x \, \rho \, (\boldsymbol{V}.\hat{n}) \, d\boldsymbol{A}$$

The external forces that act on the element are the pressure and shear forces as shown in figure (17.1).

$$pA - (p + dp)A - \tau A_{sur} = (\rho A V)(V + dV) - (\rho A V)V \qquad (18.13a)$$

$$-A dp - \tau A_{sur} = (\rho A V) dV \qquad (18.13b)$$

τ is shear stress due to wall friction and A_{sur} duct surrounding surface area. The hydraulic diameter;

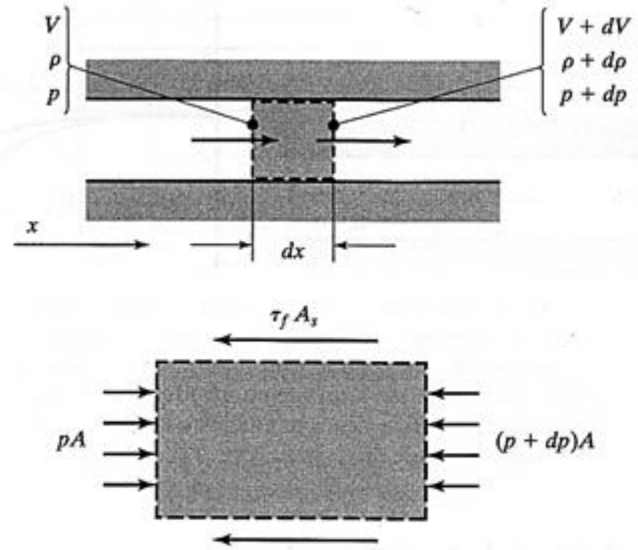


Figure 18.1 Controll volume for isolated, constant area duct with frictional flow

$$D_h = \frac{cross\ section\ area}{wetted\ perimeter} = \frac{4A}{P}$$

$$D_h = \frac{4(\pi D^2/4}{\pi D} = D$$

Surface area is

$$A_{sur} = Length * wetted\ perimeter$$

$$= dx * P = dx\frac{4A}{D}$$

Friction factor, f, is four times friction coefficient, c_f.

$$f = 4c_f = 4\tau/0.5\rho V^2$$

$$\tau = c_f * 0.5\rho V^2 = f * 0.5\rho V^2/4$$

Substitute for τ and A_{sur} in eq. (18.13)

$$-Adp - \frac{f0.5\rho V^2}{4}dx\frac{4A}{D} = (\rho AV)dV$$

$$-dp - 0.5\rho V^2 f dx\frac{dx}{D} = (\rho V)dV$$

Divided by p

$$\frac{dp}{p} + 0.5\frac{V^2}{RT}f\frac{dx}{D} + \frac{V^2}{RT}\frac{dV}{V} = 0$$

$$\frac{dp}{p} + 0.5\gamma M^2 f\frac{dx}{D} + \gamma M^2\frac{dV}{V} = 0 \qquad (18.14)$$

From state equation and the definition of Mach number

$$\rho V = \frac{p}{RT}M\sqrt{\gamma RT} = \sqrt{\frac{\gamma}{RT}}\ pM = const$$

Taking logarithmic of this expression and then differentiating gives

$$\log\sqrt{\frac{\gamma}{R}} - \frac{1}{2}\log T + \log p + \log M = \log const$$

$$\frac{dp}{p} = -\frac{dM}{M} + \frac{1}{2}\frac{dT}{T} \qquad (18.15)$$

$$V = Ma = M\sqrt{\gamma RT}$$

$$\log V = \log M + \frac{1}{2}\log\gamma R + \frac{1}{2}\log T$$

$$\frac{dV}{V} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T} \qquad (18.16)$$

Substitute for dp/p and dV/V into eq (18.15) gives

$$-\frac{dM}{M}+\frac{1}{2}\frac{dT}{T}+0.5\gamma M^2 f\frac{dx}{D}+\gamma M^2\left(\frac{dM}{M}+\frac{1}{2}\frac{dT}{T}\right)=0$$

Then

$$f\frac{dx}{D}=-\frac{2}{\gamma M^2}\left(-\frac{dM}{M}+\frac{1}{2}\frac{dT}{T}\right)-2\left(\frac{dM}{M}+\frac{1}{2}\frac{dT}{T}\right) \qquad (18.17a)$$

$$f\frac{dx}{D}=\frac{2dM}{\gamma M^3}-\frac{dM^2}{M^2}-\frac{dT}{T}-\frac{1}{\gamma M^2}\frac{dT}{T} \qquad (18.17b)$$

For this type of flow, the stagnation temperature is constant, then

$$T_o=T\left(1+\frac{\gamma-1}{2}M^2\right)=const$$

Taking logarithmic of this expression and then differentiating gives

$$\log T_o=\log T+\log\left(1+\frac{\gamma-1}{2}M^2\right)$$

$$\frac{dT}{T}=-\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)} \qquad (18.18)$$

Substitute for dT/T into eq (18.18) gives

$$f\frac{dx}{D}=\frac{2}{\gamma}\frac{dM}{M^3}-\frac{dM^2}{M^2}+\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}+\frac{1}{\gamma M^2}\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)} \qquad (18.19)$$

Eq (18.19) should be simplified further. The last term can be manipulated to be

$$\frac{1}{\gamma M^2}\frac{\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}=\frac{A}{\gamma M^2}+\frac{B}{\left(1+\frac{\gamma-1}{2}M^2\right)}=\frac{1}{\gamma M^2}+\frac{-\frac{(\gamma-1)}{2\gamma}}{\left(1+\frac{\gamma-1}{2}M^2\right)}$$

Then

$$\frac{1}{\gamma M^2}\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}=\frac{(\gamma-1)}{2\gamma}\frac{dM^2}{M^2}-\frac{(\gamma-1)}{2\gamma}\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}$$

Substitute this expression into eq (18.19) and rearrange gives

$$f\frac{dx}{D}=\left(\frac{\gamma+1}{2\gamma}\right)\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}+\frac{2}{\gamma}\frac{dM}{M^3}-\left(\frac{\gamma+1}{2\gamma}\right)\frac{dM^2}{M^2} \qquad (18.20)$$

Integration of this equation gives

$$f\frac{(x_2-x_1)}{D}=\left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)-\frac{1}{\gamma}\left(\frac{1}{M_2^2}-\frac{1}{M_1^2}\right)-\left(\frac{\gamma+1}{2\gamma}\right)\ln\frac{M_2^2}{M_1^2}$$

For Fanno flow, the integration limits are

At $x_2 = L_{max}$ $\rightarrow M_2 = 1$. This is reference length.
At $x_1 = 0$ $\rightarrow M_1 = M$. This is the section under consideration.

$$f\frac{L_{max}}{D} = \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}\right) - \frac{1}{\gamma}\left(1-\frac{1}{M^2}\right) - \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{1}{M^2}\right) \quad (18.21)$$

Eq (18.21) relates friction factor, f, to M directly. For air $\gamma = 1.4$, then;
For supersonic the value of $f\ L_{max}/D$ lies between 0 $at\ M = 1$ and 0.8215 $at\ M = \infty$
For subsonic the value of $f\ L_{max}/D$ becomes very large as M becomes very small.

18.3 Reference state and Fanno Flow Table

Eqs 18.4, 5, 6, 7, 12 and 18.21 are casted with respect to reference point $*$ where $M = 1$ and tabulated in a table called Fanno flow table.

The equations developed in this chapter are the means of computing the properties at one location in terms of those given at some other location. The key to problem solution is predicting the Mach number at the new location through the use of equation (18.21). The solution of this equation for the unknown M_2 presents a messy task, as no explicit relation is possible between $M_2\ and\ M_1$.

In $*$ reference case we imagine that we continue *by Fanno flow* (i.e., more duct is added) until the velocity reaches $M = 1$. Figure (18.2) shows a physical system together with its *T –s* diagram for a subsonic Fanno flow. We know that if we continue along the Fanno line (remember that we always move to the right), we will eventually reach the limiting point where sonic velocity exists. The dashed lines show assumed elongation duct of sufficient length to enable the flow to traverse the remaining portion of the upper branch and reach the limit point. This is the $(*)$ reference point *for Fanno flow*.

The *isentropic* $*$ reference points have also been included on the *T –s* diagram to emphasize the fact that the Fanno $*$ reference is a totally different thermodynamic state. One other fact should be mentioned. If there is any entropy difference between two points (such as points 1 and 2), their isentropic $(*)$ reference conditions are not the same $1^* \neq 2^*$. But for Fanno flow $1^* = 2^*$.

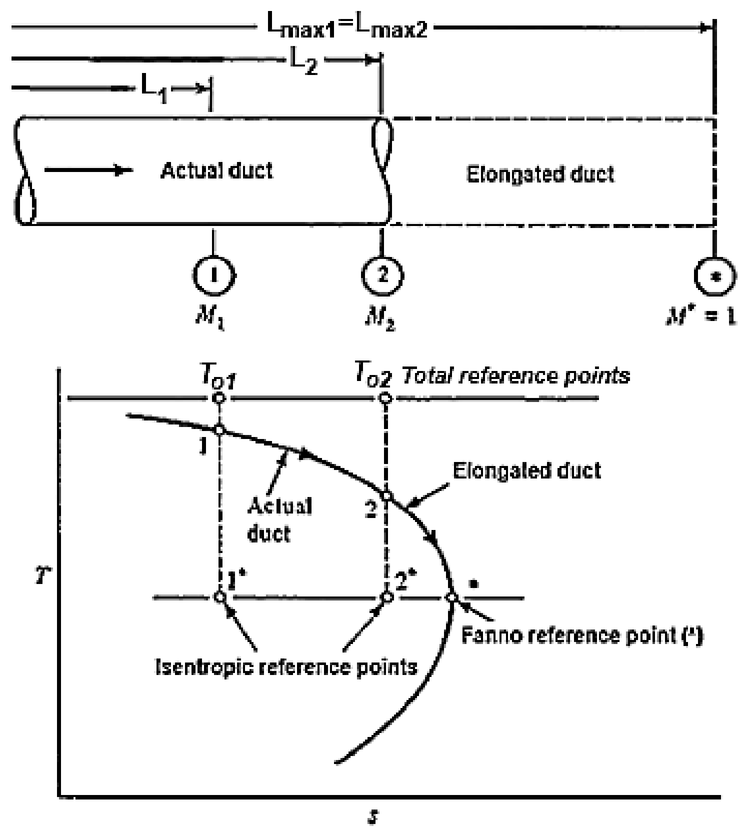


Figure 18.2 The * reference for Fanno flow.

Chapter Nineteen/Fanno Flow-Part 2

19.1 Fanno Flow line

If we want to study the behavior of Fanno Flow on T-s diagram, we must establish a relationship between entropy and temperature. From isentropic relation as T_o is constant:

$$\frac{T}{T^*}=\frac{(\gamma+1)/2}{1+[(\gamma-1)/2]M^2}$$

$$M=\left[\left(\frac{T^*}{T}\right)\left(\frac{\gamma+1}{\gamma-1}\right)-\frac{2}{\gamma-1}\right]^{1/2} \qquad (19.1)$$

Where T^* is the static temperature at $M=1$, and from eq. (17.7)

$$\frac{s-s^*}{c_p}=\ln M^2\sqrt{\left\{\frac{(\gamma+1)/2}{M^2(1+[(\gamma-1)/2]M^2)}\right\}^{(\gamma+1)/\gamma}} \qquad (18.7)$$

Substitute for M gives

$$\frac{s-s^*}{c_p}=\ln\left[\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma-1}{2\gamma}}\left(\frac{T}{T^*}\right)^{\frac{1}{\gamma}}\left(\frac{\gamma+1}{2}-\frac{T}{T^*}\right)^{\frac{\gamma-1}{2\gamma}}\right] \qquad (19.2)$$

Figure (19.1), a plot of eq. (19.2), shows the Fanno line on $T-s$ coordinates. For a perfect gas with constant specific heats, the T-s and h-s diagrams are similar. It represents the locus of states that can be obtained under the assumptions of Fanno flow for a fixed mass flow and total enthalpy. Consider the point of tangency A, where $\mathrm{ds/dt}=0$. To determine the characteristics of this point, let us starts from energy equation.

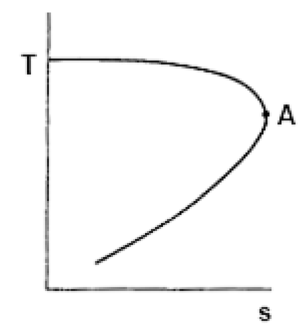


Figure 19.1 Fanno Line

$$h_o=h+\frac{V^2}{2}=const$$

$$V=\sqrt{2(h_o-h)}=\sqrt{2c_p(T_o-T)}$$

From thermodynamics relations

$$Tds=dh-vdp=dh-vd(\rho RT)$$
$$=c_pdT-RdT-vRTd\rho$$
$$=c_vdT-vRTd\rho$$

$$ds=c_v\frac{dT}{T}-R\frac{d\rho}{\rho}$$

$$s-s_1=c_v\ln\frac{T}{T_1}-R\ln\frac{\rho}{\rho_1}$$

Substitute from continuity equation for constant area duct $(\rho/\rho_1=V_1/V)$

$$s-s_1=c_v\ln\frac{T}{T_1}+R\ln\frac{V}{V_1}$$

Substitute from energy equation, $V=\sqrt{2(h_o-h)}=\sqrt{2c_p(T_o-T)}$

$$\frac{s-s_1}{c_v}=\ln\frac{T}{T_1}+\frac{\gamma-1}{2}\ln\frac{(T_o-T)}{(T_o-T_1)}$$

$$\frac{s-s_1}{c_v}=\ln T+\frac{\gamma-1}{2}\ln(T_o-T)+c$$

Differentiating with respect to dT

$$\frac{d((s-s_1)/c_v)}{dT}=0=\frac{1}{T}-\frac{\gamma-1}{2(T_o-T)}$$

$$\frac{1}{T}=\frac{\gamma-1}{2(T_o-T)}$$

Dividede by c_p and rearrange

$$Tc_p(\gamma-1)=2c_p(T_o-T)$$

$$\gamma RT=2c_p(T_o-T)$$

$$a^2=V^2$$

so means that at point A the Mach number is unity, $M=1$.

According to the energy equation, higher velocities are associated with lower enthalpies or temperatures, so the section of the Fanno line on $T-s$ coordinates that lies above (A) corresponds to subsonic flow, and the section below (A) to supersonic flow. The Fanno line becomes a most useful tool in describing the variations in properties for this frictional compressible flow.

Consider a subsonic adiabatic flow in a constant-area tube. The flow is irreversible because of friction, so for this adiabatic case, $ds>0$. In other words, the entropy increases in the flow direction.

Returning to the $T-s$ diagram in Figure 19.2, we see that for a given mass flow, the state of the fluid continually moves to the right, corresponding to an entropy rise. Thus, for subsonic flow with friction, the Mach number increases to 1. For supersonic flow, the entropy must again increase, so the flow Mach number here decreases to 1.

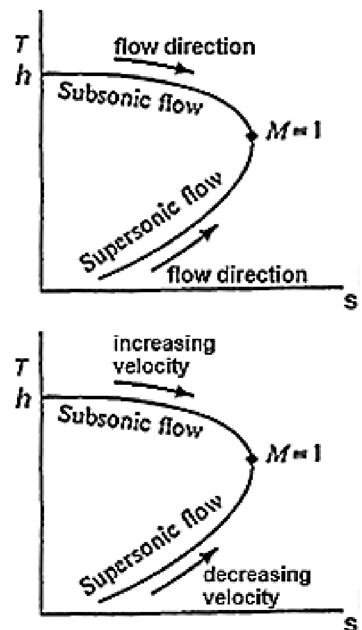


Figure 19.2

Suppose now that the duct is long enough for a flow initially subsonic to reach Mach 1, and an additional length is added, as shown in Figure (19.3). The flow Mach number for the given mass flow cannot go past 1 without decreasing

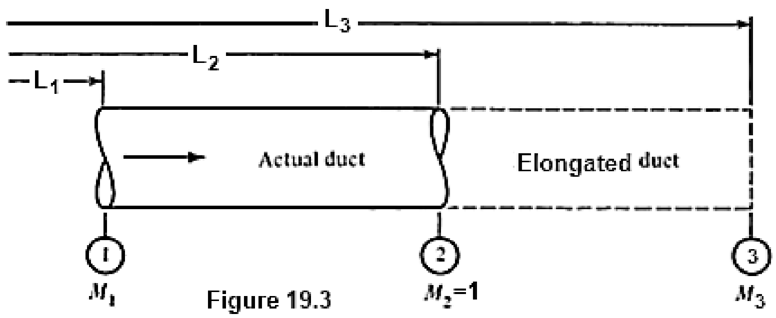


Figure 19.3

entropy. This is impossible from the second law. Hence the additional length brings about a reduction in mass flow. The flow jumps to another Fanno Line (see Figure 19.4). Essentially, the duct is choked due to friction. Corresponding to a given inlet subsonic Mach number, there is a certain maximum duct length L_{max} beyond which a flow reduction occurs.

Now suppose the inlet flow is supersonic and the duct length is made greater than L_{max} to produce Mach 1. With the supersonic flow unable to sense changes in duct length occurring ahead of it, the flow adjusts to the additional length by means of a normal shock rather than a flow reduction. The location of the shock in the duct is determined by the back pressure imposed on the duct. (This subject will be discussed in detail later)

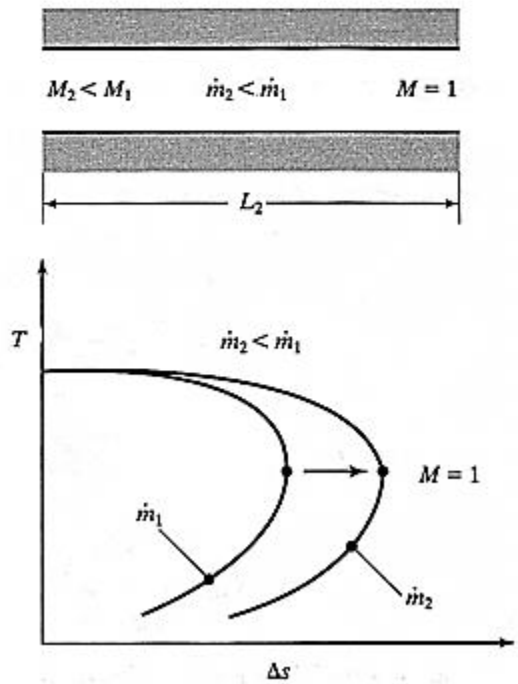


Figure 19.4 mass flow reduction

19.2 Friction factor *f*

Dimensional analysis of the fluid flow in fluid mechanics shows that the friction factor can be expressed as $f = f(Re, \varepsilon/D_e)$. Where ε/D_e is the relative roughness. The relationship among, Re, and ε/D_e is determined experimentally and plotted on a chart called a Moody chart or a Moody diagram. Typical values of ε, the *absolute roughness* are shown in Table (19.1).

Table 19.1 Absolute Roughness of Common Materials

Material	ε (ft)
Glass, brass, copper, lead	smooth < 0.00001
Steel, wrought iron	0.00015
Galvanized iron	0.0005
Cast iron	0.00085
Riveted steel	0.03

Example 19.1 for the duct in figure (19.5), given $M_1 = 1.80$, $p_1 = 275.790\ kN/m^2$, and $M_2 = 1.2$, find p_2, $f\Delta x/D$ and stagnation pressure ratio.

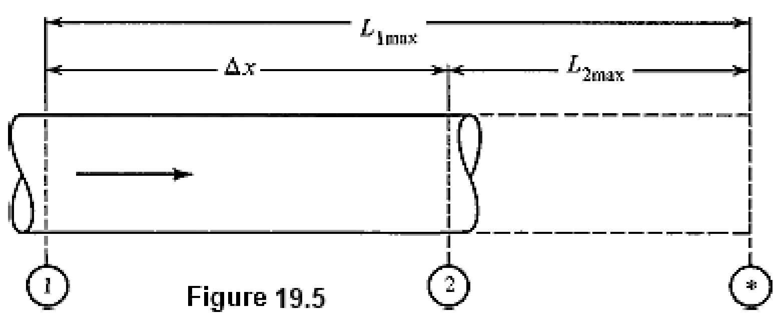


Figure 19.5

Solution

Since both Mach numbers are known, we can solve immediately.

From Fanno flow table, at $M_1 = 1.80$

$p_1/p^* = 0.47407$

$p_{o1}/p_o^* = 1.43898$

$fL_{1\,max}/D = 0.24189$

From Fanno flow table at $M_2 = 1.20$

$p_2/p^* = 0.80436$

$p_{o2}/p_o^* = 1.03044$

$fL_{2\,max}/D = 0.03364$, then

$$p_2 = \frac{p_2}{p^*} * \frac{p^*}{p_1} * p_1 = 0.80436 * \frac{1}{0.4741} * 275.790 = 467.904\ kN/m^2$$

$$\frac{p_{o2}}{p_{o1}} = \frac{p_{o2}}{p^*} * \frac{p^*}{p_{o1}} = 1.03044 * \frac{1}{1.43898} = 0.7161$$

$$\frac{f\Delta x}{D} = \frac{fL_{1\,max}}{D} - \frac{fL_{2\,max}}{D} = 0.24189 - 0.03364 = 0.2083$$

Notes that for supersonic flow, due to friction effect $p_2 > p_1$, but $p_{o2} < p_{o1}$.

Example 19.2 for frictional constant area duct, see figure (19.6), given $M_2 = 0.94$, $T_1 = 400\ K$, and $T_2 = 350\ K$, find M_1 and p_2/p_1. Also calculate stagnation pressure ratio

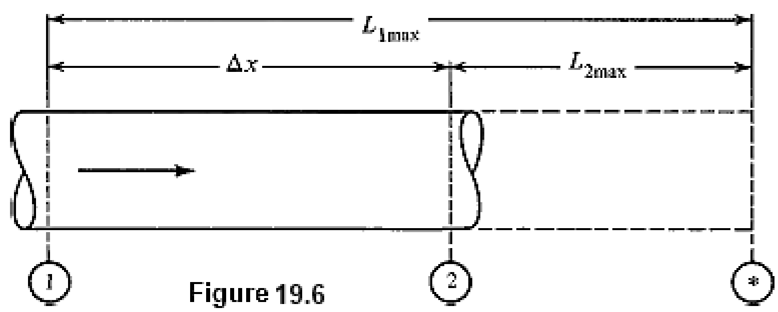


Figure 19.6

Solution

From Fanno flow table at $M_2 = 0.94$

$T_2/T^* = 1.01978$, $p_2/p^* = 1.0743$ and $p_{o2}/p_o^* = 1.00311$

To determine conditions at section 1, figure (19.6), we must establish the ratio

$$\frac{T_1}{T^*} = \frac{T_1}{T_2} * \frac{T_2}{T^*} = \frac{400}{350} * 1.01978 = 1.1655$$

From Fanno table at $T_1/T^* = 1.1655$

$M_1 = 0.385$, $p_1/p^* = 2.8046$ and $p_{o1}/p^* = 1.64105$

$$\frac{p_2}{p_1} = \frac{p_2}{p^*} * \frac{p^*}{p_1} = 1.074 * \frac{1}{2.8046} = 0.383$$

$$\frac{p_{o2}}{p_{o1}} = \frac{p_{2t}}{p^*} * \frac{p^*}{p_{o1}} = 1.00311 * \frac{1}{1.64105} = 0.61126$$

Notes that for subsonic flow, due to friction effect $p_2 < p_1$ and $p_{o2} < p_{o1}$

Notice that these examples confirm previous statements concerning static pressure changes. In subsonic flow the static pressure decreases, whereas in supersonic flow the static pressure increases, while the stagnation pressure ratio decreases in both cases due to the effect of friction losses.

Example 19.3 Air flows in a $152.4\ mm$ diameter, insulated, galvanized iron duct. Initial conditions are $p_1 = 137.895\ kN/m^2$, $T_1 = 21\ °C$, and $V_1 = 123.75\ m/s$. The absolute roughness is $\varepsilon = 0.1524\ mm$ and viscosity is $1.8 * 10^{-5}\ N.s/m^2$. After $21.34\ m$, determine the final Mach number, temperature, and pressure.

Solution

Since the duct is circular we do not have to compute an equivalent diameter. The relative roughness

$$\frac{\varepsilon}{D} = \frac{0.1524}{152.4} = 0.001$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{137.895}{0.287 * 294} = 1.6343\ {kg}/{m^3}$$

$$Re_1 = \frac{\rho_1 V_1 D}{\mu} = \frac{1.6343 * 123.75 * 152.4 * 10^{-3}}{1.8 * 10^{-5}} = 1.7 * 10^6$$

From the Moody diagram at Re = 1.7 × 106 and *ε/D* = 0.001, we determine that the friction factor is *f* = 0.0198. To use the Fanno table (or equations), we need information on Mach numbers.

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 * 287 * 294} = 343.7\ m/s$$

$$M_1 = \frac{V_1}{a_1} = \frac{123.75}{343.7} = 0.36$$

From the Fanno flow table at $M_1 = 0.36$

$p_1/p^* = 3.0042$, $T_1/T^* = 1.167$ and $fL_{1\,max}/D = 3.1801$

The key to completing the problem is in establishing the Mach number at the outlet, and this is done through the *friction length*:

$$\frac{f\Delta x}{D} = \frac{0.0198 * 21.34}{0.1524} = 2.773$$

Since f and D are assumed constant, then

$$\frac{f\Delta x}{D} = \frac{fL_{1\,max}}{D} - \frac{fL_{2\,max}}{D}$$

$$\frac{fL_{2\,max}}{D} = \frac{fL_{1\,max}}{D} - \frac{f\Delta x}{D} = 3.1801 - 2.773 = 0.408$$

From Fanno flow table at $fL_{1\,max}/D = 0.408$

$M_2 = 0.623$, $p_2/p^* = 1.6939$ and, $T_2/T^* = 1.1136$, Thus

$$p_2 = \frac{p_2}{p^*} * \frac{p^*}{p_1} * p_1 = (1.6939)\left(\frac{1}{3.0042}\right)(137.895\ ) = 77.75\ kN/m^2$$

$$T_2 = \frac{T_2}{T^*} * \frac{T^*}{T_1} * T_1 = (1.1136)\left(\frac{1}{1.1697}\right)(294) = 280\ K$$

In the example above, the friction factor was assumed constant.

Example 19.4 Flow enters a constant-area, insulated duct with a Mach number of 0.60, static pressure of 150 kPa, and static temperature of 300 K. Assume a duct length of 45 cm, duct diameter of 3 cm, and a friction coefficient of 0.02. Determine the Mach number, static pressure, and static temperature at the duct outlet

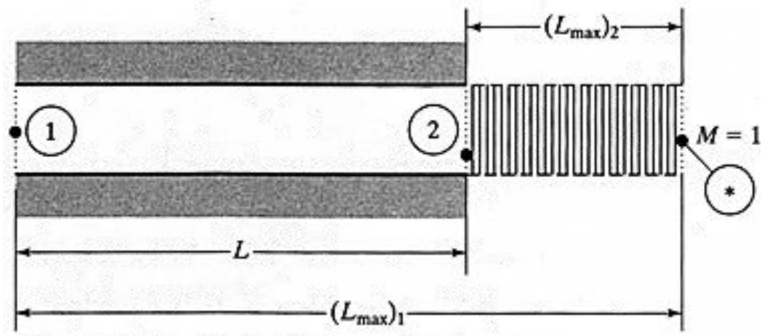


Figure 19.7 Illustarative drawing for example 19.4

Solution

From Fanno flow tables, at $M_1 = 0.60$

$fL_{1\,max}/D = 0.49081$, $p_1/p^* = 1.7634$ and $T_1/T^* = 1.1194$

The actual Fanno flow friction coefficient is

$$\frac{f\Delta x}{D} = \frac{(0.02)(45)}{3} = 0.3\ , \qquad Then$$

$$\frac{fL_{2\,max}}{D} = \frac{fL_{1\,max}}{D} - \frac{f\Delta x}{D} = 0.49081 - 0.3 = 0.19081$$

Thus from Fanno flow tables at $fL_{2\,max}/D = 0.19081$ gives

$M_2 = 0.709, p_2/p^* = 1.4728$ and, $T_2/T^* = 1.0904$, Thus

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\Big/\frac{p_1}{p^*} = \frac{1.4728}{1.7634} = 0.8349$$

$$\frac{T_2}{T_1} = \frac{T_2}{T^*}\Big/\frac{T_1}{T^*} = \frac{1.0904}{1.1194} = 0.9740\ K$$

$p_2 = 0.8349 * 150 \quad = 125.235\ \text{kPa}$

$T_2 = 0.0.9740 * 300 = 292.2\ \text{K}$

Chapter Twenty/ Fanno Flow through a Nozzle-Duct System

20.1 Converging Nozzle and Duct Combination

Very often a situation occurs where a duct is fed by a nozzle; with the back pressure and nozzle stagnation pressure are the known quantities. Consider, for example, a duct supplied by a converging nozzle, with flow provided by a reservoir at pressure p_{res} (see Figure 20.1**).** Assuming isentropic nozzle flow, with Fanno flow in the duct, the system pressure distribution ($p\ versus\ x$), can be determined for various back pressures for fixed p_{res}. As p_b is lowered below p_{res}, curves such as (a) and (b) are obtained, with pressure decreasing in both nozzle and duct. Finally, when the back pressure is decreased to that of curve (c), Mach number 1 occurs at the **duct exit** (note that the Mach number at the **nozzle exit** is still less than 1).

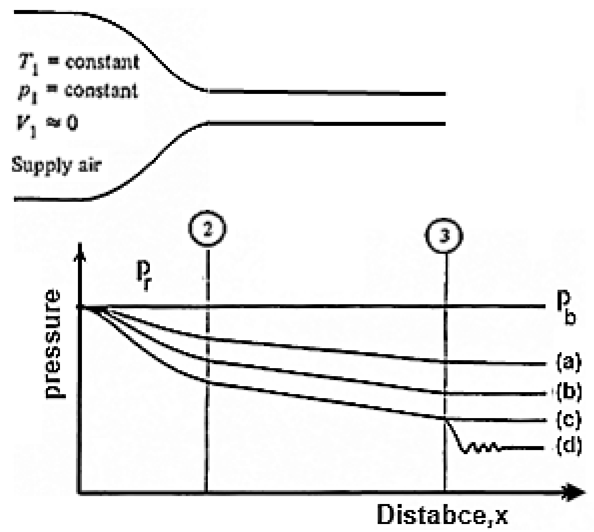


Figure 20.1 C-N and Constant Duct

Further decreases in back pressure cannot be sensed by the reservoir; for all back pressures below that of curve (c) the mass flow rate remains the same as that of curve (c); $\dot{m}$ is plotted versus p_b in Figure (20.2). The system here is ***choked*** by the **duct**, not the converging nozzle. The maximum mass flow that can be passed by this system is less for the same reservoir pressure than that for a converging nozzle with no duct.

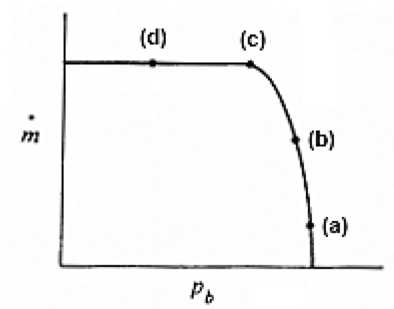


Figure 20.2

For a **subsonic Fanno flow** situation, figure (20.1) shows a given length of duct fed by a large tank and converging nozzle. If the receiver (back) pressure is below the tank pressure, flow will occur, producing a T –s diagram shown as path 1–2–3. Note that we have isentropic flow at the entrance to the duct and then we move along a Fanno line.

As the receiver pressure is lowered still more, the flow rate and exit Mach number continue to increase while the system moves to Fanno lines of higher mass velocities G (shown as path $1-2'-3'$). It is important to recognize

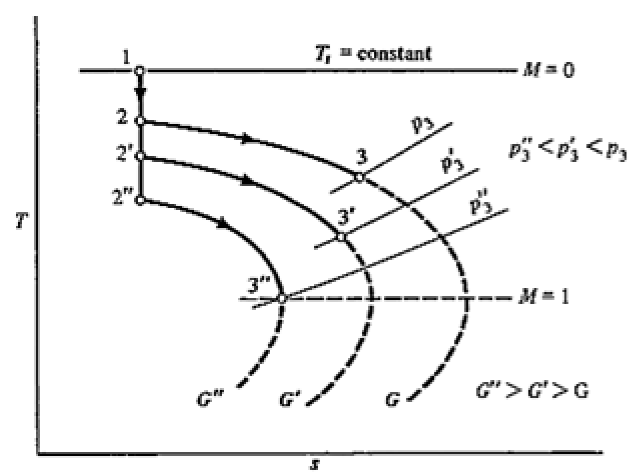


Figure 20.3 *T-s* Diagram Nozzle-duct Combination

that the receiver pressure (or more properly, the operating pressure ratio) is controlling the flow. This is because in subsonic flow the pressure at the duct exit must equal that of the receiver.

Eventually, when a certain pressure ratio is reached, the Mach number at the duct exit will be unity (shown as path $1 - 2'' - 3''$). This is called ***duct choking*** and any further reduction in receiver pressure would not affect the flow conditions *inside* the system. What would occur as the flow leaves the duct and enters a region of reduced pressure?

Let us consider this last case of choked flow with the exit pressure equal to the receiver pressure. Now *suppose that the receiver pressure is maintained is kept constant* but more duct length is added to the system. What happens? We know that we cannot move *around the Fanno line*, yet somehow we must reflect the added friction losses. This is done by moving to a new Fanno line at a *decreased* flow rate. The *T* –*s* diagram for this is shown as path $(1 - 2''' - 3''' - 4)$ in Figure (20.4). Note that pressure equilibrium is still maintained at the exit but the system is no longer choked, although the flow rate has decreased. What would occur if the receiver pressure were now lowered?

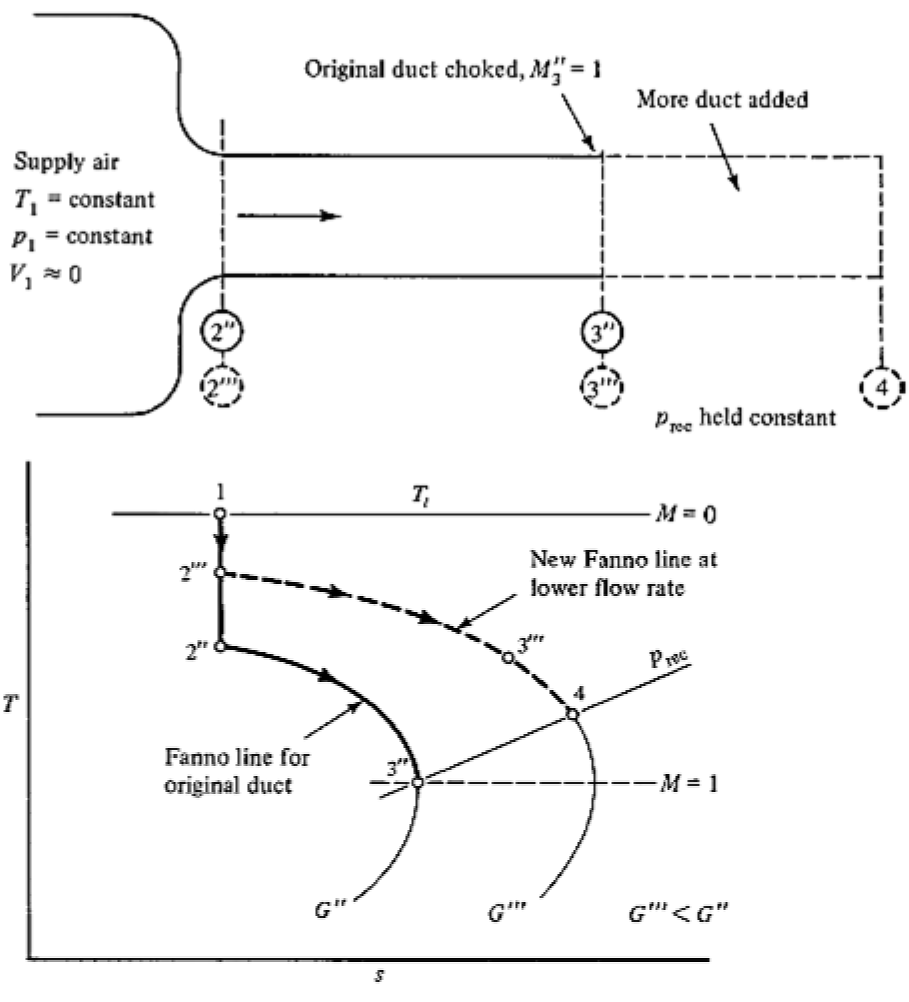


Figure 20.4 Addition of more duct when choked.

In summary, when a ***subsonic*** Fanno flow has become ***duct choked*** and more duct is added to the system, the flow rate must decrease. Just how much it decreases and whether or not the exit velocity remains sonic depends on how much duct is added and the receiver pressure imposed on the system.

Example 20.1 A constant-area duct, $20\ cm$ in length by $2\ cm$ in diameter, is connected to a reservoir through a converging nozzle, as shown in Figure (20.5a). For a reservoir pressure and temperature of $1\ MPa$ and $500\ K$. Determine the maximum air flow rate in kilograms per second through the system and the range of back pressures over which this flow is realized. Repeat these calculations for a converging nozzle with no duct. Assume $\text{f} = 0.032$

Solution

For maximum mass flow through the nozzle-duct system, $M_2 = 1$. For this condition, the actual fL/D of the duct becomes equal to fL_{max}/D, so that

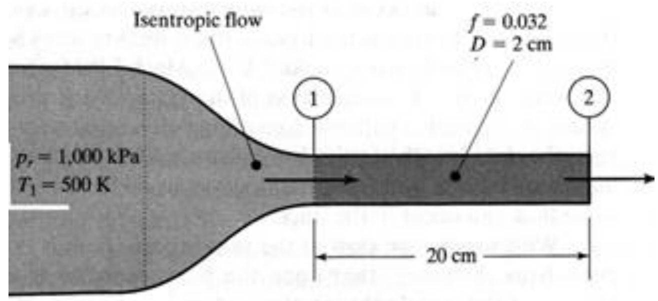


Figure 20.5a Illustrative drawing for example 20.1

$fL_{max}/D = 0.032 * 20/2 = 0.32$

From Fanno tables at $fL_{max}/D = 0.32$ gives

$M_1 = 0.652$

For isentropic nozzle flow, from isentropic flow tables at $M_1 = 0.652$ gives

$(p/p_o)_1 = 0.7515$ and $(T/T_o)_1 = 0.9217$

$p_1 = 0.7515 * 1 = 0.7515\ MPa$

$T_1 = 0.9217 * 500 = 460.9\ K$

$$\dot{m} = \rho V A = \left(\frac{p_1}{RT_1}\right) A_1 M_1 \sqrt{\gamma R T_1}$$

$$= \left[\frac{751.5}{0.287 * 460.9}\right]\left[\frac{\pi}{4}(2 * 10^{-2})^2\right]\left[0.652\sqrt{1.4 * 287 * 460.9}\right] = 0.5009\ kg/s$$

Also.

$p_1/p^* = p_1/p_2 = 1.6130$

$p_2 = 751.5 * (1/1.6130) = 465.9\ kPa$

So the system is choked over the range of back pressures from (0 to $465.9\ kPa$).

If the duct were to he removed, choking would occur with Mach 1 at the nozzle exit. For this condition

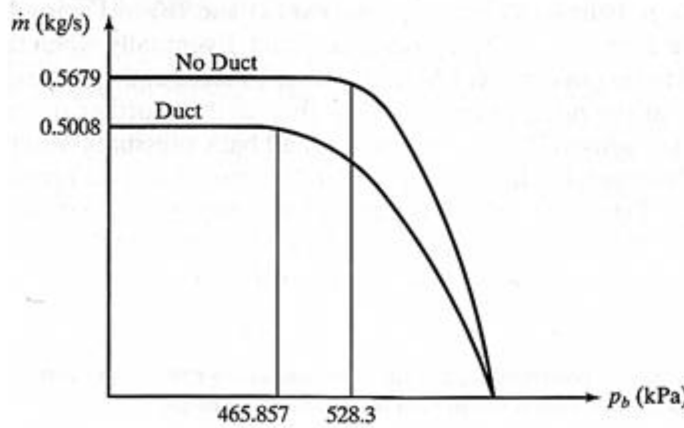


Figure 20.5b Comparison of Mass Flow Rates in a Converging Nozzle with and without a Constant-Area Duct for example 20.1

From isentropic table at $M_1 = 1$ gives

$(p/p_o)_1 = 0.5283$ and $(T/T_o)_1 = 0.8333$

$p_1 = 0.5283(1000\ kPA) = 528.3\ kPa$

$T_1 = 0.8333(500\ K) = 416.7\ K$

So the maximum mass flow (for choked flow) is

$$\dot{m}_{max} = \left[\frac{528.3}{0.287 * 416.7}\right]\left[\frac{\pi}{4}\left(\frac{4}{1000}\right)\right] * \left[1.0\sqrt{1.4 * 287 * 416.7}\right] = 0.5679\ kg/s$$

For this case, the system is choked over the back pressure range from ($0\ to\ 528.3\ kPa$) Results are shown in Figure (20.5b).

20.2 Converging–Diverging Nozzle and Duct Combination

When a duct is connected to a reservoir through a converging-diverging nozzle, the situation becomes somewhat more complex. Consider first the case of subsonic flow in both nozzle and duct. A typical pressure distribution is shown in Figure (20.6). Depending on the duct length, the minimum pressure point, or point of maximum Mach number, can occur at the **nozzle throat** or **duct exit**.

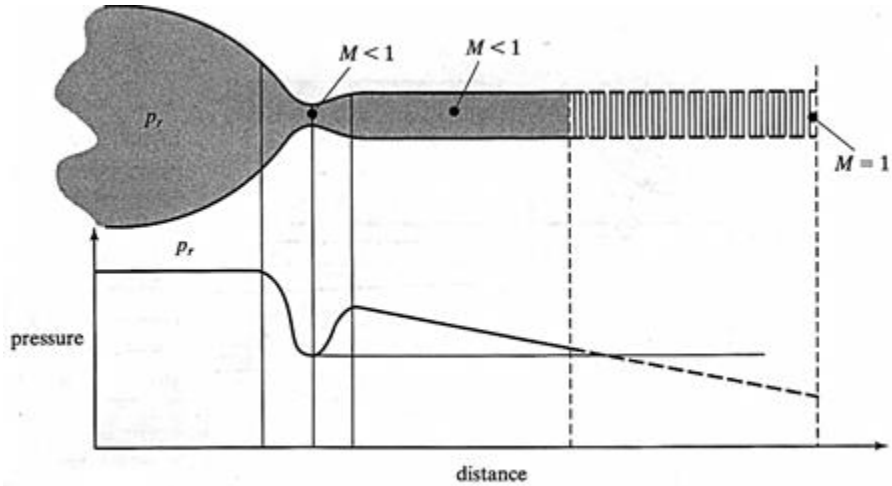


Figure 20.6 Pressure Distribution of a Subsonic Flow in a Duct Connected to a Reservoir by a C–D Nozzle

If the duct is long enough (see dashed curve), the system reaches Mach 1 first at the duct exit; in this case, the nozzle is not choked. Once Mach 1 is reached, no further increase in mass flow rate can occur by reduction of the system back pressure. Supersonic flow in this system is impossible with the converging-diverging nozzle unchoked.

Generally, however, the duct length required to cause choking is very long. For this reason, the more important case is that in which the system is choked at the nozzle throat, and supersonic flow can occur in the duct.

With supersonic flow at the nozzle exit, there is the possibility of shocks in the duct. Note, however, that once the back pressure is just low enough to produce Mach 1 at the nozzle throat, the system is choked, with no further increase in mass flow possible. Unlike the case previously discussed, in which mass flow was affected by duct length, here, once the throat velocity reaches the velocity of sound, the mass flow rate is unaffected by duct length. Now the system is **choked by the nozzle, not the duct**. Let us consider the flow pattern obtained with supersonic flow at the duct inlet.

- First, suppose the duct length is less than the maximum length corresponding to the given duct inlet supersonic Mach number M_{in} needed to reach Mach 1 at the duct exit i.e. $L < L_{max,in}$. The change in flow pattern is to be described as the back pressure p_b is increased from $0\,kPa$. A back pressure of $0\,kPa$, or a very low back pressure, implies the existence of expansion waves at the duct exit. This means that the exit Mach number must be either supersonic or unity. Since L is less than L_{max}, supersonic flow occurs at the duct exit, with the exit static pressure $p_e > p_b$,See curve (a) in Figure 20.7. When p_b is raised to a value corresponding to curve (b), $p_e = p_b$. A further increase in back pressure yields oblique shock waves at the duct exit where $p_e < p_b$,curve (c), until eventually a normal shock stands at the duct exit for a back pressure equal to that of curve (d). It can be seen that the flow described is exactly the same as that obtained at the exit of a converging-diverging nozzle. Increases in back pressure over that of curve (d) cause the shock to move into the duct. For a high-enough back pressure, the shock moves into the nozzle, thus eliminating supersonic flow in the duct. For a high enough back pressure, the shock moves into the nozzle, thus eliminating supersonic flow in the duct.

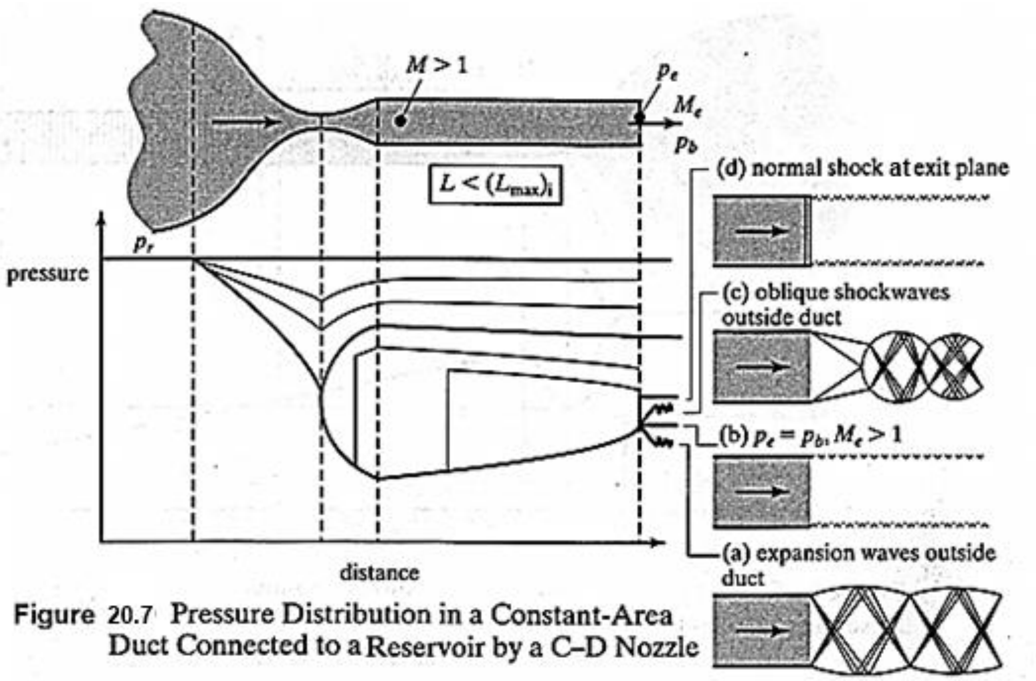


Figure 20.7 Pressure Distribution in a Constant-Area Duct Connected to a Reservoir by a C–D Nozzle

Example 20.2 A converging-diverging nozzle, with area ratio of $2:1$ is supplied by a reservoir containing air at $500\ kPa$. The nozzle exhausts into a constant-area duct of length-to-diameter ratio of 10 and friction coefficient $f = 0.02$. Determine the range of system back pressure over which a normal shock appears in the duct. Assume an isentropic flow in the nozzle and Fanno flow in the duct.

Solution

From isentropic flow tables at $A/A^* = 2.0$, gives

$M_1 = 2.197$ and $p_1/p_{o1} = 0.09393$

From Fanno flow tables at $M_1 = 2.197$, gives

$(fL_{max}/D)_1 = 0.3601$.

For the duct under consideration

$fL/D = 0.02 * 10 = 2.0$

So that $L < (L_{max})_1$. Calculations must be made for two limiting cases, one with shock at the duct inlet (Figure 20.8a), and the other with shock at the duct outlet.

(a) Shock at the duct inlet

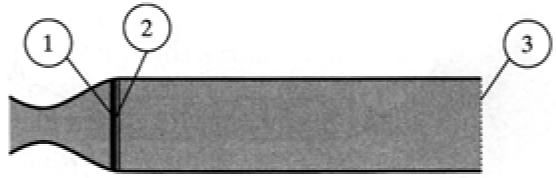


Figure 20.8a Shock at duct inlet

From normal shock tables at $M_1 = 2.197$, gives $M_2 = 0.5475$ and $p_2/p_1 = 5.4656$

From isentropic flow tables at $M_2 = 0.5475$ gives $p_2/p^* = 1.9483$

From Fanno flow tables at $M_2 = 0.5475$, gives $(fL_{max}/D = 0.7427)_2$ Thus

$$\left(\frac{fL_{max}}{D}\right)_3 - \left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL}{D}\right)_2 \equiv \left(\frac{fL}{D}\right)_1$$

$$\left(\frac{fL_{max}}{D}\right)_3 = \left(\frac{fL_{max}}{D}\right)_2 - \left(\frac{fL}{D}\right)_2$$

$$\left(\frac{fL_{max}}{D}\right)_3 = 0.7427 - 0.20 = 0.5427$$

So that from Fanno flow tables at $(fL_{max}/D)_3 = 0.527$ gives $M_3 = 0.5875$

From isentropic flow tables at $M_3 = 0.5875$ gives $p_3/p^* = 1.8071$

Then

$$p_b = p_3 = \left(\frac{p_3}{p^*}\right)\left(\frac{p^*}{p_2}\right)\left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)p_{01}$$

$$= 1.80713 * \frac{1}{1.9438} * 5.4656 * 0.09393 * 500 = 238.2\,kPa$$

(b) Shock at the duct exit

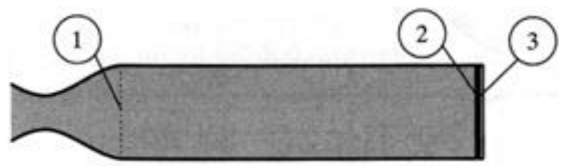


Figure 20.8b Shock at duct exit

From Fanno flow tables at, $M_1 = 2.197$, gives $p_1/p^* = 0.3557$ and $(fL_{max}/D)_1 = 0.3601$. So

$$\left(\frac{fL_{max}}{D}\right)_1 = \frac{fL}{D} + \left(\frac{fL_{max}}{D}\right)_2$$

$$\left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL_{max}}{D}\right)_1 - \frac{fL}{D} = 0.3601 - 0.20 = 0.1601$$

From Fanno flow tables at $(fL_{max}/D)_2 = 0.1601$ gives $M_2 = 1.5663$.

From isentropic table at $M_2 = 1.5663$ gives $p_2/p^* = 0.5728$

For normal wave tables, at $M_2 = 1.566$ gives $p_3/p_2 = 2.695$, then

$$p_b = p_3 = \left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)p_{o1}$$

$$= 2.695 * 0.5730 * \frac{1}{0.3557} * 0.09393 * 500 = 204.0\,kPa$$

The shock will appear in the duct over the back pressure range $204.0 \;\; to \;\; 238.2\,kPa$

- Suppose L is greater than $(L_{max})_i$, i.e. that the duct length is larger than that required to reach Mach 1 at duct exit for supersonic duct flow.

For a back pressure of $0\ kPa$ and for very low back pressures, it is evident that the back pressure is less than the exit-plane pressure, so expansion waves must occur at the duct exit, with the exit-plane Mach number equal to unity. (Flow after the shock cannot reach supersonic velocities without violating the second law of thermodynamics.) For curves (a) and (b) in Figure 20.9, therefore, a normal shock occurs inside the duct, with sonic flow at the duct exit and expansion waves outside the duct.

For curve (c), the exit- plane pressure is equal to the back pressure. It should be noted that the location of the shock is the same for curves (a), (b), and (c). For this class of problem, this location represents the farthest downstream position that the normal shock is able to reach. Finding this location is beyond our stage.

As the back pressure is raised above curve (c), the normal shock moves upstream toward the duct inlet, with the exit Mach number subsonic and the back pressure equal to the exit-plane pressure. Again, for high-enough back pressures, the shock moves into the nozzle, eliminating supersonic flow in the after-section of the duct.

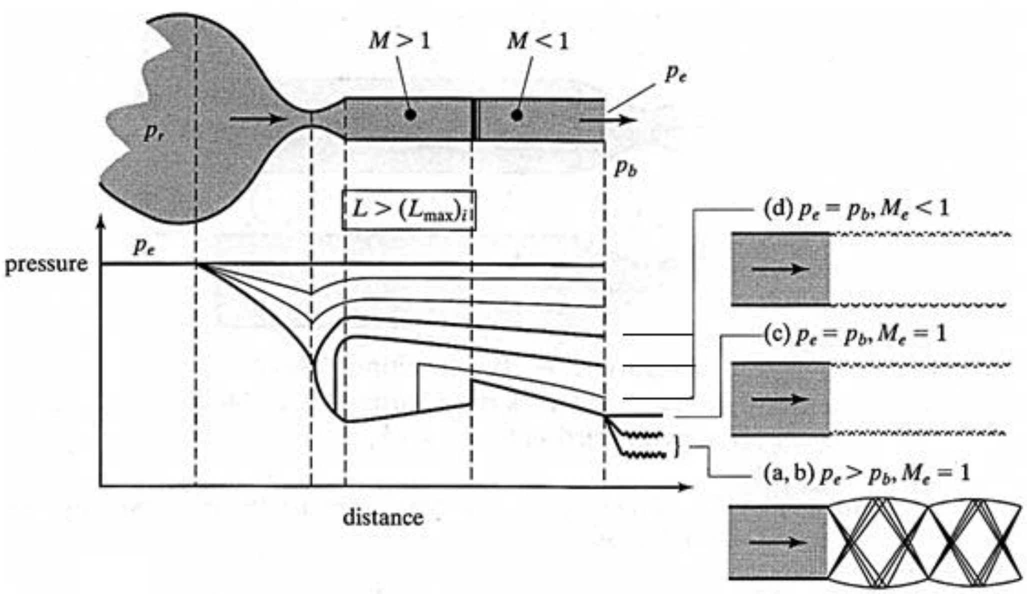


Figure 20.9 Pressure Variation in a Constant-Area Pipe Connected to a C–D Nozzle

Example 19.3 A converging-diverging nozzle, with an area ratio of $2\ to\ 1$. is supplied by a reservoir containing air at $500\ kPa$. The nozzle exhausts into a constant-area duct of length-to-diameter ratio of 25 and friction coefficient of 0.02. Determine the range of system back pressure over which a normal shock appears in the duct. Assume an isentropic flow in the nozzle and Fanno flow in the duct.

Solution

From isentropic flow table at $A/A^* = 2.0$ gives $\mathrm{M_i} = 2.197$, $p_i/p_i^* = 0.3557$ and $p_i/p_{oi} = 0.094$

From Fanno flow tables at $\mathrm{M_i} = 2.197$ gives

$$(fL_{max}/D)_i = 0.35828 + (0.36091 - 0.35828)\frac{2.197 - 2.190}{2.20 - 2.19} = 0.3601$$

For the duct fL/D $= 0.02(25) = 0.50$ which is greater than $(fL_{max}/D)_i$ i.e. $L_{duct} > L_{max}$

For this type of problem, a normal shock usually stands in the C-D nozzle-duct system. The range of back pressures over which a normal shock exists within the duct can be established as follows:

(a) <u>Shock at the duct inlet</u>

From normal shock tables at $M_1 = 2.197$, gives $\mathrm{M_2} = 0.5475$ and $p_2/p_1 = 5.5199$

From isentropic flow tables at $\mathrm{M_2} = 0.5475$ gives $p_2/p^* = 1.9483$

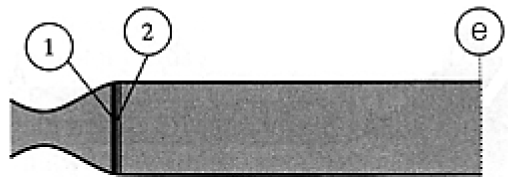


Figure 20.10a N.s at duct inlet

From Fanno flow tables at $\mathrm{M_2} = 0.5475$, gives $(fL_{max}/D = 0.7427)_2$ Thus

$$\left(\frac{fL_{max}}{D}\right)_e = \left(\frac{fL_{max}}{D}\right)_2 - \left(\frac{fL}{D}\right)_2 = 0.7427 - 0.5 = 0.2427$$

So that from Fanno flow tables at $(fL_{max}/D)_e = 0.2427$ gives

$M_e = 0.6833$ and $p_e/p_e^* = 1.5333$

Because the exit flow is subsonic, the exit pressure is equal to the back pressure, which may be computed from

$$p_b = p_e = \left(\frac{p_e}{p^*}\right)\left(\frac{p^*}{p_2}\right)\left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)p_{o1}$$

$$= 1.5334 * \frac{1}{1.9435} * 5.5199 * 0.0944 * 500 = 205.562\ kpa$$

Thus, a shock will reside within the duct for the following range of back pressures: $0 < p_b < 205.562\ kPa$

(b) <u>Shock inside the duct</u>

Since the value of $fL/D = 0.50 > (fL_{max}/D)_1$, the shock cannot exist at duct exit. When the back pressure has the lowest value, ($p_b = 0\ kPa$), the position of the normal shock is positioned far away from duct exit. As the back pressure is raised, the normal shock moves towards the duct inlet. Finding the position of the normal shock and the back pressure is left for the interested student.

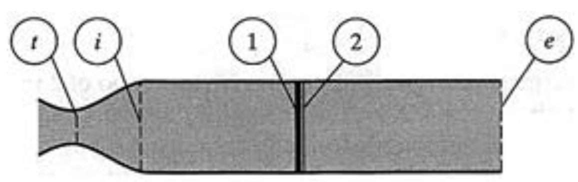


Figure 20.10b N.s inside the duct

For interested student:

Since the back pressure for the first case of this example is $0\,kPa$, the exit Mach number is clearly unity and $p_e = p^*$. However, to reach the low value of p_b, further expansion must take place outside the duct, as shown in curve (a) of Figure 20.9. To determine the location of the shock for this case, we proceed as flow; for the duct shown in Figure 20.10, the duct length can be written as:

$$L = [(L_{max})_2 - (L_{max})_e] + [(L_{max})_i - (L_{max})_1]$$

$$[(L_{max})_2 - (L_{max})_1] = L + (L_{max})_e - (L_{max})_i$$

Multiplying by the average friction coefficient, f, dividing by the hydraulic diameter, D, and rearranging yields

$$F(M_1) = \left(\frac{fL_{max}}{D}\right)_2 - \left(\frac{fL_{max}}{D}\right)_1 = \left(\frac{fL}{D}\right) + \left(\frac{fL_{max}}{D}\right)_e - \left(\frac{fL_{max}}{D}\right)_i$$

Note that because the flow between the duct inlet, station i, and the upstream side of the shock, station 1, is supersonic and because the friction decelerates supersonic flows so $M_i > M_1$ and $(L_{max})_i > (L_{max})_1$.

Also because the flow between the downstream side of the normal shock, station 2, and the duct exit, station e, is subsonic and because friction accelerates subsonic flows so $M_e > M_2$ and $(L_{max})_2 > (L_{max})_e$.

And from eq. 18.21.

$$f\frac{L_{ma}}{D} = \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}\right) - \frac{1}{\gamma}\left(1-\frac{1}{M^2}\right) - \left(\frac{\gamma+1}{2\gamma}\right)\ln\left(\frac{1}{M^2}\right) \qquad (17.21)$$

And eq. 10.7 which relates M_2 and M_1 across the normal shock

$$M_2^2 = \frac{M_1^2 + 2/(\gamma-1)}{\left[2\gamma/\left((\gamma-1)\right)\right]M_1^2 - 1} \qquad (10.7)$$

Then we have an expression to evaluate M_1

$$F(M_1) = \frac{\gamma+1}{\gamma}\ln\left[\frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right] + \frac{2(1+\gamma M_1^2)(M_1^2-1)}{\gamma M_1^2[2+(\gamma-1)M_1^2]}$$

The value of M_1 can be obtained by numerically solving this equation using the Newton-Raphson method. Because the derivative of $F(M_1)$ is complicated, it was obtained using the finite-difference approach. The solution is beyond our scope.

When M_1 is known then we find M_2, $(L_{max})_1$ and $(L_{max})_2$. This gives the position of the normal shock.

$$\left(\frac{fL}{D}\right)_{i-1} = \left(\frac{fL_{max}}{D}\right)_i - \left(\frac{fL_{max}}{D}\right)_1$$

Subject : gas Dynamic and Turbine Machine
Weekly Hours : Theoretical:2 UNITS:5
Tutorial : 1
Experimental : 1

موضوع : ديناميك غازات ومكائن تور بينية
الساعات الأسبوعية : نظري : 2 الوحدات :5
مناقشة : 1
عملي :1

week	Contents	المحتويات	الأسبوع
1.	Principles of thermodynamics	اسس ديناميك الغازات	.1
2.	Introduction to compressible flow	مقدمة للجريان المنضغط	.2
3.	Isentropic flow	الجريان الايزوتروبي	.3
4.	Choked Isentropic flow	الخنق في الجريان الايزوتروبي	.4
5.	Operation of nozzles at variable pressure ratios	عمل الفوهات عند نسب ضغط متغيرة	.5
6.	Normal shock wave	الصدمات الانضغاطية العمودية	.6
7.	Equations of Normal shock wave	العلاقات المتحكمة بالصدمة العمودية	.7
8.	Oblique shock wave	الصدمة المائلة	.8
9.	Flow in constant area duct with friction	الجريان في بحاري ثابتة المقطع مع الاحتكاك	.9
10.	Performance of long ducts at variable pressure ratios	اداء المجاري الطويلة عند نسب ضغط متغيرة	.10
11.	Isothermal flow in long ducts	الجريان ثابت درجة الحرارية في المجاري الطويلة	.11
12.	Flow ducts with heating or cooling	الجريان في مجاري مع التسخين	.12
13.	=	=	.13
14.	=	=	.14
15.	shock wave with change in stagnation temperature	الموجات الصدمية عند تغيير درجة حرارة الركود	.15
16.	Aerothermodynamics of turbomachinery	الديناميك الحراري الهوائي للمحركات التوربينية	.16
17.	Physical principle a. equation of motion b. continuity equation c. momentous equation	المبادىء الفيزياوية : أ. معادلة الحركة ب. معادلة استمرارية ج. معادلة الزخم	.17
18	Turbine momentum notation	معالم التوربين	18
19.	Efficiencies	الكفاءات	19
20	Flow in rotating blades	الجريان في الريش الدوارة	20
21.	Axial flow turbine	تربين الجريان المحوري	.21
22.	Velocity triangles	مثلثات السرع	.22
23.	Impulse turbine	التربين الدفعي	.23
24	Reaction turbine	التربين رد الفعلي	.24
25.	=	=	.25
26	Axial flow compressor	ضاغط الجريان المحوري	.25
27	Radial turbine	التربين الشعاعي	.27
28.	Centrifugal compressor	الضاغط التنابذي	.28

Chapter One

Fundamental of Fluid Dynamics

Introduction:

Gas dynamics is a branch of fluid mechanics which describe the flow of compressible fluid. Fluids which show appreciable variation in density as a results of the flow – such as gases- are called ***compressible fluids***. The variation in density is due mainly to variation in pressure and temperature.

The flow of a compressible fluid is governed by the first law of thermodynamics, which relates to energy balance, and by the second law of thermodynamics, which relates heat interaction and irreversibility to entropy. The flow is also affected by both kinetic and dynamic effects, which are described by Newton's laws of motion. An inertial frame of reference that is, a frame in which Newton's laws of motion are applicable- is generally used. In addition, the flow fulfils the requirement of conservation of mass.

These laws are not dependent on the properties of particular fluid, therefore in order to relate the motion to a particular fluid it is necessary to use subsidiary laws in addition to these fundamental principles , such as the equation of state for perfect gas.

$$p = \rho RT \quad ....(1)$$

Although the most obvious application of compressible fluid flow theory are in the design of high speed aircraft, and this remains an important application to the subject, acknowledges of compressible fluid flow theory is required in the design and operation of many devices commonly encountered in engineering practice. Among these application are:

1- Gas Turbine: the flow in the balding and nozzle is compressible.
2- Steam turbine. Here, too, the flow in the nozzles and blades must be treated as compressible.
3- Reciprocating engines, flow of gases through the valves and intake and exhaust.
4- Natural gas transmission line.
5- Combustion chambers
6- Explosive.

1.1 *Conservation of Mass:*

The principle of conservation of mass, when referred to a system of fixed identity, simple states that the mass of the system is constant. Consider an arbitrary control volume through which fluid streams Fig.1. we wish to derive the form of the law of conservation of mass as it applied to this control volume. However, in order to apply the law, we must begin with a system of fixed identity, and so we defined our system as the fluid which some instant t occupies the control volume.

Next, we consider what happens during the succeeding time interval *dt*. By definition, the control volume remains fixed in space, but the system moves in the general direction of the streamline. The two position of the system are shown in fig.1 by dashed lines. For convenience in analysis, we consider three region of space denoted bt *I,II,III* in fig.1. At time *t* the system occupies spaces *I* and *III*, and at time *t+dt* it occupies space *I* and *II*. Thus, since the mass of the system is conserved, we write.

$$m_{I\,t} + m_{III\,t} = m_{I\,t+dt} + m_{II\,t+dt} \quad \text{..................2}$$

where m_{It} means the mass of the fluid in space I at time t, and so on. A simple rearrangement then gives.

$$\frac{m_{I\,t+dt} - m_{I\,t}}{dt} = \frac{m_{III\,t}}{dt} - \frac{m_{II\,t+dt}}{dt} \quad \text{-----------------3}$$

The first term represent the time rate of change of mass within space I. But as dt goes to zero space I coincide with the control volume, and so in the limit.

$$\frac{m_{I\,t+dt} - m_{I\,t}}{dt} \rightarrow \frac{\partial}{\partial t}(m_{c.v.}) \quad \text{--------------------4}$$

where m_{cv} denoted the instantaneous mass within the control volume.

The third term may be written.

$$\frac{m_{II\,t+dt}}{dt} = \frac{\sum \delta m_{II\,t+dt}}{dt} = \sum \frac{\delta m_{II\,t+dt}}{dt} = \int dm_{out} \quad \text{-----------5}$$

where δm_{III+dt} represent the amount of mass crossing the elementary surface dA_{out} during the time dt. The ratio $\delta m_{III+dt}/dt$ is called the out going flux of mass cross the area dA_{out} Or the mass rate of flow and is denoted for convenience by dm_{out}.

similar reasoning yields for inlet.

$$\frac{m_{III\,t}}{dt} = \int dm_{in} \quad \text{---------------6}$$

and so the conservation law may now be expressed as

$$\frac{\partial}{\partial t}(m_{c.v.}) = \int dm_{in} - \int dm_{out} \quad \text{----------------7}$$

for detailed computation we note that at any instant

$$m_{c.v} = \int \delta m_{c.v} = \int_{c.v} \rho dv \quad \text{-------------------8}$$

where dv is an element of control volume, ρ is the local mass density of that element and the integral is to be taken over the entire control volume.

$$\frac{\partial m_{c.v}}{dt} = \frac{\partial}{dt}\int_{c.v} \rho dv = \int_{c.v} \frac{\partial \rho}{\partial t} dv \quad \text{---------------------9}$$

with the help of fig.1 we may express the mass rate of flow in the form.

$$dm_{out} = \frac{\delta m_{II\,t+dt}}{dt} = \frac{\rho(dA_{out})(V_n dt)}{dt} = \rho V_n dA_{out} \quad \text{---------10}$$

where ρ is the local instantaneous mass density in the neighbourhood of dA_{out} and V_n is the corresponding local instantaneous component of velocity normal to dA_{out}, with the forgoing expression equation 7 may now written.

$$\int_{c.v} \frac{\partial \rho}{\partial t} dv = \int \rho V_n dA_{in} - \int \rho V_n dA_{out} \quad \text{-----------------------11}$$

a form which is usually called the equation of continuity.

When the flow is steady, the identity of the fluid within the control; volume changes continuously, but the total mass remains constant or mathematically $\partial\rho/\partial t$ is zero for each element of control volume . Then equation 11 state that the incoming and outgoing mass rate of flow are identical.

$$\int \rho V_n dA_{in} = \int \rho V_n dA_{out} \quad \text{---------------------} 12$$

For one dimensional steady state flow equation 12 for the inlet and outlet condition become.

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \quad \text{---------------------} 13$$

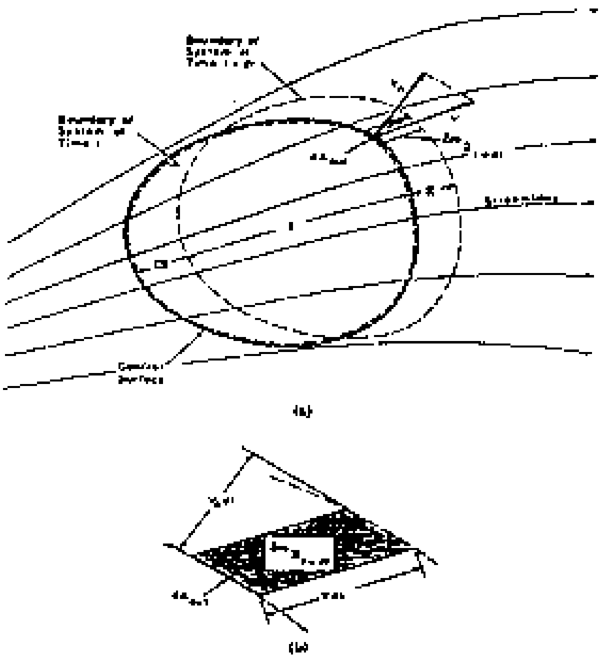


fig.(1) Flow through a control volume(continuity equation)

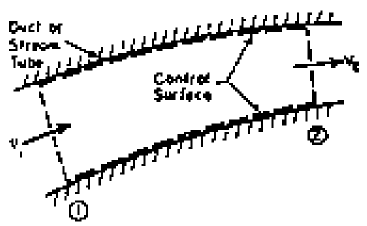


fig.2 One dimensional flow

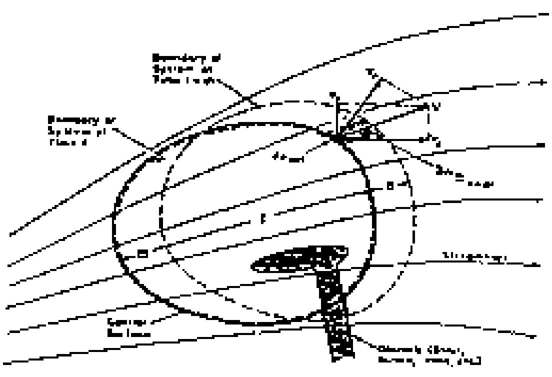


fig.3 flow through control volume with obstacle(momentum equation)

Example: I

Ten kg/sec of air enters a tank of $10m^3$ in volume while 2 kg/sec is discharge from the tank as show in fig. If the temperature of the air inside the tank remains constant at 300K°. Find the rate of pressure rise inside the tank.

Solution:

Appling continuity equation

$$\int_v \frac{\partial \rho}{\partial t} dv = \int \rho V_n dA_{in} - \int \rho V_n dA_{out}$$

$$10\frac{\partial \rho}{\partial t} = 10-2, \qquad \text{but } p = \rho RT \qquad \text{so } \frac{\partial p}{\partial t} = RT\frac{\partial \rho}{\partial t}$$

$$\frac{\partial p}{\partial t} = 287 * 300 * \frac{8}{10} = 68880 \ P_a/sec$$

Example: 2

A tank 1 m^3 in volume contains air at an initial pressure of 6 atm (606.95 kPa) and an initial temperature of 25°C. Air is discharged isothermally from the tank at the rate of 0.1 m^3/s. Assuming that the discharged air has the same density as that of the air in the tank, find an expression for the time rate of change of density of the air in the tank. What would be the rate of pressure drop in the tank after 5 seconds?

solution:

Appling continuity equation $\int_v \frac{\partial \rho}{\partial t} dv = \int \rho V_n dA_{in} - \int \rho V_n dA_{out}$

$$1.0 \ \frac{\partial \rho}{\partial t} = -0.1\rho$$

or

$$\frac{\partial \rho}{\partial t} = -0.1\rho$$

Separating variables and integrating gives:

$$\rho = \rho_1 e^{-0.1t} = \left(\frac{p_1}{RT_1}\right) e^{-0.1t}$$

where subscript 1 refers to initial conditions in the tank. Pressure change may be expressed in terms of density change according to the relation:

$$p = \rho RT$$

so that:

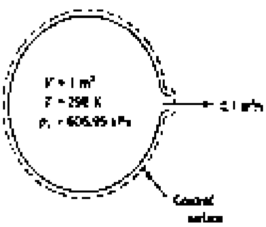
$$\frac{dp}{dt} = RT\frac{d\rho}{dt} = RT(-0.1\rho)$$

$$= -0.1RT\frac{p_1}{RT_1}e^{-0.1t}$$

$$= -0.1p_1e^{-0.1t}$$

Substituting numerical values gives:

$$\frac{dp}{dt} = -0.1 \times 606.95 \times e^{-0.1(5)} = -102.3 \text{ kPa/s}$$



1.2- *Momentum conservation theorem.*

The fundamental principle of dynamics is Newton's law of motion, and according to this law the resultant of force applied to a particle which may be at rest or in motion is equal to the rate of change of momentum of the particle in the direction of the resultant force. Newton's second law is vector relation. Consider the x-direction we write for the system.

$$\sum F_x = \frac{d}{dt}(mV_x) \qquad \text{----------14}$$

Where the left hand side represent the algebraic sum of the X-force acting on the system during the time interval dt, and the right hand side represent the time of change of the total momentum of the system see fig.3.

$$\frac{d}{dt}(mV_x) = \frac{(mV_x)_{I\,t+dt} + (mV_x)_{II\,t+dt} - (mV_x)_{I\,t} - (mV_x)_{III\,t}}{dt} \qquad \text{-----15}$$

$\dfrac{(mV_x)_{I\,t+dt} - (mV_x)_{I\,t}}{dt}$ -------- as dt goes to zero this term represent the time rate of change of the X-momentum within the control volume. $= \dfrac{\partial}{\partial t}(mV_x)_{c.v}$

so that :

$$\sum F_x = \frac{\partial}{\partial t}(mV_x)_{c.v} + \int V_x dm_{out} - \int V_x dm_{in} \qquad \text{-----------------16}$$

or

$$\sum F_x = \int_{cv} \frac{\partial \rho V_x dv}{\partial t} + \int \rho V_n V_x dA_{out} - \int \rho V_n V_x dA_{in} \qquad \text{---------17}$$

Example:3

Air flowing isentropically in a nozzle strikes a stationary blade when it leaves the nozzle as shown in fig. Determine ;

1- The magnitude of the reaction in the x-direction and in the y-direction needed to hold the blade in place.
2- The magnitude of the reaction in the x-direction and in the y-direction of the blade moves towed the nozzle at 80m/sec.

Solution:

$$T_2 = T_1\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} = 308\left(\frac{1}{1.5}\right)^{0.4/1.4} = 274.3\ \text{K}$$

The gas velocity at this section is obtained from the energy equation:

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

Therefore

$$\frac{V_2^2}{2} = c_p(T_1 - T_2) + \frac{V_1^2}{2}$$

$$= 1000(308 - 274.3) + \frac{(60)^2}{2}$$

from which $V_2 = 266.46$ m/s. The mass rate of flow is:

$$\dot{m} = \rho_1 A_1 V_1 = \left(\frac{p_1}{RT_1}\right)A_1 V_1$$

$$= \left(\frac{1.5 \times 1.013 \times 10^5}{287 \times 308}\right)(25 \times 10^{-4})(60)$$

$$= 0.258\ \text{kg/s}$$

Applying the momentum equation to the control volume shown gives:

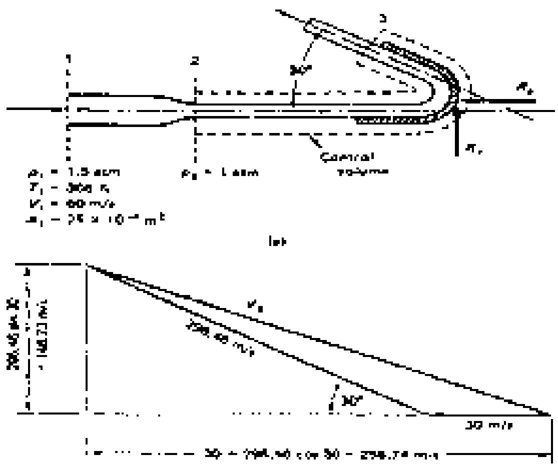
$$R_x = \dot{m}(V_{3x} - V_{2x}) = 0.258(V_3 \cos 30 + V_2)$$

$$= 0.258(266.46 \cos 30 + 266.46) = 128.28\ \text{N}$$

and

$$R_y = \dot{m}(V_{3y} - V_{2y}) = 0.258(V_3 \sin 30 - 0)$$

$$= 0.258(266.46 \sin 30) = 34.37\ \text{N}$$



(b) When the blade moves toward the nozzle, the relative velocity is 266.46 + 30 = 296.46 m/s. The mass striking the blade per unit time now becomes:

$$\dot{m} = 0.258\left(\frac{296.46}{266.46}\right) = 0.287 \text{ kg/s}$$

From the velocity diagram shown:

$$V_{3x} = 256.74 \text{ m/s} \quad \text{and} \quad V_{3y} = 148.23 \text{ m/s}$$

The momentum equation then gives:

$$R_x = \dot{m}(V_{3x} - V_{2x}) = 0.287(256.14 + 266.46) = 149.7 \text{ N}$$

and

$$R_y = \dot{m}(V_{3y} - V_{2y}) = 0.287(148.23 - 0) = 42.54 \text{ N}$$

Example:4

An airplane is traveling at a constant speed of 200 m/s. Air enters the jet engine's inlet at the rate of 40 kg/s while the combustion products are discharged at an exit velocity of 600 m/s relative to the airplane. The intake area is 0.3 m² and the exit area 0.6 m². The ambient pressure is 0.7 atm, and the pressure at the exit is 0.72 atm. Calculate the net thrust developed by the engine. Assume uniform steady conditions at the inlet and exit planes and the properties of the products of combustion to be the same as those of air.

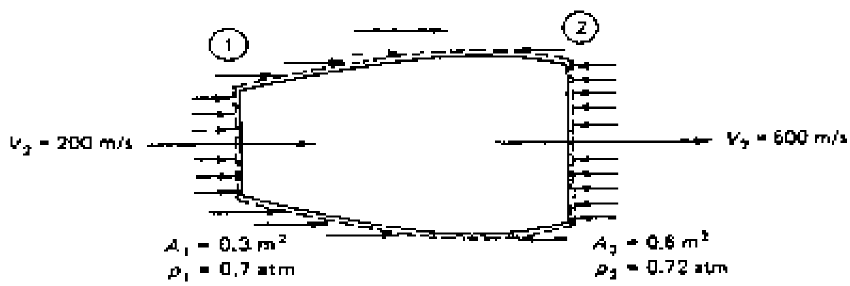
Solution: consider the jet engine as a control volume as in fig. the air enters the engine with a speed of 200m/s. assuming horizontal flight and neglecting the momentum of the fuel, the net force opposite to thrust is:

Appling momentum equation:

$$\sum F_x = \int_v \frac{\partial \rho dv}{\partial t} + \int V_x dm_{out} - \int V_x dm_{in}$$

since the case is steady state thus mean that $\partial\rho/\partial t=0$ therefore the momentum equation become

$$\begin{aligned} F &= (p_2A_2 + \dot{m}V_2) - (p_1A_1 + \dot{m}V_1) \\ &= [(0.72 - 0.7)1.013 \times 10^5 \times 0.6 + 40 \times 600] - (0 + 40 \times 200) \\ &= 17{,}215.6 \text{ N} \end{aligned}$$



1.3 The First Law of thermodynamic: (Energy Equation)

Energy is conveyed a cross the boundary of control volume in he form of heat and work. Consider the flow through the control volume with of fig., with the system defined as the material occcuping the control volume at time t. We consider what happens during the time interval dt. Passing through the control surface are a stationary strut and a rotating shaft attached to a turbo-machine, perhaps a compressor or turbine. The energy equation in a simple form can be written as following.

$$\frac{\delta Q}{dt}=\frac{dE}{dt}+\frac{\delta W}{dt}$$

Rate of change of total energy E:

$$\frac{dE}{dt}=\frac{(E_{I\,t+dt}+E_{II\,t+dt})-(E_{I\,t}+E_{III\,t})}{dt}$$

$$\frac{dE}{dt}=\frac{E_{I\,t+dt}-E_{I\,t}}{dt}+\int\frac{e\,\delta m_{out}}{dt}-\int\frac{e\,\delta m_{in}}{dt}$$

δ

$$\frac{dE}{dt}=\left(\frac{\partial E}{\partial t}\right)_{cv}+\int e\,dm_{out}-\int e\,dm_{in}$$

$$\frac{dE}{dt}=\int_{cv}\frac{\partial e\rho dv}{\partial t}+\int e\,dm_{out}-\int e\,dm_{in}$$

Rate of work done.

Omitting from our consideration capillary, magnetic, and electrical force, the work done during the processes is the result of normal and shear stresses at the moving boundaries of the system.

A- Work Done by Normal Stresses.

Taking the normal stress at the boundary of the system as the hydrostatic pressure, the work done by the system owing to normal force at an element of area dA_{out} is $pdA_{out}dx$, where dx is the component of distance moved normal to dA_{out}. But $dA_{out}dx$ is the volume of the mass element $\delta m_{II\,t+dt}$ which volume may be writeen as $v\delta m_{II\,t+dt}$. The total rate of work done by normal stresses during the process may now be set down, with the aid of the foregoing, as

$$\left(\frac{\delta W}{dt}\right)_{normal}=\frac{\int p\,v\,\delta m_{II\,t+dt}}{dt}-\frac{\int p\,v\,\delta m_{III}}{dt}$$

$$=\int p\,v\,dm_{out}-\int p\,v\,dm_{in}$$

B- Work Done by Shear Stresses: This work may be conveniently divided into two categories (i) the work done by the part of the shaft inside the system on the part outside the system, owing to the torque in the rotating shaft resulting from the shear stresses. (ii) the shear work done at the boundaries of the system on adjacent fluid which is in motion. Therefore the rate change of work can be written as follow.

$$\frac{\delta W}{dt}=W_{shaft}+W_{shear}+\int p\,v\,dm_{out}+\int p\,v\,dm_{in}$$

The total fluid energy per mass flow e is

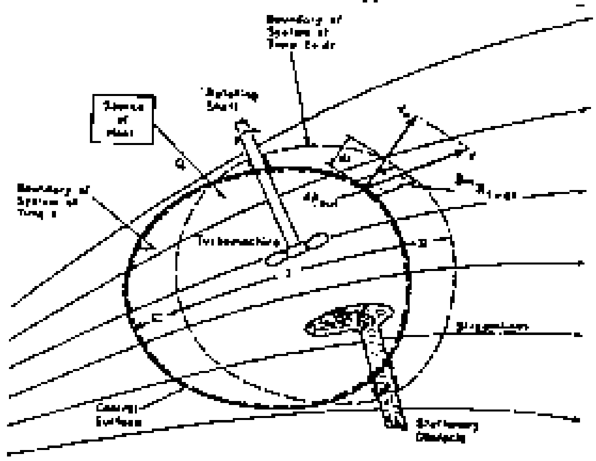
Total fluid energy= internal energy + kinetic energy + potential energy

$$e = u + \frac{V^2}{2} + gz$$ ----------------

$$u = h - pv = h - \frac{p}{\rho}$$ ----------------

Substitute these equations into the energy equation results

$$\frac{\delta Q}{dt} = W_{shaft} + W_{shear} + \int_{cv} \frac{\partial e \rho dv}{dt} + \int (h + \frac{V^2}{2} + gz) dm_{out} - \int (h + \frac{V^2}{2} + gz) dm_{in}$$ ----------



1.4 The second Law of Thermodynamics:

In a fixed-mass system entropy change occurs as a result of irreversible events or as a result of interaction with the environment in which there is heat transfer.

$$\oint_{cs} \frac{dQ}{T} \le (\frac{\partial S}{\partial t})_{cv} + \int s dm_{out} + \int s dm_{in}$$

$$\oint_{cs} \frac{dQ}{T} \le \int_{cv} \frac{\partial s \rho dv}{\partial t} + \oint_{cs} s \rho V dA$$

for steady –one dimension flow

$$m(s_2 - s_1) \ge \oint_{cs} \frac{dQ}{T}$$

for adiabatic flow $dQ=0$ therefore

$s_2 - s_1 \ge 0$ or $ds \ge 0$ for isentropic flow $ds = 0$ and flow adiabatic irreversible flow $ds>0$

1.5 The perfect Gas:

For most problem in gas dynamics, the assumption of perfect gas law is sufficiently in accord with the properties of real gases as to be a acceptable. We shall therefore set down here the special thermodynamics relations which apply to perfect gas.

1- *Equation of state:*

$$pv = \frac{p}{\rho} = RT = \frac{\Re}{M} T$$ --------------

Where T is the absolute temperature (K^o), R is the gas constant($J/kg.mol.K^o$), $\Re$ is the universal gas constant and is equal to 8134.3 $J/kg.mol.K^o$, and M is the molecular weight kg/kg.mol. For atmospheric air between 0 and 100 km, M=28.966, therefore the air gas constant is 287.04 J/kg.K°

When a perfect gas undergoes a thermodynamic process between to equilibrium state.

$$u_2 - u_1 = \int_1^2 cv.dT \quad \text{and} \quad h_2 - h_1 = \int_1^2 cp.dT$$

$$cv = (\frac{\partial u}{\partial T})_v = \frac{du}{dT} \quad \text{and} \quad Cp = (\frac{\partial h}{\partial T})_p = \frac{dh}{dT} \quad \text{for perfect gas}$$

$$Cp - Cv = \frac{dh}{dT} - \frac{du}{dT} = \frac{d(u+pv)}{dt} - \frac{du}{dT} = \frac{d(RT)}{dT}$$

$$Cp - Cv = R \text{ --}$$

The specific heat ratio γ is $\gamma = \frac{Cp}{Cv}$ therefore $Cp = \frac{\gamma R}{\gamma - 1}$ and $Cv = \frac{R}{\gamma - 1}$

Changes of Entropy : Applying the special relation of a perfect gas to the general relation between s,u,v, we get

$$ds = \frac{du}{T} + \frac{pdv}{T} = Cv\frac{dT}{T} + R\frac{dv}{v}$$

and, upon integration

$$S_2 - S_1 = Cv \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1} = Cv\ln(\frac{T_2}{T_1}).(\frac{v_2}{v_1})^{\gamma-1} \text{ ----------------------}$$

Alternatively, we may eliminate either T or v from this express the aid of $pv=RT$,and so obtain

$$S_2 - S_1 = Cv\ln\frac{p_2}{p_1} + Cp\ln\frac{v_2}{v_1} = Cv\ln(\frac{p_2}{p_1})(\frac{v_2}{v_1})^{\gamma} \text{ ----------------------}$$

$$S_2 - S_1 = Cp\ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} = Cv\ln(\frac{T_2}{T_1})^{\gamma}.(\frac{p_2}{p_1})^{-\gamma-1} \text{ ----------------------}$$

The isentropic. Often the isentropic process is taken as a model or as a limit for real adiabatic processes. If entropy is constant at each step of the processes, it follows from equation that T and v,p and v, and T and p are connected with each other during the processes by the following laws:

$$Tv^{\gamma-1} = const. \qquad pv^{\gamma} = \frac{p}{\rho^{\gamma}} = const. \qquad \frac{T^{\frac{\gamma}{\gamma-1}}}{p} = const. \text{ ------------------}$$

For isentropic flow process the enthalpy change is important. It is calculated in terms of the initial temperature and the pressure ratio as follows:

$$(\Delta h)_s = Cp(T_2 - T_1) = Cp.T_1\left[(\frac{T_2}{T_1}) - 1\right] = CpT_1\left[(\frac{p_2}{p_1})^{\frac{\gamma-1}{\gamma}} - 1\right] \text{ --------------------}$$

Chapter Two

Wave Propagation in Compressible flow

2.1 Introduction:

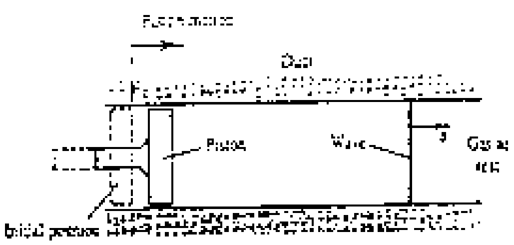
The term compressible flow implies variation in density through the field of flow. These variations are, in many cases, the result principally of pressure changes from one point to another. The rate of change of density with respect to pressure is, therefore, an important parameter in the analysis of compressible flow, and, as we shall see, it is closely connected with the velocity of propagation of small pressure disturbance, i.e. with the velocity of sound.

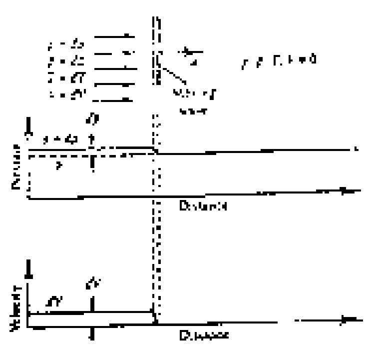
2.2 Wave Propagation in Elastic Media:

let us examine what happens when a solid elastic object such as steel bar is subjected to a sudden uniform distributed compressive stress applied at one end. in the first instant of time, a thin layer next to the point of application is compressed, while the remainder of the bar is unaffected. This compression is then transmitted to the next layer, and so on down the bar. Thus a disturbance created at the left side is eventually sensed at the opposite end. The compression wave initiated at the left side of the bar takes a finite time to travel to the right side, the wave velocity being dependent on the elasticity and density of the media.

Gases and liquid also are elastic substance and longitudinal wave can be propagated through these media in the same way that waves propagated through solid. Let a gas be confined in a long tube with a piston at the left hand. The piston is given a sudden push to the right. In the first instant a layer of gas piles up next to the piston and is compressed, the reminder of the gas is unaffected. The compression wave created by the piston then moves through the gas until eventually all the gas is able to sense the movement of the piston. If the impulse given to the gas is infinitesimally small, the wave is called a sound wave and the resultant compression wave move through the gas at velocity equal to the velocity of sound.

Let the pressure change across the wave be dp and let the corresponding density and temperature change be $d\rho$ and dT respectively. The gas into which the wave is propagated is assumed to be at rest. The wave will then induce a gas velocity dV behind it as it move through the gas. The changes across the wave are, therefore as shown in fig.2.2. In order to analyze the flow through the wave and thus to determine (a), it is convenient to use a coordinated system that is attached to the wave, i.e. is moving with the wave. In this coordinate system, the wave will of course be at rest and the gas will effectively flow through it with the velocity ,a, ahead of the wave and a velocity, a-dV, behind the wave. In this coordinate system, then, the changes through the wave are shown in fig.2.3. The pressure, temperature and density change, of course, independent of the coordinate system used.





The continuity and momentum equation are applied to a control volume of unit area across the wave as indicated in fig. For steady state the continuity equation for the control volume is:

$m^* = \rho a = (\rho + d\rho)(a - dV)$ ------------------2.1

where m is the mass low rate per unit area through the wave. Since the case of a very weak is being consider, the second order term, $d\rho dV$ that arises in equation can be neglected and this equation then gives:

$$d\rho = \frac{\rho}{a} dV \text{ ------------------------2.2}$$

Conservation of momentum is next considered. The only force acting on the control volume are the pressure force. The momentum equation for steady state become:

$pA - (p + dp)A = m^*[(a - dV) - a]$ ----------------------2.3

which lead to:

$$Adp = m^* dV \quad \text{or} \quad dp = \rho a dV \text{ -------- 2.4}$$

Substitute equation 2.2 into equation 2.4 gives:

$$\frac{dp}{d\rho} = a^2 \quad \text{or} \quad a = \sqrt{\frac{dp}{d\rho}} \text{ --------2.5}$$

In order to evaluate a using the above equation, it is necessary to know the process that the gas undergoes in passing through the wave. Because a very weak wave is being considered, the temperature and velocity changes through the wave are very small and the gradient of temperature and velocity within the wave remain small. For this reason, heat transfer and viscous effect for flow through the wave are assumed to be negligible. Hence, is passing through the wave, the gas is assumed to undergo an isentropic process. The flow through the wave is, therefore, assumed to satisfy:

$$\frac{p}{\rho^\gamma} = const. \text{ -------------------2.6}$$

putting this into logarithmic form, and differentiating the equation:

$\ln p - \gamma \ln \rho = const.$

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \quad \text{or} \quad \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \text{ -------------- 2.7}$$

noting that the fluid is compressible and is perfect gas, therefore $p = \rho RT$ substituting this into equation 2.7 and equation 2.5.

$$a = \frac{dp}{d\rho} = \sqrt{\gamma RT} \text{ ---------------2.8}$$

2.3 Pressure Field Created by a Moving Point Disturbance:

In order to illustrate the effect of the velocity of the body relative to the speed of sound on the flow field, consider the small body, i.e. essentially a point source of disturbance, to be moving at a uniform liner velocity, through the gas and let the speed of sound in the gas be a. Although the body is essentially emitted wave continuously, a series of wave emitted at time interval t will be consider. Since the body is moving through the gas, the origin of these waves will be continually changing. Wave generated at time 0,t,2t, and 3t will be considered. First, consider the case where the speed of the body is very small compared to the speed of sound. The pressure pattern which exists at any instant is then found by superposition of all the pressure pulses which were previously emitted. Fig. shows several pressure pulse pattern for different value of the speed of the source compared with the speed of sound in the fluid.

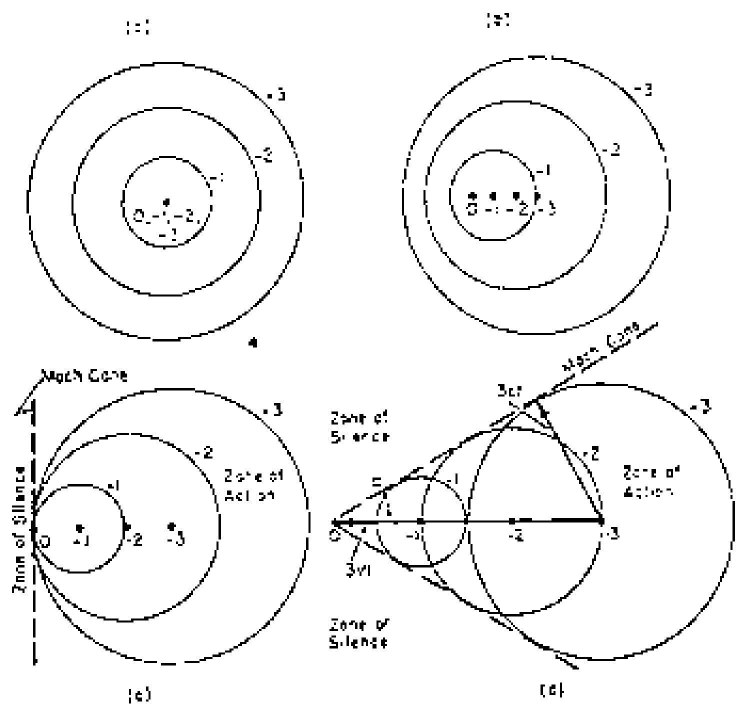


Fig (2.3). Pressure field produced by a point source of disturbance moving at uniform speed leftwards.

(a) Incompressible fluid ($V/c = 0$).
(b) Subsonic motion ($V/c = \frac{1}{2}$).
(c) Transonic motion ($V/c = 1$).
(d) Supersonic motion, illustrating Karman's three rules of supersonic flow ($V/c = 2$).

*- **Incompressible Flow:** When the medium is incompressible (fig.2.3a). or when the speed of the moving point disturbance is small compared with the speed of sound. the pressure pulse spread uniformly in all direction.

- Subsonic Flow : When the source move at subsonic speeds, Fig.2.3b, the pressure disturbance is felt in all direction and at all points in space, but the pressure pattern is no longer symmetrical.

- Supersonic Flow: For supersonic speed Fig.2.3d indicates that the phenomena are entirely different from those at subsonic speed. All the pressure disturbance are included in a cone which has the point source of disturbance. The cone within which the disturbances are confined is called the Mach cone. Fig.2.3c shows the pressure pattern at the boundary between subsonic and supersonic, that is, for the case where the stream velocity is identical with the sonic velocity; here the wave front is a plane.

Karman's Rules of Supersonic Flow : Fig 2.3 illustrates the three rules of supersonic flow proposed by Van Karman's .

1- **The Rules of Forbidden Signals.** The effect of pressure change produced by a body moving at a speed faster than sound cannot reach point ahead of the body.
2- **The Zone of Action and the Zone of Silences.** A stationary point source in a supersonic stream produces effect only on point that lie on or inside the Mach cone extending downstream from the point source. Conversely, the pressure and velocity at an arbitrary point of the stream can be influenced only by disturbance acting at point that lies on or inside a cone extending upstream from the point considered and having the same vertex angle as the Mach cone.
3- **The Rule of Concentrated Action.** The pressure disturbance is largely concentrated in the neighbourhood of the Mach cone that forms the outer limit of the zone of action.

2.4 The Mach Number and the Mach Angle:

It was shown that the nature of the flow pattern depends on the comparative magnitudes of the stream velocity and the sonic velocity. The ratio of these two velocity is called the Mach Number. Thus,

$$M = \frac{V}{a} \quad \text{.......................2.9}$$

The semi-angle of the Mach cone is related to the Mach number by

$$\sin\alpha = \frac{1}{M} \quad \text{.......................2-10}$$

Note that the mach angle is imaginary for subsonic flow.

Example:

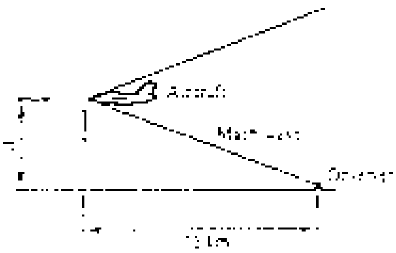
An observer on the ground finds that an airplane flying horizontally at an altitude of 5000 m has traveled 12 km from the overhead position before the sound of the airplane is first heard. Estimate the speed at which the airplane is flying.

Solution

It is assumed that the net disturbance produced by the aircraft is weak, i.e., that, as indicated by the wording of the question, basically what is being investigated is how far the aircraft will have traveled from the overhead position when the sound waves emitted by the aircraft are first heard by the observer. If the discussion of Mach waves given above is considered, it will be seen that, as indicated in Fig. E3.9, the aircraft will first be heard by the observer when the Mach wave emanating from the nose of the aircraft reaches the observer.

Now, since the temperature varies through the atmosphere, the speed of sound varies as the sound waves pass down through the atmosphere which means that the Mach waves from the aircraft are actually curved. This effect is, however, small and will be neglected here, the sound speed at the average temperature between the ground and the aircraft being used to describe the Mach wave.

Now as discussed in Example 3.3, for altitudes, H, of from 0 m (sea-level) to 11019 m, the temperature in the atmosphere is given by $T = 288.16 - 0.0065H$ so, at the mean altitude of 2500 m, the temperature is $288.16 - 0.0065 \times 2500 = 271.9$ K. Hence, the mean speed of sound is given by:



$$a = \sqrt{\gamma RT} = \sqrt{1.4 \times 287.04 \times 271.9} = 330.6 \text{ m/s}$$

From the above figure it will be seen that if α is the Mach angle based on the mean speed of sound then

$$\tan\alpha = 5000/12\,000 = 0.417$$

But since $\sin\alpha = 1/M$, it follows that $\tan\alpha = 1/\sqrt{M^2 - 1}$ so

$$M = \sqrt{(1/0.417)^2 - 1} = 2.6$$

Hence, it follows that:

$$\text{Velocity of aircraft} = 2.6 \times 330.6 = 859.6 \text{ m/s}$$

Problem:

2.1 Air at a temperature of 25°C is flowing with a velocity of 180 m/s. A projectile is fired into the air stream with a velocity of 800 m/s in the opposite direction to that of the air flow. Calculate the angle that the Mach waves from the projectile make to the direction of motion.

2.2 An observer at sea level does not hear an aircraft that is flying at an altitude of 7000 m until it is a distance of 13 km from the observer. Estimate the Mach number at which the aircraft is flying. In arriving at the answer, assume that the average temperature of the air between sea level and 7000 m is −10°C.

2.3 An observer on the ground finds that an airplane flying horizontally at an altitude of 2500 m has traveled 6 km from the overhead position before the sound of the airplane is first heard. Assuming that, overall, the aircraft creates a small disturbance, estimate the speed at which the airplane is flying. The average air temperature between the ground and the altitude at which the airplane is flying is 10°C. Explain the assumptions you have made in arriving at the answer.

In the absent of electromagnetic force and with friction negligible, the only force acting on the control surface are pressure force. Assume that a pressure $p-dp/2$ acts on the side surface of the control volume.

$$pA + (p + \frac{dp}{2})dA - (p + dp)(A + dA) = (\rho AV)(V + dV - V),$$

Simplifying yields.

$$dp + \rho VdV = 0 \quad \text{------------------------3.2}$$

The energy equation with no external heat transfer and no work, for stady one-dimensional flow become.

$$\int_s (h + \frac{V^2}{2})(\rho VdA) = 0 \quad \text{------------------------}$$

or $\quad dh + d\frac{V^2}{2} = 0$

An expression for the second law of thermodynamic is given :

$Tds = dh - \frac{dp}{\rho}$ and for isentropic flow $ds = 0$ therefore $dh = \frac{dp}{\rho}$

Combining these equation we obtain:

$\frac{dp}{\rho} = -d\frac{V^2}{2}$ or $dp + \rho VdV = 0$ which is the same as the momentum equation

3.3 Isentropic flow Through a Varying Area Channel.

Combining the continuity and momentum equation for isentropic flow result in.

$$dp + \rho V^2\left[-\frac{d\rho}{\rho} - \frac{dA}{A}\right] = 0$$

But

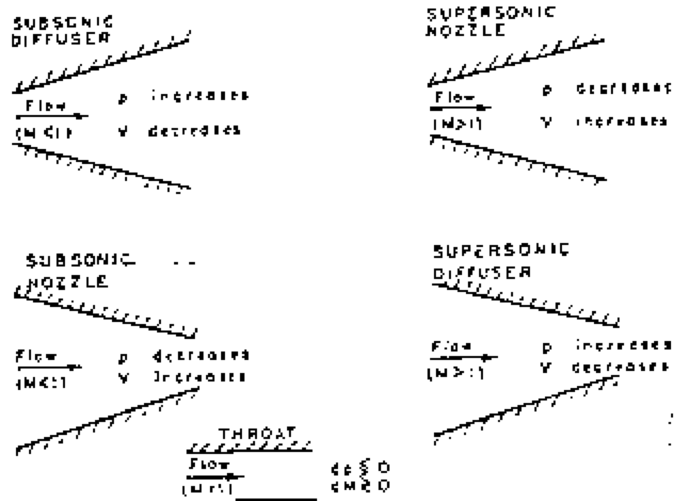
$$\frac{\partial p}{\partial \rho} = a^2 \quad \text{Therefore, for isentropic flow}$$

$$dp - \rho V^2(-\frac{dp}{\rho a^2} - \frac{dA}{A}) = 0 \quad \text{and} \quad M = \frac{V}{a}$$

$$dp(1 - M^2) = \rho V^2\frac{dA}{A} \quad \text{-------------------3.3}$$

Equation 3.3 demonstrates the influence of Mach number on that flow. For M<1, subsonic flow, the term $1-M^2$ is positive. Therefore, an increase in area result in an increase in pressure and from equation 3.2 a decrease in velocity. Likewise, a decrease in area results in decrease in pressure and an increase in velocity. For supersonic flow, the term $1-M^2$ in equation 3.3 is negative, and opposite variation occur. The result illustrate in fig have ramifications. Subsonic flow cannot be accelerated to a velocity greater than the velocity of sound in a converging nozzle. This is true irrespective of the pressure difference imposed on the flow through the nozzle. If it is desired to accelerate a stream from negligible velocity to supersonic velocity. A convergent-divergent channel must be used as show in fig.

Fig 3.2 Show the variation of the pressures and velocity in different shape of area change for subsonic and supersonic flow.



3.4 Stagnation ~~and~~ Properties:

Stagnation properties are useful in that they define a reference state for compressible flow. Stagnation enthalpy or total enthalpy, at appoint in flow is defined as the enthalpy attained by bringing the flow adiabatically to rest at that point. For adiabatic process energy equation become

$$h_o = h + \frac{V^2}{2}$$

Where h_o is the stagnation or total enthalpy per unit mass. Likewise, stagnation temperature T_o or T_o can be defined as the temperature measured by bringing a flow adiabatically to rest at a point. For a perfect gas with constant specific heats the energy equation becomes:

$$c_p T_o + \frac{V_o^2}{2} = c_p T + \frac{V^2}{2}$$ since V_o=0, therefore

$$c_p T_o = c_p T + \frac{V^2}{2} \quad \text{or} \quad T_o = T + \frac{V^2}{2c_p} \quad \text{or} \quad \frac{T_o}{T} = \left(1 + \frac{V^2}{2c_pT}\right) \quad \text{since } c_p = \frac{\gamma R}{\gamma - 1}$$

Therefore. $\frac{T_o}{T} = \left(1 + \frac{(\gamma-1)V^2}{2\gamma RT}\right)$ whereas $a = \sqrt{\gamma RT}$, and $M = \frac{V}{a}$ Therefore.

$$\frac{T_o}{T} = \left(1 + \frac{\gamma - 1}{2} M^2\right) \quad \text{-------------------3.4}$$

For isentropic flow the relation between pressure, temperature and density of perfect gas are: $\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma}$ and $\frac{T}{T_o} = \left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}}$ Therefore the pressure and density relation become.

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \quad \text{-----------------------3.5}$$

$$\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}} \quad \text{-----------------------3.6}$$

3.5 Flow per Unit Area.

Next we will derive a useful relation between the flow per unit area, stagnation temperature, pressure and Mach number for perfect gas. Starting with the equation of continuity we make the following arrangements:

$$\frac{m^*}{A} = \rho V = \frac{p}{RT}V = \frac{pV}{\sqrt{\gamma RT}}\sqrt{\frac{\gamma}{R}}\sqrt{\frac{T_o}{T}}\frac{1}{\sqrt{T_o}}$$

Substitute equation 3.4 for adiabatic flow

$$\frac{m^*}{A} = \sqrt{\frac{\gamma}{R}}\frac{p}{\sqrt{T_o}}M\sqrt{1-\frac{\gamma-1}{2}M^2} \quad \text{--------------3.7}$$

To find a conventional formula for the mass flow per unit area in terms of M, we eliminate p in the equation above by means of the isentropic law relation , or substitute equation 3.5.

$$\frac{m^*}{A} = \sqrt{\frac{\gamma}{R}}\frac{p_o}{\sqrt{T_o}}\frac{M}{(1-\frac{\gamma-1}{2}M^2)^{\frac{\gamma+1}{2(\gamma-1)}}} \quad \text{--------------3.8}$$

3.6 Maximum Flow per Unit Area: To find the condition of maximum flow per unit area we could differentiate equation 3.8 with respect to M and set this derivative equal to zero. At this condition, we would find that M=1. Therefore to find m/A_{max} we need only set M=1 in equation 3.8 , thus we find.

$$(\frac{m^*}{A})_{max} = \frac{m^*}{A^*} = \sqrt{\frac{\gamma}{R}}(\frac{2}{\gamma+1})^{\frac{\gamma+1}{2(\gamma-1)}}\frac{p_o}{\sqrt{T_o}} \quad \text{----------------3.9}$$

For a given gas, therefore, the maximum flow per unit area depends only on the ratio $p_o/\sqrt{T_o}$. For a given value of the stagnation pressure and stagnation temperature and for a passage with minimum area. Equation 3.9 shows that maximum flow which can be passed is relatively large for gases of high molecular weight and relatively small for gases of low molecular weight. Doubling the pressure level doubles the maximum flow, whereas doubling the absolute temperature level reduce the maximum flow by a bout 29 per cent. For air with γ=1.4 and R=287 J/kg.K° the maximum mass flow per unit area is:

$$\frac{m^*\sqrt{T_o}}{A^* p_o} = 0.04042$$

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The particular value of the temperature, pressure and density ratios at the critical state (i.e at the minimum area) are found by setting M=1 in equations 3.4, 3.5, 3.6. We will refer to the critical properties by superscript asterisk (*).

$$\frac{T^*}{T_o} = (\frac{2}{\gamma+1}) \quad \text{for air} = 0.833$$

$$\frac{p^*}{p_o} = (\frac{2}{\gamma+1})^{\frac{\gamma}{\gamma-1}} \quad \text{for air} = 0.5283 \qquad \frac{\rho^*}{\rho_o} = (\frac{2}{\gamma+1})^{\frac{1}{\gamma-1}} \quad \text{for air} = 0.6339$$

$P_B < P$

3.7 The area Ratio.

Just as we have found it convenient to work with the dimensionless ratio p/p_o etc. it is convenient to introduce a dimensionless area ratio. Obviously the appropriate reference area is A^*, and so we compute from equation 3.8 and 3.9 the formula.

$$\frac{A}{A^*} = \frac{m}{m}\frac{A^*}{A} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma+1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad \text{.............. 3.10}$$

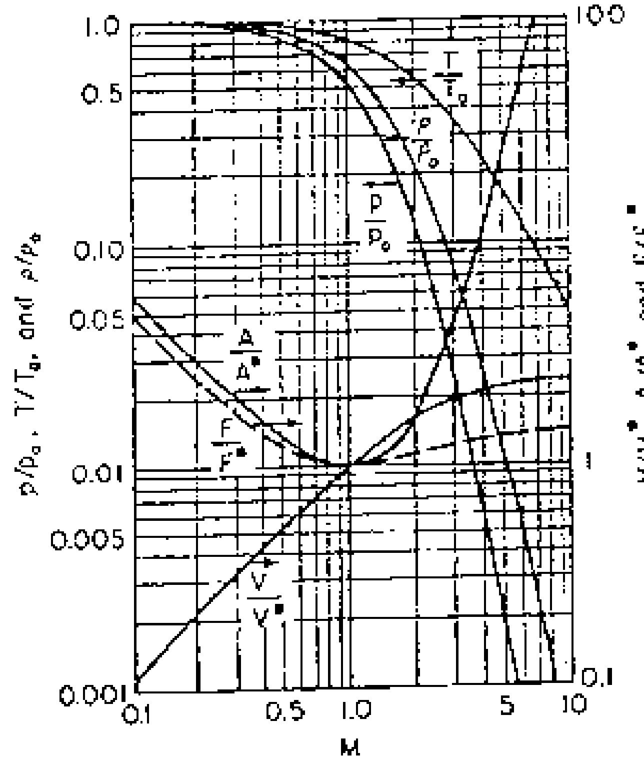
The area ration is always grater than unity, and for any given value of A/A^* there always correspond two value of M, one for subsonic flow and the other for supersonic flow

3.8 Working Charts and Tables for isentropic Flow

Since the formulas thus fae derived lead to tedious numerical calculation , of the of a trial-error natural, practical computation are greatly facilitated by working chart and tables.

Chart for Isentropic Flow.

Fig. represent in graphical form the various dimensionless ratio foe isentropic flow with M as independent variable. Since changes of fluid properties in isentropic flow are brought a bout through change in cross-sectional area, the key curve on this chart is that of A/A^*. The effects of change in area on other properties may easily be found by tracing the curve of A/A^*, keeping in mind that A^*, p_o ,etc. are all constant reference value for a given problem. For example, an increase in area at subsonic speed produces a decrease in velocity, an increase in p, T, ρ.



$\rho_o = 5$...

Working Tables.

For accurate or extensive calculation tables is available. lists the various isentropic function for γ=1.4 with Mach number as independent argument.

M	p/p_o	T/T_o	ρ/ρ_o	A/A^*	$\frac{A}{A^*}\cdot\frac{p}{p_o}$
0.5	0.843	0.95238	0.8893	1.3398	1.1295
2.0	0.1278	0.55556	0.2300	1.6875	0.21567

3.9 Isentropic Flow in Convergent Nozzle:

Consider a fluid stored in a large reservoir is to be discharge through a converging nozzle to region where the back pressure P_b is controllable by means of a valve. For a constant reservoir pressure P_o it is desired to study the effects of the variations in back pressure on the rat of mass flow through the nozzle, the pressure distribution along the passage and on the exit-plane pressure P_e. These effect are portrayed graphically in Fig am b. and c, respectively.

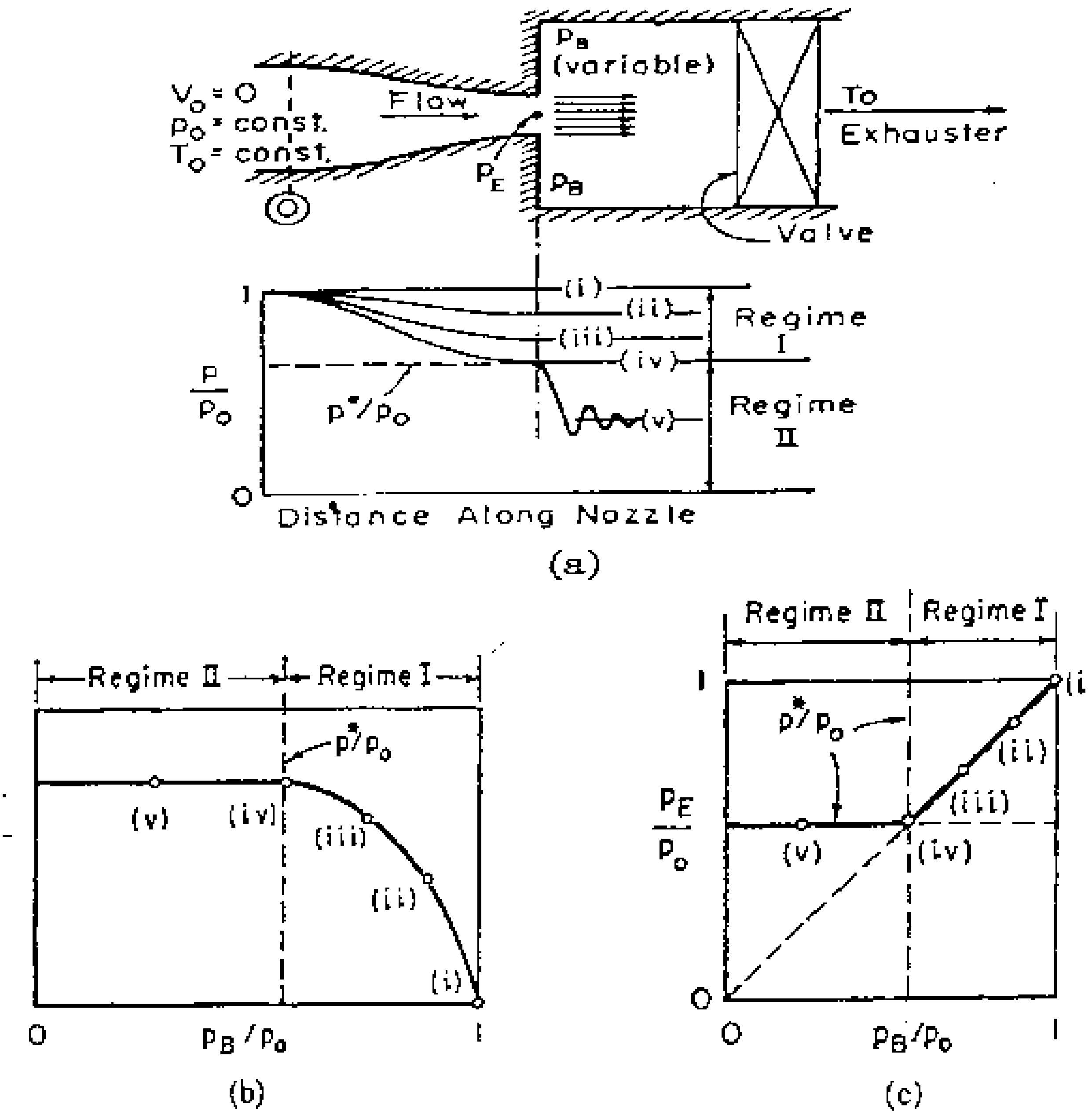


Fig. operation of converging nozzle at various back pressure.

To begin with, suppose that $P_b/P_o=1$, shown as condition (i) in fig.. The pressure is then constant through the nozzle, and there is no flow. If P_b is now reduced to a value slightly less than P_o as shown by condition (ii), there will be flow with a constantly decreasing pressure through the nozzle. Because the exit flow is subsonic, the exit-plan pressure Pe must be the same as the back pressure P_b. A further reduction in P_b to condition(iii) acts to increase the flow rate and to change the pressure distribution , but there is no qualitative change in performance. Similar consideration apply until condition (v) is reach at which point P_b/P_o equal the critical pressure ratio and the value of Me equal unity. Further reduction in Pb/Po, say to condition (v), cannot produce further change in condition within the nozzle, for the value of Pe/Po cannot be made less than the critical pressure ratio unless there is a throat upstream of the exit section (it is assumed here that the stream fills the passage). Consequently at condition(v), the pressure distribution within the nozzle, the value of Pe/Po , and the flow rate are all identical with the corresponding quantities for condition (iv). When the flow reach the condition the flow is called to be chocked.

To summarize the proceeding discussion, the two different type of flow will be denoted as regime I and regime II. These regimes may be compared as follows.

Regime I	Regime II
$Pb/Po > P^*/Po$	$Pb/Po < P^*/Po$
$Pe/Po=Pb/Po$	$Pb/Po=P^*/Po$
M<1	M=1
$\frac{m\sqrt{To}}{Ae.Po}$ dependent on Pb/Po	$\frac{m\sqrt{To}}{Ae.Po}$ independent on Pb/Po

3-10 Convergent-Divergent Nozzles:

Consider an experiment similar to the one describe, except that a converging –diverging nozzle is to be used. Fig. With Pb less than Po by a small amount , the flow is similar to that through a venture passage, and it may be treated approximately as incompressible. The corresponding pressure distribution is shown by curve(i) and (ii) in fig. When Pb/Po is reduced to the value corresponding to curve (iii). The Mach Number at the throat is unity, and no further reduction in Pt/Po are possible if the stream fills the passage. We consider next the operation when the flow is entirely supersonic, corresponding to curve(iv). The value of Pb/Po for curve(iv) corresponds exactly to the area ratio of the nozzle. Ae/At, as given by isentropic table(in this case At=A*, since Mt=1). This is often called the ***design pressure ratio of the nozzle.***

- No flow pattern fulfilling the condition of isentropic and one-dimensional flow can be found which will correspond to values of *Pb/Po* between those of curves (iii) and (iv) in fig. One method of finding solutions foe these boundary condition is to suppose that irreversible discontinuity involving entropy increase occur somewhere within the passage.

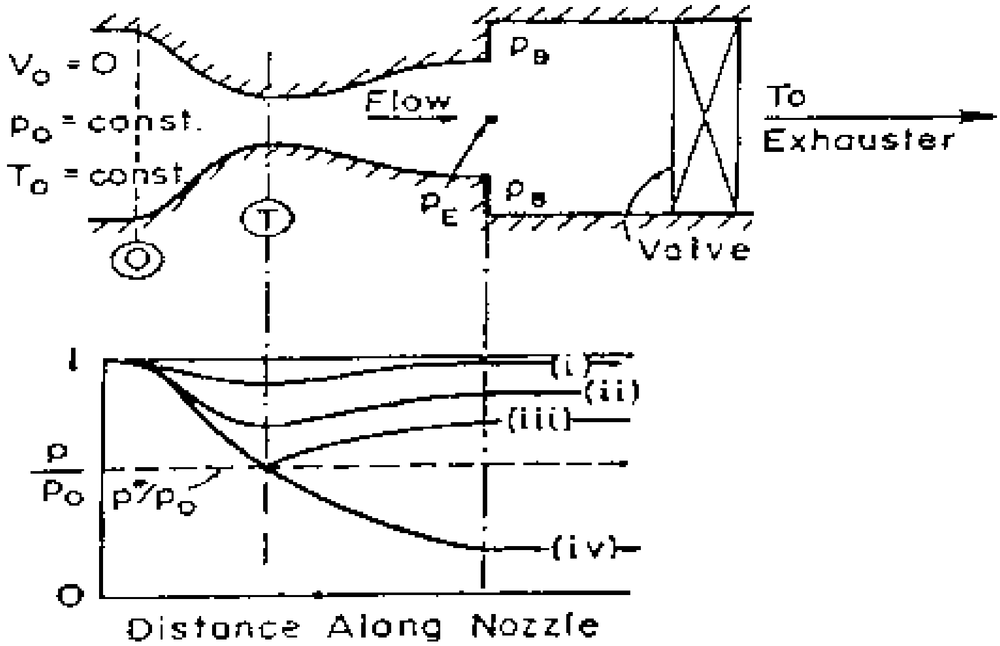


Fig. Operation of converging-diverging nozzle at various back pressure.

3-11 Some Application of Isentropic Flow.

Thrust of Rocket Motor. Rocket motor is generally consist of two parts, the combustion chamber which is a container where the fuel is burn and the thrust unit where the thrust is develop. The thrust unit is almost a convergent-divergent nozzle. The combustion chamber is generate gasses steadily at a stagnation pressure of Po and stagnation temperature of To and then the gas is expanded isentopically in the thrust unit as show in fig.

The converging-diverging nozzle has a throat area of At and exit area of Ae. The generated gases discharge to the atmosphere at pressure of Pa. Most rocket engine gases at about 3600kPa and operate in atmospheres with pressure of 101,3kPa or less, therefore, such a reduction in pressure is only possible by converging-diverging nozzle. The net thrust acting on the rocket

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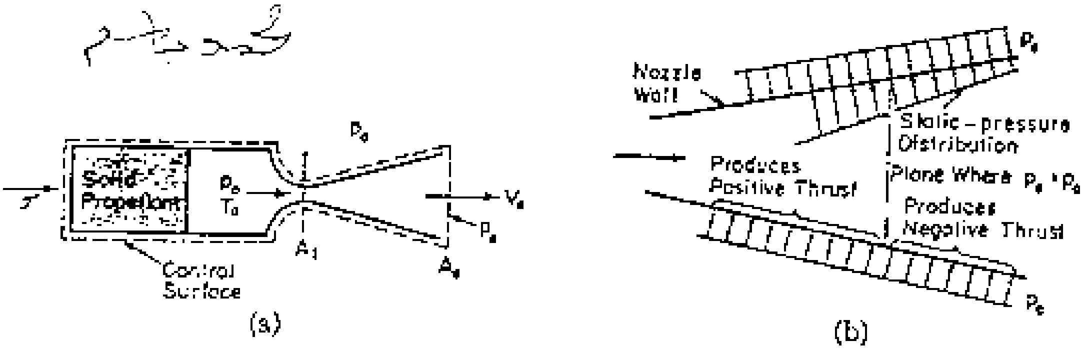


Fig Isentopic flow in rocket motor.

engine may now be obtained by applying the momentum equation on the free body diagrams of the control volume.

$$\Im = \dot{m}Ve + Ae(Pe - Pa) \quad \text{------------------------} \; 3.11$$

which is then put into dimensionless for through division by $Po.At$.

$$\frac{\Im}{Po.At} = \frac{\dot{m}}{Po.At} Ve + \frac{Ae}{At}\left(\frac{Pe}{Po} - \frac{Pa}{Po}\right) \quad \text{-----------------} \; 3.12$$

From choked flow equation

$$\frac{\dot{m}}{Po.At} = \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \frac{1}{\sqrt{To}} \quad \text{------------------------}$$

and from the energy equation :

$$Ve = \sqrt{2Cp(To - Te)} = \sqrt{2CpTo}\sqrt{1 - \frac{Te}{To}} = \sqrt{2CpTo}\sqrt{1 - \left(\frac{Pe}{Po}\right)^{\frac{\gamma-1}{\gamma}}} \quad \text{---------------}$$

Subsisting these into the thrust equation and rearranging, there results.

$$\frac{\Im}{Po.At} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}\sqrt{1 - \left(\frac{Pe}{Po}\right)^{\frac{\gamma-1}{\gamma}}} + \frac{Ae}{At}\left(\frac{Pe}{Po} - \frac{Pa}{Po}\right) \quad \text{-------------------} \; 3.13$$

Since the pressure ratio Pe/Po depends only on the area ratio equation 3.13 , indicates that the trust for a nozzle of given size and geometry depends only on Po and the ratio $Pe.Po$ and is independent of the temperature To.

Effect of Area Ration

We now ask, for given value of At, Po and Pa what exit area should be used in order to obtain maximum thrust?. By appling the calculus to equation 3.13 it may be shown after a laborious calculation that $\Im$ is a maximum when the area ratio is chosen in such a way to make the pressure in the exit plane exactly equal to Pa. Therfore equation 3.13 become.

$$\frac{G_{max}}{Po.At} = \gamma\sqrt{\frac{2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}\sqrt{1-\left(\frac{Pe}{Po}\right)^{\frac{\gamma-1}{\gamma}}} \quad \text{-------------------- 3.14}$$

Performance of Real Nozzle:

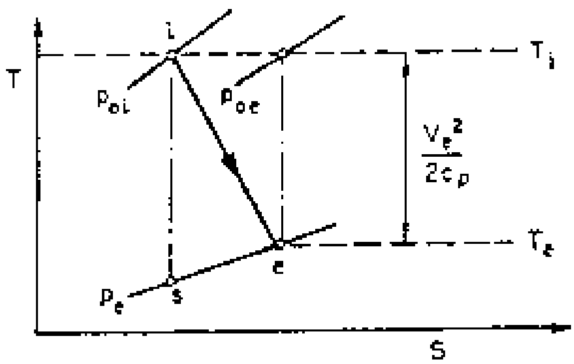
The performance of real nozzle differs slightly from that computed by isentropic flow owing to the friction effect. Since departure from isentropic flow are usually small, tha usual design procedure is based on the use of isentropic flow function which then modified by empirically determined coefficient. These coefficient are the nozzle efficiency and the nozzle discharge coefficient.

The nozzle efficiency η_N may defined as the rate of the exit kinetic energy to the kinetic energy which may be obtained by expanding the gas isentropically to the same final pressure.

$$\eta_N = \frac{V_e^2/2}{V_{es}^2/2} \quad \text{----------------------- 3.15}$$

The nozzle discharge coefficient Cd is defined as the rate of the actual mass flow rate m to the isentropic mass flow rate m_{is} which would be obtained by expanding the gas isentropically to the same final pressure.

$$Cd = \frac{m^*}{m_{is}^*} \quad \text{-----------------3.16}$$



The figure at the right hand side shows the isentropic and the real expansion process through the nozzle. When the first law of thermodynamic applying at the expansion process for both isentropic and the real process.

$h_{oi} = h_{es} + \frac{V_{es}^2}{2}$ and $h = c_p.T$, therefore

$V_{es}^2 = 2.c_p.T_{oi}(1-\frac{T_s}{T_{oi}})$ and for isentropic process $\frac{T_s}{T_{oi}} = (\frac{P_e}{P_{oi}})^{\frac{\gamma-1}{\gamma}}$ therefore ,

$$V_{es}^2 = 2c_p.T_{oi}\left[1-\left(\frac{P_e}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad \text{-----------------3.17}$$

similarly one might consider the imaginary isentropic process between the actual exit state and its stagnation state oe.

$h_{oe} = h_e + \frac{V_e^2}{2}$ and $h = c_p.T$, therefore

$V_e^2 = 2.c_p.T_{oe}(1-\frac{T_e}{T_{oe}})$ and for isentropic process $\frac{T_e}{T_{oe}} = (\frac{P_e}{P_{oe}})^{\frac{\gamma-1}{\gamma}}$ therefore ,

$$V_e^2 = 2c_p T_{oe}\left[1-\left(\frac{P_e}{P_{oe}}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad \text{------------------3.18}$$

The process within the nozzle is adiabatic this mean that $T_{oi} = T_{oe}$, substitute equation 3.18 and 3.17 into equation 3.16 and simplifying.

$$\frac{P_e}{P_{oe}} = \left[1-\eta_N\left(1-\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}\right)\right]^{\frac{\gamma}{\gamma-1}} \quad \text{----------3.19}$$

The mass per unit area for isentropic flow can be evaluated as a function of pressure ratio instated of Mach Number. If one can substitute equation 3.5 into equation 3.7.

$$\frac{m_{ist}^{\cdot}}{A} = P_o\sqrt{\frac{\gamma}{RT_o}}\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_s}{P_{oi}}\right)^{\frac{2}{\gamma}}-\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma+1}{\gamma}}\right]} \quad \text{-------------3.20}$$

Similarly the actual mass flux may be obtain.

$$\frac{m^{\cdot}}{A} = P_o\sqrt{\frac{\gamma}{RT_o}}\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_e}{P_{oe}}\right)^{\frac{2}{\gamma}}-\left(\frac{P_e}{P_{oe}}\right)^{\frac{\gamma+1}{\gamma}}\right]} \quad \text{---------------3.21}$$

Substituting equation 3.21, 3.20 into equation 3.16 to find the discharge coefficient in term of pressure ratio.

$$Cd = \frac{\left(\frac{P_e}{P_{oe}}\right)^{\frac{\gamma-1}{\gamma}}}{\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}}\sqrt{\frac{1-\left(\frac{P_e}{P_{oe}}\right)^{\frac{\gamma-1}{\gamma}}}{1-\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}}} \quad \text{---------------- 3.22}$$

Substitute equation 3.19 into the above equation to find the discharge coefficient in term of isetropic pressure ratio and nozzle efficiency.

$$Cd = \frac{\sqrt{\eta_N\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}}}{1-\eta_N\left[1-\left(\frac{P_s}{P_{oi}}\right)^{\frac{\gamma-1}{\gamma}}\right]} \quad \text{---------------------3.23}$$

PROBLEMS

3.1. Air flows at the rate of 1 kg/s through a convergent-divergent nozzle. The entrance area is 2×10^{-3} m^2 and the inlet temperature and pressure are 438 K and 580 kPa. If the exit pressure is 140 kPa and the expansion is isentropic, find:

(a) The velocity at entrance.

(b) The stagnation temperature and stagnation pressure.
(c) The throat and exit areas.
(d) The exit velocity.

3.2. A convergent nozzle has an exit area 6.5×10^{-4} m^2. Air enters the nozzle at $p_0 = 680$ kPa, $T_0 = 370$ K. If the flow is isentropic, determine the mass rate of flow for back pressure of:
(a) 359 kPa.
(b) 540 kPa.
(c) 200 kPa.

3.3. A convergent-divergent steam nozzle has an exit area of 3.2×10^{-4} m^2 and an exit pressure of 270 kPa. The inlet conditions are 1 MPa and 590 K with negligible velocity. Assume ideal flow, i.e., no losses, and

$$\frac{p^*}{p_0} = 0.545$$

Find:
(a) The mass rate of flow for this nozzle.
(b) The throat area.
(c) The sonic velocity at the throat.

3.4. Air flows isentropically through a convergent-divergent passage with inlet area 5.2 cm^2, minimum area 3.2 cm^2 and exit area 3.87 cm^2. At the inlet the air velocity is 100 m/s, pressure is 680 kPa, and temperature 345 K. Determine:
(a) The mass rate of flow through the nozzle.
(b) The Mach number at the minimum-area section.
(c) The velocity and the pressure at the exit section.

3.5. Air is flowing in a convergent nozzle. At a particular location within the nozzle the pressure is 280 kPa, the stream temperature is 345 K, and the velocity is 150 m/s. If the cross-sectional area at this location is 9.29×10^{-3} m^2, find:
(a) The Mach number at this location.
(b) The stagnation temperature and pressure.
(c) The area, pressure, and temperature at the exit where $M = 1.0$.
(d) The mass rate of flow for the nozzle.
Indicate any assumptions you may make and the source of data used in the solution.

3.6. Air flows isentropically at the rate of 0.5 kg/s through a supersonic convergent-divergent nozzle. At the inlet, the pressure is 680 kPa, the temperature 295 K, and the area is 6.5 cm^2. If the exit area is 13 cm^2, calculate:
(a) The stagnation pressure and temperature.
(b) The exit Mach number.
(c) The exit pressure and temperature.
(d) The area and the velocity at the throat.
(e) What will be the maximum rate of flow and the corresponding exit Mach number if the flow is completely subsonic in the nozzle?

3.7. A stream of carbon dioxide is flowing in a 7.5 cm I.D. pipe at a stream pressure of 680 kPa and a stream temperature of 365 K. A 7.5 cm $\times$ 5 cm venturimeter installed in this pipe shows a pressure differential reading of 1.68 mm Hg. Assuming ideal flow, determine:
(a) The mass rate of flow of CO_2. Compare your answer with that obtained if the gas is considered incompressible.

(b) If the mass rate of flow of CO_2 were to be doubled, what would be the new pressure differential reading for the venturimeter?

(c) If the fluid were hydrogen instead of CO_2, other conditions being the same as given in the problem statement, what would be the mass rate of flow?

(d) If the temperature of the CO_2 were 440 K instead of 365 K, other conditions being the same as given in the problem statement, what would be the mass rate of flow for the CO_2?

3.8. A 0.14 m^3 tank of compressed air discharges through a 2.2 cm diameter converging nozzle located in the side of the tank. If the mass flow coefficient of the nozzle based on isentropic flow through it is 0.95 and the gas within the tank expands isothermally from 1 MPa to 350 kPa, plot the pressure in the tank versus elapsed time as the pressure decreases. Assume the temperature of the tank is 295 K and the surrounding pressure is 101.3 kPa.

علامة 3.9. Air at stagnation conditions of 2 MPa and 750 K flows isentropically through a converging-diverging nozzle. If the maximum flow rate is 5.4 kg/s, determine:

(a) The throat area in m^2.

(b) The velocity, pressure, and temperature at the nozzle exit if the exit area is three times as large as the throat area.

3.10. Find the throat and exit areas in m^2 for a critical-flow nozzle handling air at the rate of 6.7 kg/s when the desired exit velocity is 1100 m/s with the stream at $p = 170$ kPa and $T = 310$ K. Assume isentropic flow and $\gamma = 1.4$.

3.11. Air flows reversibly and adiabatically in a nozzle. At section 1 of the nozzle the velocity, pressure, temperature, and area are 165 m/s, 350 kPa, 480 K, and 13×10^{-4} m^2. At section 2 in nozzle the area is 26×10^{-4} m^2. Find:

(a) The mass flow rate in the nozzle.

(b) V_2, M_2, p_2, t_2 and v_2.

(*Note:* There are two independent answers for this condition. Calculate both cases. If there is a throat, determine its area.)

3.12. Air at a pressure of 680 kPa and a temperature of 833 K enters a converging-diverging nozzle through a line of 4.6×10^{-3} m^2 area and expands to a delivery-region pressure of 33 kPa. Assuming isentropic expansion and a mass rate of flow of 1 kg/s, find:

(a) The stagnation enthalpy.

(b) The temperature and enthalpy at discharge.

(c) The Mach number and velocity of the air stream at discharge.

(d) The maximum mass flow rate per unit area.

3.13. Air flows isentropically at the rate of 1 kg/s through a duct. At one section of the duct the cross-sectional area is 9.3×10^{-3} m^2, static pressure is 200 kPa, and stagnation temperature is 550 K. Determine the velocity of the stream and the minimum area at the exit of the duct that causes no reduction in the mass rate of flow.

3.14. Air flows isentropically through a converging nozzle. At the inlet of the nozzle the pressure $p_1 = 340$ kPa, the temperature T_1 is 550 K, the velocity V_1 is 200 m/s, and the cross-sectional area A_1 is 9.3×10^{-3} m^2. Consider air to be an ideal gas with $\gamma = 1.4$ and find:

(a) The stagnation temperature and pressure.

(b) The sonic velocity and the Mach number at the inlet.

(c) The area, pressure, temperature, and velocity at the exit if $M = 1$ at exit.

(d) Draw graphs of G, M, V, and v versus pressure, indicating the values at the inlet and exit of the nozzle.

3.15. Superheated steam expands isentropically in a convergent-divergent nozzle from an initial state in which the pressure is 2.0 MPa and the superheat is 378 K to a pressure of 680 kPa. The rate of flow is 0.5 kg/s.

(a) Find the velocity of the steam and the cross-sectional area of the nozzle at the sections where the pressures are 1.0 MPa and 1.2 MPa.

(b) Determine the pressure, velocity, and cross-sectional area at the throat.

(c) Determine the velocity and cross-sectional area at discharge.

Assume $\dfrac{p^*}{p_0} = 0.55$.

3.16. A convergent nozzle receives steam at a pressure of 3.4 MPa and a temperature of 640 K with negligible velocity. The nozzle discharges into a chamber at which the pressure is maintained at 1.36 MPa. If the throat area of the nozzle is 2.3×10^{-4} m^2 and the discharge chamber area is 0.056 m^2, find

(a) The velocity at the throat.

(b) The mass rate of flow.

Assume $\dfrac{p^*}{p_0} = 0.55$ and the flow is isentropic.

3.17. Air flows isentropically through the convergent-divergent nozzle shown in Fig. 3.24. The inlet pressure is 80 kPa, the inlet temperature 295 K, and the back

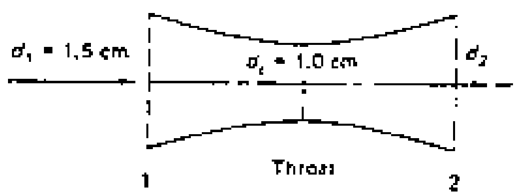


FIGURE 3.24

pressure 1.013 kPa. What should be the exit diameter of the nozzle which corresponds to the maximum obtainable value of Mach number at the exit? What are the mass rate of flow, the exit Mach number, and the exit temperature?

3.18. A rocket motor is fitted with a convergent-divergent nozzle having a throat diameter 2.5 cm. If the chamber pressure is 1 MPa and the chamber temperature is 2200 K, determine:

(a) The mass flow rate through the nozzle.

(b) The Mach number at the exit ($p_{back} = 101.3$ kPa).

(c) The thrust developed at sea level.

Assume that the products of combustion behave like a perfect gas ($\gamma = 1.4$, $R = 540$ J/kg K) and the expansion through the nozzle is isentropic.

3.19. Air is flowing through a section of a straight convergent nozzle. At the entrance to the nozzle section the area is 4×10^{-3} m^2, the velocity is 100 m/s, the air pressure is 680 kPa, and the air temperature is 365 K. At the exit of the section the area is 2×10^{-3} m^2. Assume reversible adiabatic flow. Calculate the magnitude and direction of the force exerted by the fluid upon the given nozzle section.

Chapter Four

Normal Shock Waves

Introduction:

The shock process represent an abrupt change in fluid properties, in which finite variation in pressure temperature and density occur over a shock thickness comparable to the mean free path of the gas molecules. It has been established that supersonic flow adjust to the pressure of a body by mean of such shock wave, whereas subsonic flow can adjust by gradual change in flow properties. Shock may also occur in the flow through nozzle or duct and have a decisive effect on these flow.

How Shock Wave Take Place:

Consider a piston in a tube and its given a steady velocity to the right of magnitude dv. A sound wave travels a head of the piston through the medium in the tube. Suppose the piston is now given a second increment of velocity dv, casing a second wave to move into the compressed gas behind the first wave. The location of the wave and the pressure distribution in the tube after a time t are shown in figure. Each wave travel at the velocity of sound with respect to the gas into which its moving, since the second wave is moving into a gas that is already moving to the right with velocity dv. The second wave is moving into a compressed gas having a slightly elevated temperature, therefore the second wave travel with a greater absolute velocity than the first wave and gradually over take it. A series of this induced wave after its over take each other will produce a shock wave or a sudden change in pressure and other properties.

$a \propto \sqrt{T}$ $\quad$ $T3>T2>T1$ $\quad$ therefore $\quad$ $a3>a2>a1$

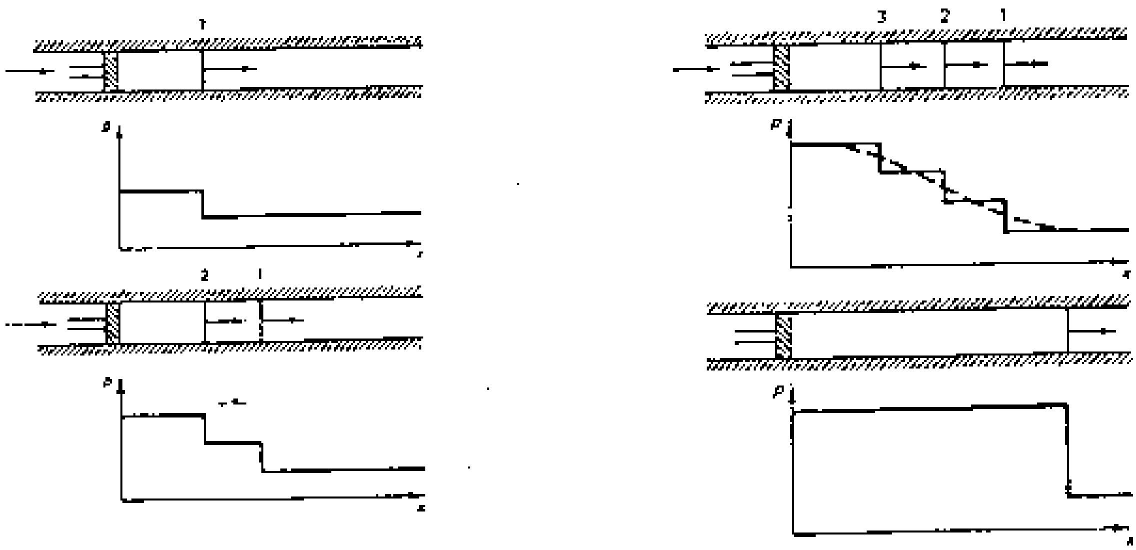


Fig shows one and two, three and the over take of the sound wave propagate a head of the piston

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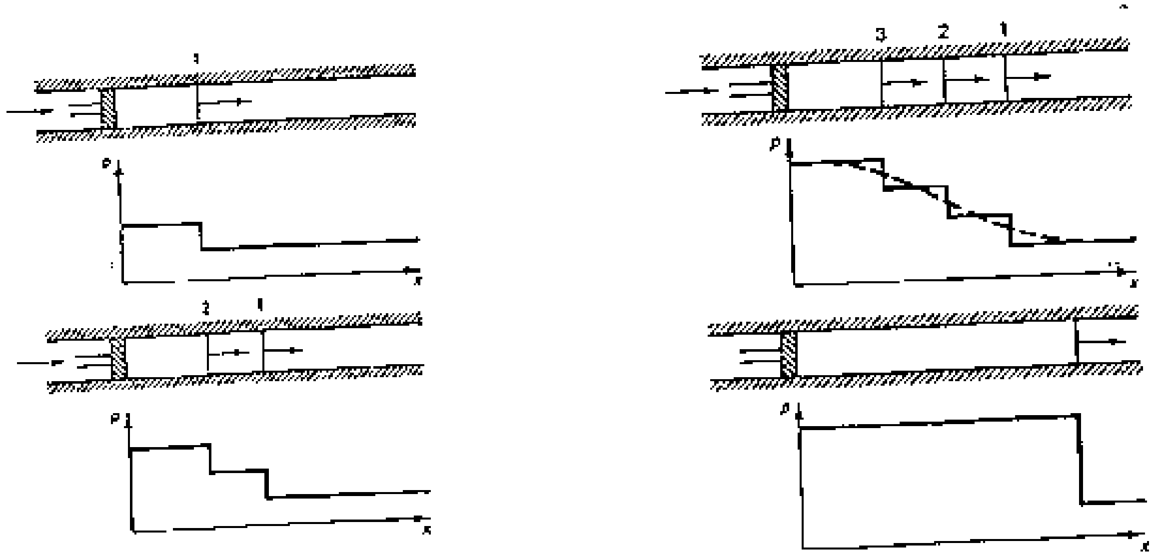

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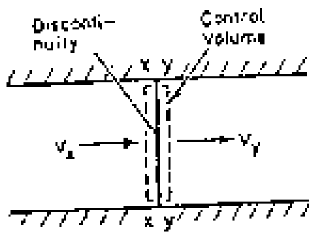
Stationary Normal Shock Waves:

In order to analyze the flow through a stationary normal shock wave, consider a control volume of the form shown. This control volume has cross sectional area S normal to the flow direction. The shock wave relations are obtained by applying the laws of conservation of mass, momentum and energy to the control volume for steady state flow. We will refer to the properties of the flow upstream of the shock by subscript "x" and downstream by "y".



$m^{\cdot} = \rho_x V_x A_x = \rho_y V_y A_y$

The shock wave thickness is very small therefore $A_x = A_y$.

$$\rho_x V_x = \rho_y V_y \quad \text{------------- } 4.1$$

For perfect gas

$$\frac{P_x}{RT_x} M_x \sqrt{\gamma R T_x} = \frac{P_y}{RT_y} M_y \sqrt{\gamma R T_y} \quad \text{-------}4.2$$

Since the only force acting on the control volume in the flow direction are the pressure force, conservation of momentum is.

$$P_x A_x - P_y A_y = m^{\cdot}(V_y - V_x)$$

Combine of equation 4.1 into the above equation, where $m^{\cdot} = \rho_x V_x A_x = \rho_y V_y A_y$

$$P_x + \rho_x V_x^2 = P_y + \rho_y V_y^2 \quad \text{-------}4.3$$

For perfect gas $P = \rho . R . T$

$$P_x + \rho_x V_x^2 = P_x(1 + \gamma M_x^2)$$

$$P_y + \rho_y V_y^2 = P_y(1 + \gamma M_y^2)$$

$$P_x(1 + \gamma M_x^2) = P_y(1 + \gamma M_y^2) \quad \text{------------}4.4$$

The flow through the control volume is adiabatic and the energy equation become.

$$c_p T_x + \frac{V_x^2}{2} = c_p T_y + \frac{V_y^2}{2} = c_p + T_o$$ For adiabatic flow the stagnation temperature does nit change across the shock wave this mean that $T_{ox} = T_{oy}$.

$$T_x\left(1 + \frac{\gamma + 1}{2} M_x^2\right) = T_y\left(1 + \frac{\gamma + 1}{2} M_y^2\right) \quad \text{---------------}4.5$$

Substitute energy equation 4.5 and momentum equation 4.4 into the continuity equation 4.2.

$$\frac{M_x}{1 + \gamma M_x^2} \sqrt{1 + \frac{\gamma + 1}{2} M_x^2} = \frac{M_y}{1 + \gamma M_y^2} \sqrt{1 + \frac{\gamma + 1}{2} M_y^2} \quad \text{----------------}4.6$$

By inspection its evident that one solution 4.6 is the trivial one, $M_x = M_y$. This solution involving no change in properties in constant area flow corresponding to isentropic flow and that is not of interest for the irreversible of normal shock. Equation 4.6 can be solve to yield M_y in term of M_x.

$$M_y^2 = \frac{M_x^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma}{\gamma - 1} M_x^2 - 1} \quad \text{-------------------4.7}$$

Now to find the pressure ratio after and before the shock, substitute equation 4.7 into equation 4.4 .

$$\frac{P_y}{P_x} = \frac{2\gamma M_x^2 - (\gamma - 1)}{\gamma + 1} \quad \text{-----------------4.8}$$

also to find the temperature ratio after and before the shock, one may substitute equation 4.7 into equation 4.5

$$\frac{T_y}{T_x} = \frac{\left[2\gamma M_x^2 - (\gamma - 1)\right]\left[2 + (\gamma - 1)M_x^2\right]}{(\gamma + 1)M_x^2} \quad \text{---------------4,9}$$

and if we substitute equation 4.7 into equation 4.1 we can find the density and the velocity ratio.

$$\frac{\rho_y}{\rho_x} = \frac{V_x}{V_y} = \frac{(\gamma + 1)M_x^2}{2 + (\gamma - 1)M_x^2} \quad \text{-------------------4.10}$$

The ratio of stagnation pressure is a measure of the irreversibility in the shock process. It may be found by observing that:

$$\frac{P_{oy}}{P_{ox}} = \frac{P_{oy}}{P_y}\frac{P_y}{P_x}\frac{P_x}{P_{ox}}$$

Now P_y/P_x is given by Eq. 4.8, and P_{oy}/P_y and P_x/P_{ox} may be found from Eq.3.5. Using Eq. 4.7 for the value of M_y we get after algebraic simplification.

$$\frac{P_{oy}}{P_{ox}} = \left[\frac{2\gamma}{\gamma + 1} M_x^2 - \frac{\gamma - 1}{\gamma + 1}\right]^{-\frac{1}{\gamma - 1}} \left[\frac{(\gamma + 1)M_x^2}{2 + (\gamma + 1)M_x^2}\right]^{\frac{\gamma}{\gamma - 1}} \quad \text{---------------------4.11}$$

To evaluate the entropy change across the shock, we employ the perfect gas formula,

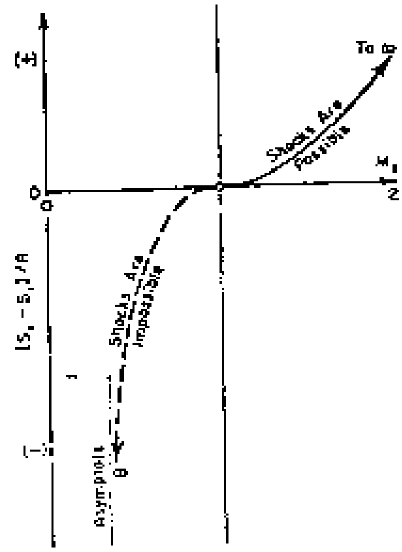
$$S_y - S_x = c_p \ln\frac{T_y}{T_x} - R\ln\frac{P_y}{P_x} \quad \text{---------------- 4.12}$$

substitute Eq. 4.8 and 4.9 into Eq. 4.12 then.

$$\frac{S_y - S_x}{R} = \frac{1}{\gamma - 1}\ln\left[\frac{2\gamma}{\gamma + 1} M_x^2 - \frac{\gamma - 1}{\gamma + 1}\right] + \frac{\gamma}{\gamma - 1}\ln\left[\frac{(\gamma + 1)M_x^2}{2 + (\gamma + 1)M_x^2}\right] \quad \text{-----4.12}$$

Impossibility of a Rerefaction Shock.

Carful study of Eq.4.12 indicate that for gases with $1<\gamma<1.67$ the entropy change is always positive when Mx is greater than unity, and is always negative when Mx is less than unity. The general form of Eq.4.12 is shown in Fig. It is proven rigorously that for perfect gas only the shock from supersonic to subsonic is possible. Since the shock process is adiabatic and according to second law of thermodynamic the entropy change must be positive.



Comparing Eq. 4.12 for entropy change and Eq.4.11 for stagnation pressure ration, one can conclude the following correlation:

$$\frac{S_y - S_x}{R} = -\ln\frac{P_{oy}}{P_{ox}} \quad \text{------------4.13}$$

According to the second law of thermodynamic the rate of change of entropy is positive $ds>0$, and referring to Eq.4.13 this mean that P_{oy} is less than P_{ox}.

The shock wave take place in-order to keep the flow continuation this mean that the flow is steady and the mass flow does not change across the shock.

$$m^*_x = m^*_y$$

we have seen from the previous chapter that the maximum mass flow rate can be achieved at the choked condition and the mass flow rate in term of stagnation properties and the critical area is,

$$\frac{P_{ox}A_x^* constant}{\sqrt{T_{ox}}} = \frac{P_{oy}A_y^* constant}{\sqrt{T_{oy}}}$$

the flow through the shock is adiabatic therefore $T_{ox}=T_{oy}$

$P_{ox}A^*_x=P_{oy}A^*_y$ or $\frac{P_{oy}}{P_{ox}} = \frac{A_x^*}{A_y^*}$ since $P_{oy}< P_{ox}$ this mean that $A^*_y>A^*_x$

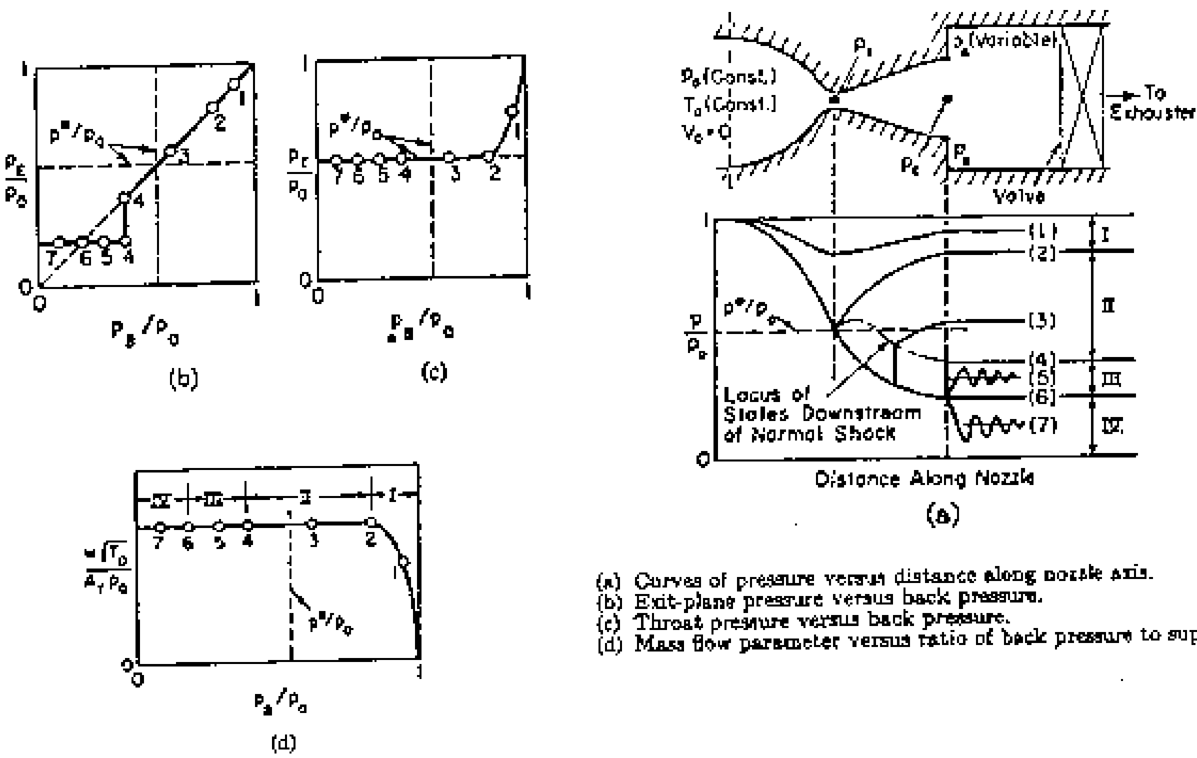
Normal Shock Table:

Table is available which list the ratio of the various flow variable such as pressure, temperature , and density across the normal shock wave and the downstream Mach Number as a function of the upstream Mach Number.

M_x M_y P_y/P_x T_y/T_x ρ_y/ρ_x P_{oy}/P_{ox} or A^*_x/A^*_y

Convergent-Divergent Nozzle:
We return now to the problem of the operating characteristics of converging-diverging nozzle under pressure ratio, discussed previously in chapter two. *Fig.* show the characteristic performance of convergent divergent nozzle with various back pressure to the supply pressure.

Four different regimes are possible . In regime I the flow is entirely subsonic, and the passage behave like a conventional venture tube. The flow rate is sensitive to change in back pressure. At condition *2*, which forms the dividing line between *I* and *II*, the Mach Number at the throat is unity. As regime *II* is entered, a normal shock appears down stream of the throat, and the process aft of the shock comprises subsonic deceleration. As the back pressure is lowered, the shock move down the nozzle until, at condition *4* it appears in the exit plane of the nozzle. In regime *II*, as in regime *I*, the exit plane pressure P_E is virtually identical with the back pressure P_B. On the other hand, the flow rate in regime *II* is constant and is unaffected by the back pressure. This is in accord



(a) Curves of pressure versus distance along nozzle axis.
(b) Exit-plane pressure versus back pressure.
(c) Throat pressure versus back pressure.
(d) Mass flow parameter versus ratio of back pressure to supply

with the fact that throughout regime *II* all stream properties at the throat section are constant.

In regime *III*. As for condition *5*, the flow within the entire nozzle is supersonic, and the pressure in the exit plane is lower than the back pressure. The compression which subsequently occurs outside the nozzle involve oblique shock wave which cannot be treated on one-dimensional grounds. Condition *6* is termed the design condition for the nozzle under supersonic condition, since the exit-plane pressure is then identical with the back pressure. A reduction in the back pressure below that corresponding to condition 6 has no effect whatsoever on the flow pattern within the nozzle. In regime *IV* the expansion from the exit-plane pressure to the back pressure occurs outside the nozzle in the form of oblique expansion waves which also cannot be studied by one-dimensional analysis.

In both regimes *III* and *IV* the flow pattern within the nozzle is independent of back pressure, and corresponds to the flow pattern for the design condition. Adjustment to the back pressure are made outside the nozzle.

For subsonic flow, there are an infinite number of possible pressure distance curves. For the supersonic region of flow, however, the pressure-distance curve is unique. To put it differently, in subsonic flow the pressure ratio does not depend solely on the area ratio; in supersonic flow the pressure ratio does depend solely on the area ratio.

Only over a narrow range of back pressure ratio, namely, the range covered by regime *I*. does the flow rate depend on the back pressure. For regime *II, III, IV*, the flow rate is independent of the back pressure, since $M=1$ at the throat, may be computed form choked flow equation.

Converging- Diverging Supersonic Diffuser.

A diffuser is a device that cause the static pressure of a gas to rise while the gas is decelerating. When deceleration is isentropic, the maximum pressure that can be attained is the isentropic stagnation pressure. Diffusers are either subsonic or supersonic, depending on the Mach Number of the approaching stream. In a subsonic diffuser the cross-sectional area increases in the direction of flow, while in a supersonic diffuser the cross sectional area first decrease and then increases.

A supersonic diffuser is located at the inlet to such air-breathing engines as the supersonic turbojet and the ramjet,. The high velocity air is decelerated by the diffuser before it is compressed in the axial flow compressor of the turbojet or before it undergoes combustion in the ramjet. An ideal supersonic diffuser consists of a convergent-divergent passageway in which the flow is shock-free and isentropic. Deceleration of the flow to M=1 at the throat is followed by a further deceleration to subsonic speed downstream of the throat. In real application, however, starting transients and off-design interfere in establishing the desired flow pattern. The maximum pressure that can be achieved in the diffuser is the isentropic stagnation pressure. Any loss in available energy (or stagnation pressure) in the diffuser will have a harmful effect on the operation of the engine as a whole. For a supersonic diffuser it would be highly desirable to provide shock free isentropic flow.

For any configuration of the converging-diverging diffuser, there are two values of Mach number in which the flow is isentropically compressed, this will called subsonic design Mach number(M_{Dsub})and supersonic design Mach number(M_{Dsup}).The following cases will show how the flow is established from the starting-up to the design flying Mach number.

1- When the flying Mach Number is below M_{Dsub} value , this mean that the actual throat area is grater than the critical area, therefore the flow at the throat is subsonic and the flow is continue to compressed at the divergent part as show in fig.a .
2- When the flying Mach number reach the M_{Dsub} value, this mean that the actual throat area is equal to the critical area of the flying Mach number, therefore the flow at the throat is sonic M=1 and the flow is continue to compressed at the divergent part and the exit Mach number will be subsonic fig.b.

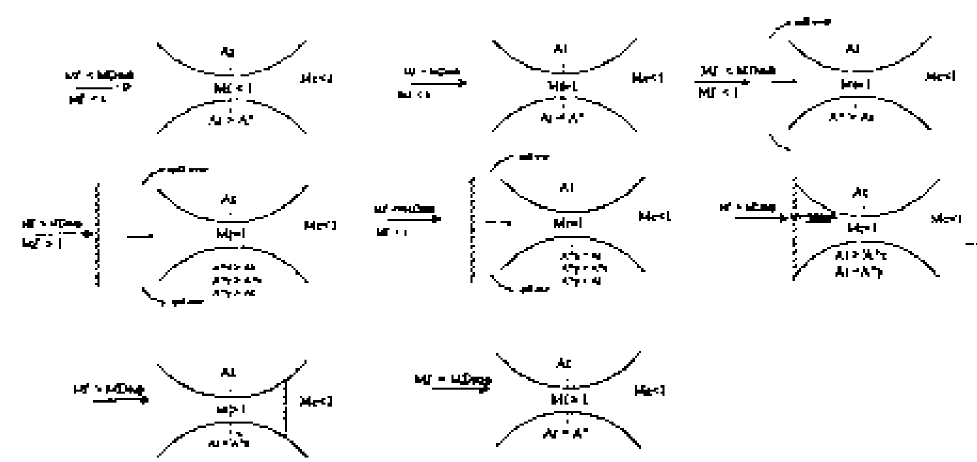
3- When the flying Mach number is grater than M_{Dsub} value, this mean that the actual throat area is less than the critical area this mean that the throat area is too small to accommodate the flow. The pressure is instantaneously increased at the throat area and part of the incoming flow is divert or spill over the inlet cowl of the diffuser as show in fig.c This mean that as the flying Mach number increase the different between the throat area and the required area increase and hence mass spill over is increase.

4- When the flying Mach number is grater than one but is less than the M_{Dsup} , in this case the throat area is less than the critical area or the required area to accommodate the flow. Therefore the instantaneously pressure built up at the throat area. A curved or normal shock is appears in the front of the diffuser inlet. The subsonic flow downstream of the shock is partially spilled over the diffuser inlet, reducing the mass flow through the inlet. this will lower the combustion pressure and a loss in thrust.

5- When the flying Mach number is equal to the M_{Dsup} value, in this case the existing of the shock wave will caused of stagnation pressure loss. The critical area behind the existing shock is increased and this mean that the critical area upstream of the shock is equal to the throat area but the area down stream of the shock is still grater than the throat area. Therefore the normal shock is still existing and the flow spill over is continue as show in fig. d.

6- To over come the existing shock the engine have to speed over the design supersonic Mach number until the shock located at the diffuser inlet. At this case the Mach number down stream of the shock wave is equal to the M_{Dsub} so that the mach number at the throat is equal to sonic. A little increase in speed will make the shock wave to swallowed and stand at the divergent part of the diffuser as show in fig. e.

7- To return back to the design condition the engine have to slow down to the design supersonic flying Mach number, in this case the shock wave is drawn back toward the throat and it strength will reduce gradually until it vanished at the throat when the flying Mach number is equal to the M_{Dsup} as show in fig.f



PROBLEMS

4.1. Air with initial stagnation conditions of 700 kPa and 330 K passes through a convergent-divergent nozzle at the rate of 1 kg/s. At the exit area of the nozzle the stagnation pressure is 550 kPa and the stream pressure is 500 kPa. The nozzle is insulated and there is no irreversibility except for the occurrence of a shock.

(a) What is the nozzle throat area?
(b) What is the Mach number before and after the shock?
(c) What is the nozzle area at the point of shock and at the exit?
(d) What is the stream density at the exit?

4.2. A perfect gas ($\gamma = 1.4$) enters a converging-diverging nozzle with a Mach number of 0.50 and local pressure and temperature values of 280 kPa and 280 K, respectively. The nozzle throat area is 6.5×10^{-4} m^2 and the nozzle exit area is 26×10^{-4} m^2. The nozzle exit pressure is 170 kPa.

(a) What are the values of the Mach number and the stream temperature at the exit?
(b) At what area does the shock occur?

Show your method of solution on a skeleton flow chart.

4.3. An air nozzle has an exit area 1.6 times the throat area. If a normal shock occurs at a plane where the area is 1.2 times the throat area, find the pressure, temperature, and Mach number at the exit. The stagnation temperature and pressure before the shock are 310 K and 700 kPa.

4.4. Air enters a supersonic nozzle with inlet conditions $A_1 = 6.5 \times 10^{-4}$ m^2, $M_1 = 1.8$, $p_1 = 35$ kPa, and $T_1 = 260$ K. A normal shock occurs in the nozzle resulting in an increase in entropy of $\Delta s = 113$ J/kg K. If the Mach number at the exit $M_2 = 0.3$, find:

(a) The area of the normal shock A_{xy}.
(b) The Mach numbers before and after the shock M_x, M_y.
(c) The pressure at the exit p_2.
(d) The mass rate of flow per unit area at exit.
(e) Show the process on a schematic flow chart and a Fanno-Rayleigh plot.

Assume isentropic flow except for the normal shock.

4.5. An impact (stagnation) tube in an air stream reads 186 kPa. If the local temperature is 293 K and the local Mach number is 0.8, determine:

(a) The local pressure.
(b) The mass rate of flow per unit area.

4.6. A Pitot tube and a thermocouple give the following measurements pertaining to air flow in a duct:

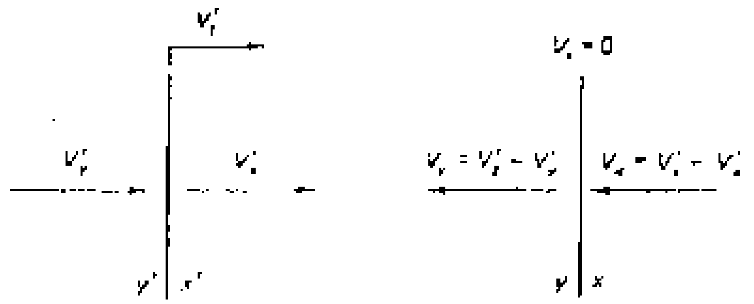
$$p_0 = 180 \text{ kPa}, \qquad p = 157 \text{ kPa}, \qquad T_0 = 1250 \text{ K}$$

Moving Shock Wave:

Previous section have dealt with the fixed normal shock wave. However, many physical situation arise in which a normal shock is moving. When an explosive occurs, a shock propagates through the atmosphere from the point of the explosion. As a blunt body re-enters the atmosphere from space, a shock travels a short distance a head of the body. When a valve in a gas line is suddenly closed, a shock propagates back through the gas. To treat these cases, it is necessary to extend the procedures already develop for the fiqed normal shock wave.

Consider a normal shock moving at constant velocity into still air as show in fig. Let Vs= absolute shock velocity and Vg= velocity of the gases behind the wave, both velocities are measured with respect to a fixed observer. For a fixed observer, the flow is not steady, since condition at a point are dependent on whether or not the shock has passed over that point.

Now consider the same physical situation with an observer moving at the shock-wave velocity, a situation, for instant, with the observer "sitting on the shock wave". The shock is now fixed with respect to the observer as shown in fig. But this the same case already covered in the normal shock section. Relation have been derived and result tabulated for the fixed normal shock. To apply these result to the moving shock, consideration must be given to the effect of observer velocity on static and stagnation properties.



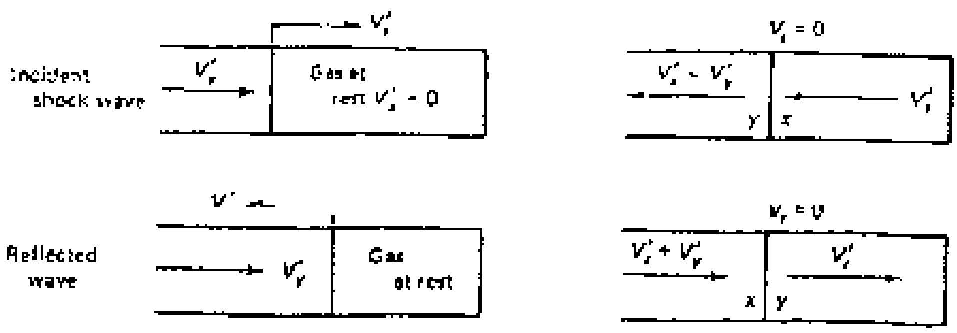
(a) Moving wave (b) Stationary wave

Since static properties are independent of the observer velocity, the transformation of the coordinate system has no effect on static properties. Stagnation properties on the other hand depend on the observer velocity and consequently are affected by the choice of the coordinate system. Table 4.1 show properties in a fixed coordinate system and I a moving coordinate system.

TABLE 4.1

Static properties:	
$p_x = p'_x$	$p_y = p'_y$
$T_x = T'_x$	$T_y = T'_y$
$c_x = c'_x$	$c_y = c'_y$
Mach numbers:	
$M_x = \dfrac{V_x}{c_x} = \dfrac{V'_s - V'_x}{c_x}$	$M_y = \dfrac{V_y}{c_y} = \dfrac{V'_s - V'_y}{c_y}$
$M'_x = \dfrac{V'_x}{c'_x} = \dfrac{V'_s - V_x}{c_x}$	$M'_y = \dfrac{V'_y}{c'_y} = \dfrac{V'_s - V_y}{c_y}$
Stagnation properties:	
$T_{0x} = T_x\left(1 + \dfrac{\gamma - 1}{2} M_x^2\right)$	$T'_{0x} = T_x\left(1 + \dfrac{\gamma - 1}{2} M_x'^2\right)$
$T_{0y} = T_y\left(1 + \dfrac{\gamma - 1}{2} M_y^2\right)$	$T'_{0y} = T_y\left(1 + \dfrac{\gamma - 1}{2} M_y'^2\right)$
$p_{0x} = p_x\left(1 + \dfrac{\gamma - 1}{2} M_x^2\right)^{\gamma/(\gamma-1)}$	$p'_{0x} = p_x\left(1 + \dfrac{\gamma - 1}{2} M_x'^2\right)^{\gamma/(\gamma-1)}$
$p_{0y} = p_y\left(1 + \dfrac{\gamma - 1}{2} M_y^2\right)^{\gamma/(\gamma-1)}$	$p'_{0y} = p_y\left(1 + \dfrac{\gamma - 1}{2} M_y'^2\right)^{\gamma/(\gamma-1)}$

When a normal shock wave travels in a closed-end, the gas between the shock wave and the closed end remains at rest. The gas behined the shock, however, moves at a velocity V_y' as shown in fig. The incident shock is reflec)ed at the closed end of the tube and propagates back through the incoming gas. For an observer moving with the wave the velocity appear as shown in fig. Since the gas velocity decres across the reflected wave, the incident shock wave is reflected at the end of the tube as a shock wave.



(a) Velocities relative to a fixed coordinate system

(b) Velocities relative to a system moving with the wave

Chapter 9

Fanno Flow

9.1 INTRODUCTION

At the start of Chapter 1 we mentioned that area changes, friction, and heat transfer are the most important factors affecting the properties in a flow system. Up to this point we have considered only one of these factors, that of variations in area. However, we have also discussed the various mechanisms by which a flow adjusts to meet imposed boundary conditions of either flow direction or pressure equalization. We now wish to take a look at the subject of friction losses.

To study only the effects of friction, we analyze flow in a constant-area duct without heat transfer. This corresponds to many practical flow situations that involve reasonably short ducts. We consider first the flow of an arbitrary fluid and discover that its behavior follows a definite pattern which is dependent on whether the flow is in the subsonic or supersonic regime. Working equations are developed for the case of a perfect gas, and the introduction of a reference point allows a table to be constructed. As before, the table permits rapid solutions to many problems of this type, which are called *Fanno flow*.

9.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. List the assumptions made in the analysis of Fanno flow.
2. *(Optional)* Simplify the general equations of continuity, energy, and momentum to obtain basic relations valid for any fluid in Fanno flow.
3. Sketch a Fanno line in the h–v and the h–s planes. Identify the sonic point and regions of subsonic and supersonic flow.
4. Describe the variation of static and stagnation pressure, static and stagnation temperature, static density, and velocity as flow progresses along a Fanno line. Do for both subsonic and supersonic flow.

5. (*Optional*) Starting with basic principles of continuity, energy, and momentum, derive expressions for property ratios such as T_2/T_1, p_2/p_1, and so on, in terms of Mach number (M) and specific heat ratio (γ) for Fanno flow with a perfect gas.
6. Describe (include T–s diagram) how the Fanno table is developed with the use of a * reference location.
7. Define *friction factor, equivalent diameter, absolute* and *relative roughness, absolute* and *kinematic viscosity*, and *Reynolds number*, and know how to determine each.
8. Compare similarities and differences between Fanno flow and normal shocks. Sketch an h–s diagram showing a typical Fanno line together with a normal shock for the same mass velocity.
9. Explain what is meant by *friction choking*.
10. (*Optional*) Describe some possible consequences of adding duct in a choked Fanno flow situation (for both subsonic and supersonic flow).
11. Demonstrate the ability to solve typical Fanno flow problems by use of the appropriate tables and equations.

9.3 ANALYSIS FOR A GENERAL FLUID

We first consider the general behavior of an arbitrary fluid. To isolate the effects of friction, we make the following assumptions:

Steady one-dimensional flow	
Adiabatic	$\delta q = 0, ds_e = 0$
No shaft work	$\delta w_s = 0$
Neglect potential	$dz = 0$
Constant area	$dA = 0$

We proceed by applying the basic concepts of continuity, energy, and momentum.

Continuity

$$\dot{m} = \rho A V = \text{const}$$

but since the flow area is constant, this reduces to

$$\rho V = \text{const} \tag{9.1}$$

We assign a new symbol G to this constant (the quantity ρV), which is referred to as the *mass velocity*, and thus

$$\boxed{\rho V = G = \text{const}} \tag{9.2}$$

What are the typical units of G?

Energy

We start with

$$h_{t1} + q = h_{t2} + w_s$$

For adiabatic and no work, this becomes

$$h_{t1} = h_{t2} = h_t = \text{const} \tag{9.3}$$

If we neglect the potential term, this means that

$$h_t = h + \frac{V^2}{2g_c} = \text{const} \tag{9.4}$$

Substitute for the velocity from equation (9.2) and show that

$$\boxed{h_t = h + \frac{G^2}{\rho^2 2g_c} = \text{const}} \tag{9.5}$$

Now for any given flow, the constant h_t and G are known. Thus equation (9.5) establishes a unique relationship between h and ρ. Figure 9.1 is a plot of this equation in the h–v plane for various values of G (but all for the same h_t). Each curve is called a *Fanno line* and represents flow at a particular *mass velocity*. Note carefully that this is constant G and not constant $\dot{m}$. Ducts of various sizes could pass the same mass flow rate but would have different mass velocities.

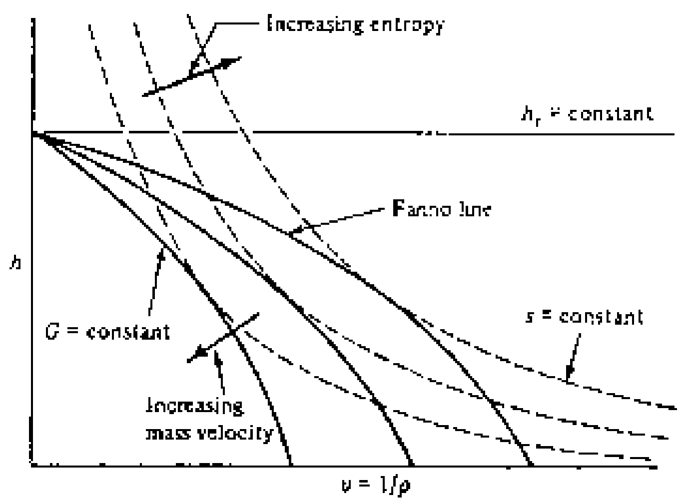


Figure 9.1 Fanno lines in h–v plane.

Once the fluid is known, one can also plot lines of constant entropy on the h–v diagram. Typical curves of s = constant are shown as dashed lines in the figure. It is much more instructive to plot these Fanno lines in the familiar h–s plane. Such a diagram is shown in Figure 9.2. At this point, a significant fact becomes quite clear. Since we have assumed that there is no heat transfer ($ds_e = 0$), the *only* way that entropy can be generated is through irreversibilities (ds_i). Thus *the flow can only progress toward increasing values of entropy*! Why? Can you locate the points of maximum entropy for each Fanno line in Figure 9.1?

Let us examine one Fanno line in greater detail. Figure 9.3 shows a given Fanno line together with typical pressure lines. All points on this line represent states with the same mass flow rate per unit area (mass velocity) and the same stagnation enthalpy. Due to the irreversible nature of the frictional effects, the flow can only proceed to the right. Thus the Fanno line is divided into two distinct parts, an upper and a lower branch, which are separated by a limiting point of maximum entropy.

What does intuition tell us about adiabatic flow in a constant-area duct? We normally feel that frictional effects will show up as an internal generation of "heat" with a corresponding reduction in density of the fluid. To pass the same flow rate (with constant area), continuity then forces the velocity to increase. This increase in kinetic energy must cause a decrease in enthalpy, since the stagnation enthalpy remains constant. As can be seen in Figure 9.3, this agrees with flow along the *upper branch* of the Fanno line. It is also clear that in this case both the static and stagnation pressure are decreasing.

But what about flow along the *lower branch*? Mark two points on the lower branch and draw an arrow to indicate proper movement along the Fanno line. What is happening to the enthalpy? To the density [see equation (9.5)]? To the velocity [see equation (9.2)]? From the figure, what is happening to the static pressure? The stagnation pressure? Fill in Table 9.1 with *increase*, *decrease*, or *remains constant*.

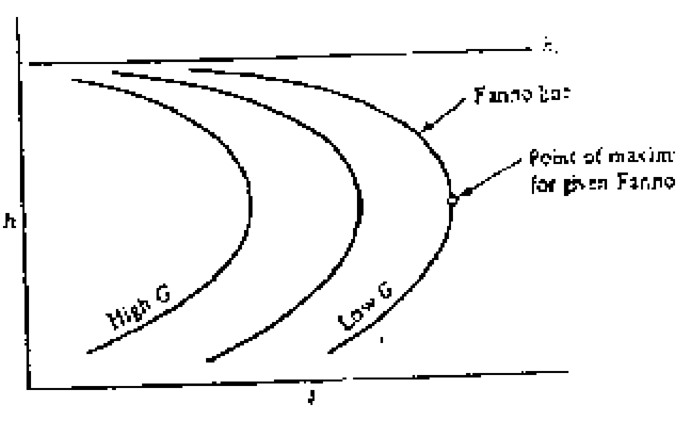


Figure 9.2 Fanno lines in h–s plane.

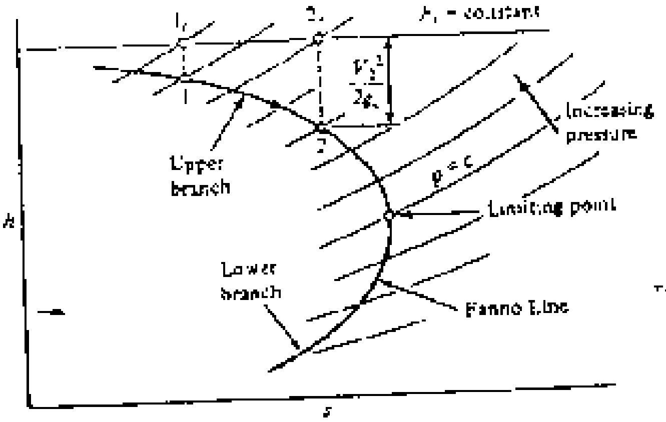


Figure 9.3 Two branches of a Fanno line.

Table 9.1 Analysis of Fanno Flow for Figure 9.3

Property	Upper Branch	Lower Branch
Enthalpy		
Density		
Velocity		
Pressure (static)		
Pressure (stagnation)		

Notice that on the lower branch, properties do not vary in the manner predicted by *intuition*. Thus this must be a flow regime with which we are not very familiar. Before we investigate the limiting point that separates these two flow regimes, let us note that these flows do have one thing in common. Recall the stagnation pressure energy equation

STAGNATION PRESSURE–ENERGY EQUATION

Consider the two section locations on the physical system shown in Figure. If we let the distance between these locations approach zero, we are dealing with an infinitesimal control volume with the thermodynamic states differentially separated, as shown in Figure below. Also shown are the corresponding stagnation states for these two locations.

We may write the following property relation between points 1 and 2:

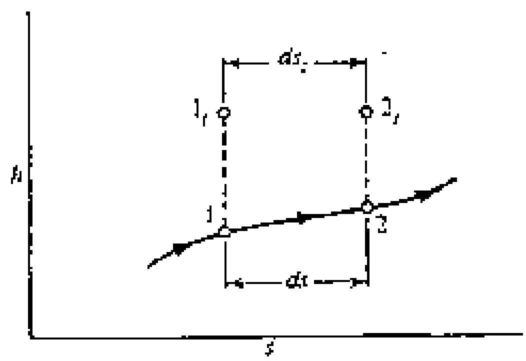


Figure Infinitesimally separated static states with associated stagnation states.

$$T\,ds = dh - v\,dp \tag{A.1}$$

Note that even though the stagnation states do not actually exist, they represent legitimate thermodynamic states, and thus any valid property relation or equation may be applied to these points. Thus we may also apply equation (A.1) between states 1_t and 2_t:

$$T_t\,ds_t = dh_t - v_t\,dp_t \tag{A.2}$$

However,

$$ds_t = ds \tag{A.3}$$

and

$$ds = ds_e + ds_i \tag{A.4}$$

Thus we may write

$$T_t(ds_e + ds_i) = dh_t - v_t\,dp_t \tag{A.5}$$

Recall the energy equation written in the form

$$\delta q = \delta w_s + dh_t \tag{A.6}$$

By substituting dh_t from equation (A.5) into (A.6), we obtain

$$\delta q = \delta w_s + T_t(ds_e + ds_i) + v_t\,dp_t \tag{A.7}$$

Now also recall that

$$\delta q = T\,ds_e \tag{A.8}$$

Substitute equation (A.8) into (A.7) and note that $v_t = 1/\rho_t$ and you should obtain the following equation, called the *stagnation pressure–energy equation*:

$$\boxed{\frac{dp_t}{\rho_t} + ds_e(T_t - T) + T_t\,ds_i + \delta w_s = 0} \tag{A.9}$$

For Fanno flow, $ds_e = \delta w_s = 0$.

Thus any frictional effect must cause a decrease in the total or stagnation pressure! Figure 9.3 verifies this for flow along both the upper and lower branches of the Fanno line.

Limiting Point

From the energy equation we had developed,

$$h_t = h + \frac{V^2}{2g_c} = \text{constant} \tag{9.4}$$

Differentiating, we obtain

$$dh_t = dh + \frac{V\,dV}{g_c} = 0 \tag{9.6}$$

From continuity we had found that

$$\rho V = G = \text{constant} \tag{9.2}$$

Differentiating this, we obtain

$$\rho\,dV + V\,d\rho = 0 \tag{9.7}$$

which can be solved for

$$dV = -V\frac{d\rho}{\rho} \tag{9.8}$$

Introduce equation (9.8) into (9.6) and show that

$$dh = \frac{V^2\,d\rho}{g_c\,\rho} \tag{9.9}$$

Now recall the property relation

$$T\,ds = dh - v\,dp$$

which can be written as

$$T\,ds = dh - \frac{dp}{\rho} \tag{9.10}$$

Substituting for dh from equation (9.9) yields

$$\boxed{T\,ds = \frac{V^2\,d\rho}{g_c\,\rho} - \frac{dp}{\rho}} \tag{9.11}$$

We hasten to point out that this expression is valid for *any* fluid and between two differentially separated points *anyplace* along the Fanno line. Now let's apply equation (9.11) to two adjacent points that surround the limiting point of maximum entropy. At this location s = const; thus $ds = 0$, and (9.11) becomes

$$\frac{V^2\,d\rho}{g_c} = dp \qquad \text{at limit point} \tag{9.12}$$

or

$$V^2 = g\left(\frac{dp}{d\rho}\right)_{\text{at limit point}} = g_c\left(\frac{\partial p}{\partial \rho}\right)_{s=\text{const}} \tag{9.13}$$

This should be a familiar expression ($dp/d\rho = \sqrt{\gamma RT}$) and we recognize that *the velocity is sonic at the limiting point.* The upper branch can now be more significantly called the *subsonic branch*, and the lower branch is seen to be the *supersonic branch*.

Now we begin to see a reason for the failure of our intuition to predict behavior on the lower branch of the Fanno line. From our privous studies, it shows that fluid behavior in supersonic flow is frequently contrary to our expectations. This points out the fact that we live most of our lives "subsonically," and, in fact, our knowledge of fluid phenomena comes mainly from experiences with incompressible fluids. It should be apparent that we cannot use our intuition to guess at what might be happening, particularly in the supersonic flow regime. We must learn to get religious and put faith in our carefully derived relations.

Momentum

The foregoing analysis was made using only the continuity and energy relations. We now proceed to apply momentum concepts to the control volume shown in Figure 9.4. The x-component of the momentum equation for steady, one-dimensional flow is

$$\sum F_x = \frac{\dot{m}}{g_c}\left(V_{\text{out}_x} - V_{\text{in}_x}\right)$$

From Figure 9.4 we see that the force summation is

$$\sum F_x = p_1 A - p_2 A - F_f \tag{9.14}$$

where F_f represents the total wall frictional force on the fluid between sections 1 and 2. Thus the momentum equation in the direction of flow becomes

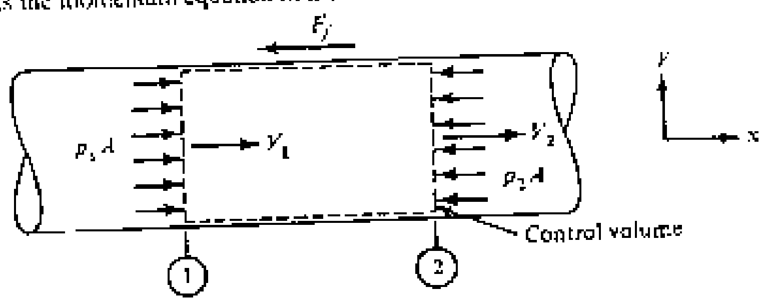


Figure 9.4 Momentum analysis for Fanno flow.

$$(p_1 - p_2)A - F_f = \frac{\dot{m}}{g_c}(V_2 - V_1) = \frac{\rho A V}{g_c}(V_2 - V_1) \tag{9.15}$$

Show that equation (9.15) can be written as

$$p_1 - p_2 - \frac{F_f}{A} = \frac{\rho_2 V_2^2}{g_c} - \frac{\rho_1 V_1^2}{g_c} \tag{9.16}$$

or

$$\boxed{\left(p_1 + \frac{\rho_1 V_1^2}{g_c}\right) - \frac{F_f}{A} = p_2 + \frac{\rho_2 V_2^2}{g_c}} \tag{9.17}$$

In this form the equation is not particularly useful except to bring out one significant fact. *For the steady, one-dimensional, constant-area flow of any fluid*, the value of $p + \rho V^2/g_c$ *cannot* be constant if frictional forces are present. This fact will be recalled later in the chapter when Fanno flow is compared with normal shocks.

Before leaving this section on fluids in general, we might say a few words about Fanno flow at low Mach numbers. A glance at Figure 9.3 shows that the upper branch is asymptotically approaching the horizontal line of constant total enthalpy. Thus the extreme left end of the Fanno line will be nearly horizontal. This indicates that flow at very low Mach numbers will have almost constant velocity. This checks our previous work, which indicated that we could treat gases as incompressible fluids if the Mach numbers were very small.

9.4 WORKING EQUATIONS FOR PERFECT GASES

We have discovered the general trend of property variations that occur in Fanno flow, both in the subsonic and supersonic flow regime. Now we wish to develop some specific working equations for the case of a perfect gas. Recall that these are relations between properties at arbitrary sections of a flow system written in terms of Mach numbers and the specific heat ratio.

Energy

We start with the energy equation developed in Section 9.3 since this leads immediately to a temperature ratio:

$$h_{t1} = h_{t2} \tag{9.3}$$

But for a perfect gas, enthalpy is a function of temperature only. Therefore,

$$T_{t1} = T_{t2} \tag{9.18}$$

Now for a perfect gas with constant specific heats,

$$T_t = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

Hence the energy equation for Fanno flow can be written as

$$T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right) = T_2\left(1 + \frac{\gamma - 1}{2} M_2^2\right) \tag{9.19}$$

or

$$\boxed{\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2}} \tag{9.20}$$

Continuity

From Section 9.3 we have

$$\rho V = G = \text{const} \tag{9.2}$$

or

$$\rho_1 V_1 = \rho_2 V_2 \tag{9.21}$$

If we introduce the perfect gas equation of state

$$p = \rho R T$$

the definition of Mach number

$$V = Ma$$

and sonic velocity for a perfect gas

$$a = \sqrt{\gamma g_c R T}$$

equation (9.21) can be solved for

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{1/2} \tag{9.22}$$

Can you obtain this expression? Now introduce the temperature ratio from (9.20) and you will have the following working relation for static pressure:

$$\boxed{\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2}\right)^{1/2}} \tag{9.23}$$

The density relation can easily be obtained from equations (9.20), (9.23), and the perfect gas law

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{1/2}} \tag{9.24}$$

Entropy Change

We start with an expression for entropy change that is valid between any two points:

$$\Delta s_{1-2} = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{1.53}$$

Equation (4.15) can be used to substitute for c_p and we nondimensionalize the equation to

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma - 1} \ln \frac{T_2}{T_1} - \ln \frac{p_2}{p_1} \tag{9.25}$$

If we now utilize the expressions just developed for the temperature ratio (9.20) and the pressure ratio (9.23), the entropy change becomes

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma - 1} \ln \left(\frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2}\right) - \ln \frac{M_1}{M_2}\left(\frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2}\right)^{1/2} \tag{9.26}$$

Show that this entropy change between two points in Fanno flow can be written as

$$\frac{s_2 - s_1}{R} = \ln \frac{M_2}{M_1}\left(\frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2}\right)^{(\gamma+1)/2(\gamma-1)} \tag{9.27}$$

Now recall that in Section 4.5 we integrated the stagnation pressure–energy equation for adiabatic no-work flow of a perfect gas, with the result

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

and equation (9.2[illegible]) becomes

$$\frac{T}{T^*} = \frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2} = f(M, \gamma) \tag{9.41}$$

We see that $T/T^* = f(M, \gamma)$ and we can easily construct a table giving values of T/T^* versus M for a particular γ. Equation (9.25) can be treated in a similar fashion. In this case

$$p_2 \Rightarrow p \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$p_1 \Rightarrow p^* \qquad M_1 \Rightarrow 1$$

and equation (9.25) becomes

$$\frac{p}{p^*} = \frac{1}{M}\left(\frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2}\right)^{1/2} = f(M, \gamma) \tag{9.42}$$

The density ratio can be obtained as a function of Mach number and γ from equation (9.24). This is particularly useful since it also represents a velocity ratio. Why?

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{1/2} = f(M, \gamma) \tag{9.43}$$

Apply the same techniques to equation (9.28) and *show* that

$$\frac{p_t}{p_t^*} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M, \gamma) \tag{9.44}$$

We now perform the same type of transformation on equation (9.40); that is, let

$$x_2 \Rightarrow x \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$x_1 \Rightarrow x^* \qquad M_1 \Rightarrow 1$$

with the following result:

$$\frac{f(x - x^*)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right) - \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) - \frac{\gamma + 1}{2\gamma} \ln M^2 \tag{9.45}$$

But a glance at the physical diagram in Figure 9.5 shows that $(x^* - x)$ will always be a negative quantity; thus it is more convenient to change all signs in equation (9.45) and simplify it to

$$\ln T - \ln\left(1 + \frac{\gamma - 1}{2}M^2\right) = \ln \text{ const} \tag{9.32}$$

and then differentiating, we obtain

$$\frac{dT}{T} - \frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} = 0 \tag{9.33}$$

which can be used to substitute for dT/T in (9.30).

The continuity relation [equation (9.21)] put in terms of a perfect gas becomes

$$\frac{pM}{\sqrt{T}} = \text{const} \tag{9.34}$$

By logarithmic differentiation (take the natural logarithm and then differentiate) *show* that

$$\frac{dp}{p} + \frac{dM}{M} - \frac{1}{2}\frac{dT}{T} = 0 \tag{9.35}$$

We can introduce equation (9.33) to eliminate dT/T, with the result that

$$\frac{dp}{p} = -\frac{dM}{M} + \frac{1}{2}\frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} \tag{9.36}$$

which can be used to substitute for dp/p in (9.30).

Make the indicated substitutions for dp/p and dT/T in the momentum equation, neglect the potential term, and *show* that equation (9.30) can be put into the following form:

$$f\frac{dx}{D_e} = \frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} - \frac{dM^2}{M^2} + \frac{2}{\gamma}\frac{dM}{M^3} + \frac{1}{\gamma M^2}\frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} \tag{9.37}$$

The last term can be simplified for integration by noting that

$$\frac{1}{\gamma M^2}\frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} = \frac{(\gamma - 1)}{2\gamma}\frac{dM^2}{M^2} - \frac{(\gamma - 1)}{2\gamma}\frac{d\{1 + [(\gamma - 1)/2]M^2\}}{1 + [(\gamma - 1)/2]M^2} \tag{9.38}$$

The momentum equation can now be written as

$$f\frac{dx}{D_e} = \frac{\gamma+1}{2\gamma}\frac{d\{1+[(\gamma-1)/2]M^2\}}{1+[(\gamma-1)/2]M^2} - \frac{2}{\gamma}\frac{dM}{M^3} - \frac{\gamma+1}{2\gamma}\frac{dM^2}{M^2} \tag{9.39}$$

Equation (9.39) is restricted to steady, one-dimensional flow of a perfect gas, with no heat or work transfer, constant area, and negligible potential changes. We can now integrate this equation between two points in the flow and obtain

$$\frac{f(x_2-x_1)}{D_e} = \frac{\gamma+1}{2\gamma}\ln\frac{1+[(\gamma-1)/2]M_2^2}{1+[(\gamma-1)/2]M_1^2} + \frac{1}{\gamma}\left(\frac{1}{M_2^2}-\frac{1}{M_1^2}\right) - \frac{\gamma+1}{2\gamma}\ln\frac{M_2^2}{M_1^2} \tag{9.40}$$

Note that in performing the integration we have held the friction factor constant. Some comments will be made on this in a later section. If you have forgotten the concept of equivalent diameter, you may want to review the last part of Section 3.8 and equation (3.61).

9.5 REFERENCE STATE AND FANNO TABLE

The equations developed in Section 9.4 provide the means of computing the properties at one location in terms of those given at some other location. The key to problem solution is predicting the Mach number at the new location through the use of equation (9.40). The solution of this equation for the unknown M_2 presents a messy task, as no explicit relation is possible. Thus we turn to a technique similar to that used with isentropic flow in Chapter.

We introduce *another* * reference state, which is defined in the same manner as before (i.e., "that thermodynamic state which would exist if the fluid reached a Mach number of unity *by a particular process*"). In this case we imagine that we continue *by Fanno flow* (i.e., more duct is added) until the velocity reaches Mach 1. Figure 9.5 shows a physical system together with its T–s diagram for a subsonic Fanno flow. We know that if we continue along the Fanno line (remember that we always move to the right), we will eventually reach the limiting point where sonic velocity exists. The dashed lines show a hypothetical duct of sufficient length to enable the flow to traverse the remaining portion of the upper branch and reach the limit point. This is the * reference point *for Fanno flow*.

The *isentropic* * reference points have also been included on the T–s diagram to emphasize the fact that the Fanno * reference is a totally different thermodynamic state. One other fact should be mentioned. If there is any entropy difference between two points (such as points 1 and 2), their isentropic * reference conditions are not the same and we have always taken great care to label them separately as 1^* and 2^*.

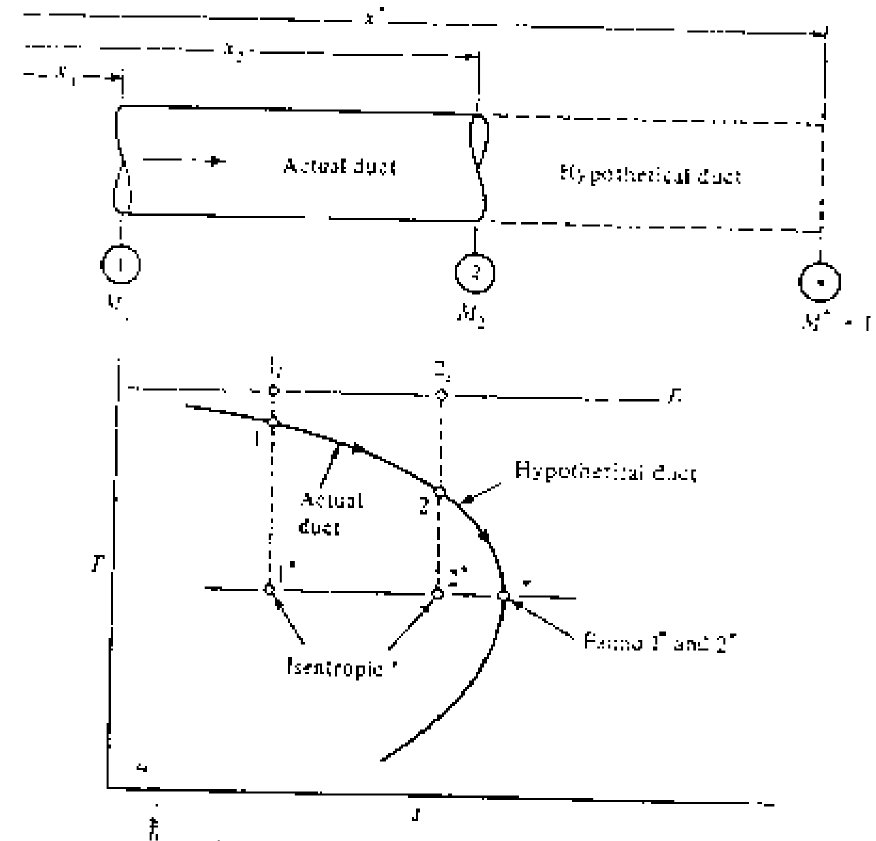


Figure 9.5 The * reference for Fanno flow.

However, proceeding from either point 1 or point 2 *by Fanno flow* will ultimately lead to the same place when Mach 1 is reached. Thus we do not have to talk of 1* or 2* but merely * in the case of Fanno flow. Incidentally, why are all three * reference points shown on the same horizontal line in Figure 9.5? (You may need to review Section 4.6.)

We now rewrite the working equations in terms of the Fanno flow * reference condition. Consider first

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2} \tag{9.20}$$

Let point 2 be an arbitrary point in the flow system and let its Fanno * condition be point 1. Then

$$T_2 \Rightarrow T \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$T_1 \Rightarrow T^* \qquad M_1 \Rightarrow 1$$

and equation (9.20) becomes

$$\frac{T}{T^*} = \frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2} = f(M,\gamma) \qquad (9.41)$$

We see that $T/T^* = f(M,\gamma)$ and we can easily construct a table giving values of T/T^* versus M for a particular γ. Equation (9.23) can be treated in a similar fashion. In this case

$$p_2 \Rightarrow p \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$p_1 \Rightarrow p^* \qquad M_1 \Rightarrow 1$$

and equation (9.23) becomes

$$\frac{p}{p^*} = \frac{1}{M}\left(\frac{(\gamma + 1)/2}{1 + [(\gamma - 1)/2]M^2}\right)^{1/2} = f(M,\gamma) \qquad (9.42)$$

The density ratio can be obtained as a function of Mach number and γ from equation (9.24). This is particularly useful since it also represents a velocity ratio. Why?

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{1/2} = f(M,\gamma) \qquad (9.43)$$

Apply the same techniques to equation (9.28) and *show* that

$$\frac{p_t}{p_t^*} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M,\gamma) \qquad (9.44)$$

We now perform the same type of transformation on equation (9.40); that is, let

$$x_2 \Rightarrow x \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$x_1 \Rightarrow x^* \qquad M_1 \Rightarrow 1$$

with the following result:

$$\frac{f(x - x^*)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)$$

$$- \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) - \frac{\gamma + 1}{2\gamma} \ln M^2 \qquad (9.45)$$

But a glance at the physical diagram in Figure 9.5 shows that $(x^* - x)$ will always be a negative quantity; thus it is more convenient to change all signs in equation (9.45) and simplify it to

$$\frac{f(x^* - x)}{D_e} = \left(\frac{\gamma + 1}{2\gamma}\right) \ln\left(\frac{[(\gamma + 1)/2] M^2}{1 + [(\gamma - 1)/2] M^2}\right)$$

$$+ \frac{1}{\gamma}\left(\frac{1}{M^2} - 1\right) = f(M, \gamma) \qquad (9.46)$$

The quantity $(x^* - x)$ represents the amount of duct that would have to be added to cause the flow to reach the Fanno * reference condition. It can alternatively be viewed as the maximum duct length that may be added without changing some flow condition. Thus the expression

$$\frac{f(x^* - x)}{D_e} \quad \text{is called} \quad \frac{fL_{max}}{D_e}$$

and is listed in table along with the other Fanno flow parameters: T/T^*, p/p^*, V/V^*, and p_t/p_t^*. In the next section we shall see how this table greatly simplifies the solution of Fanno flow problems. But first, some words about the determination of friction factors.

Dimensional analysis of the fluid flow problem shows that the friction factor can be expressed as

$$f = f(\text{Re}, \varepsilon/D) \qquad (9.47)$$

where Re is the *Reynolds number*,

$$\text{Re} \equiv \frac{\rho V D}{\mu g_c} \qquad (9.48)$$

and

$$\varepsilon/D \equiv \textit{relative roughness}$$

Typical values of ε, the *absolute roughness* or average height of wall irregularities, are shown in Table 9.2.

The relationship among f, Re, and ε/D is determined experimentally and plotted on a chart similar to Figure 9.6, which is called a *Moody diagram*.
If the flow rate is known together with the duct size and

Table 9.2 Absolute Roughness of Common Materials

Material	ε (ft)
Glass, brass, copper, lead	smooth < 0.00001
Steel, wrought iron	0.00015
Galvanized iron	0.0005
Cast iron	0.00085
Riveted steel	0.03

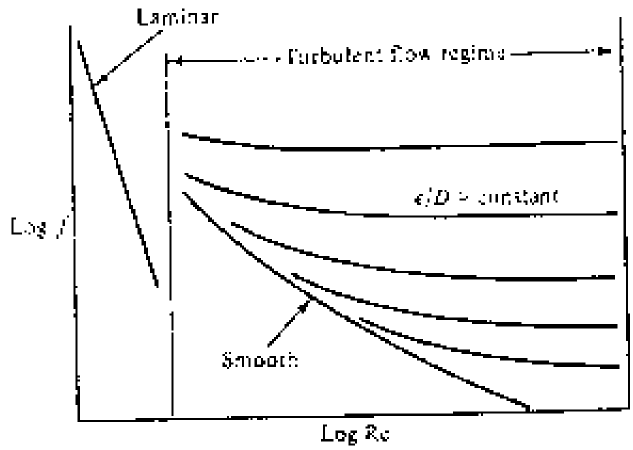


Figure 9.6 Moody diagram for friction factor in circular ducts.

material, the Reynolds number and relative roughness can easily be calculated and the value of the friction factor is taken from the diagram. The curve in the laminar flow region can be represented by

$$f = \frac{64}{\text{Re}} \tag{9.49}$$

For noncircular cross sections the *equivalent diameter* as described in Section 3.8 can be used.

$$D_e = \frac{4A}{P} \tag{3.61}$$

This equivalent diameter may be used in the determination of relative roughness and Reynolds number, and hence the friction factor. However, care must be taken to work with the *actual* average velocity in all computations. Experience has shown that the use of an equivalent diameter works quite well in the turbulent zone. In the laminar flow region this concept is not sufficient and consideration must also be given to the aspect ratio of the duct.

In some problems the flow rate is not known and thus a trial-and-error solution results. As long as the duct size is given, the problem is not too difficult; an excellent approximation to the friction factor can be made by taking the value corresponding to where the ϵ/D curve begins to *level off*. This converges rapidly to the final answer, as most engineering problems are well into the turbulent range.

9.6 APPLICATIONS

The following steps are recommended to develop good problem-solving technique:

1. Sketch the physical situation (including the hypothetical * reference point).
2. Label sections where conditions are known or desired.
3. List all given information with units.
4. Compute the equivalent diameter, relative roughness, and Reynolds number.
5. Find the friction factor from the Moody diagram.
6. Determine the unknown Mach number.
7. Calculate the additional properties desired.

The procedure above may have to be altered depending on what type of information is given, and occasionally, trial-and-error solutions are required. You should have no difficulty incorporating these features once the basic straightforward solution has been mastered. In complicated flow systems that involve more than just Fanno flow, a T–s diagram is frequently helpful in solving problems.

For the following examples we are dealing with the steady one-dimensional flow of air ($\gamma = 1.4$), which can be treated as a perfect gas. Assume that $Q = W_s = 0$ and negligible potential changes. The cross-sectional area of the duct remains constant. Figure E9.1 is common to Examples 9.1 through 9.3.

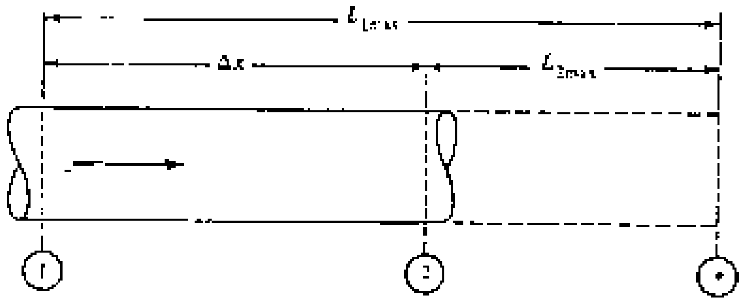


Figure E9.1

Example 9.1 Given $M_1 = 1.80$, $p_1 = 40$ psia, and $M_2 = 1.20$, find p_2 and $f\,\Delta x/D$.

Since both Mach numbers are known, we can solve immediately for

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (0.8044)\left(\frac{1}{0.4742}\right)(40) = 67.9 \text{ psia}$$

Check Figure E9.1 to see that

$$\frac{f\,\Delta x}{D} = \frac{fL_{1max}}{D} - \frac{fL_{2max}}{D} = 0.2419 - 0.0336 = 0.208$$

Example 9.2 Given $M_2 = 0.94$, $T_1 = 400$ K, and $T_2 = 350$ K, find M_1 and p_2/p_1.

To determine conditions at section 1 in Figure E9.1, we must establish the ratio

$$\frac{T_2}{T^*} = \frac{T_2}{T_1}\frac{T_1}{T^*} = \left(\frac{400}{350}\right)(1.0198) = 1.1655$$

↑ Given ↑ From Fanno table at $M = 0.94$

Look up $T/T^* = 1.1655$ in the Fanno table and determine that $M_2 = 0.385$. Then

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1} = (1.0743)\left(\frac{1}{2.8046}\right) = 0.383$$

Notice that these examples confirm previous statements concerning static pressure changes. In subsonic flow the static pressure decreases, whereas in supersonic flow the static pressure increases. *Compute* the stagnation pressure ratio and show that the friction losses cause p_{t2}/p_{t1} to decrease in each case.

For Example 9.1:

$$\frac{p_{t2}}{p_{t1}} = \qquad (p_{t2}/p_{t1} = 0.7161)$$

For Example 9.2:

$$\frac{p_{t2}}{p_{t1}} = \qquad (p_{t2}/p_{t1} = 0.611)$$

Example 9.3 Air flows in a 6-in.-diameter, insulated, galvanized iron duct. Initial conditions are $p_1 = 20$ psia, $T_1 = 70°F$, and $V_1 = 406$ ft/sec. After 70 ft, determine the final Mach number, temperature, and pressure.

Since the duct is circular we do not have to compute an equivalent diameter. From Table 9.2 the absolute roughness ε is 0.0005. Thus the relative roughness

$$\frac{\varepsilon}{D} = \frac{0.0005}{0.5} = 0.001$$

We compute the Reynolds number at section 1 (Figure E9.1) since this is the only location where information is known.

$$\rho_1 = \frac{p_1}{RT_1} = \frac{(20)(144)}{(53.3)(530)} = 0.102 \text{ lbm/ft}^3$$

$$\mu_1 = 3.8 \times 10^{-7} \text{ lbf-sec/ft}^2 \text{ (Air properties table)}$$

Thus

$$\text{Re}_1 = \frac{\rho_1 V_1 D_1}{\mu_1 g_c} = \frac{(0.102)(406)(0.5)}{(3.8 \times 10^{-7})(32.2)} = 1.69 \times 10^6$$

From the Moody diagram at $\text{Re} = 1.69 \times 10^6$ and $\varepsilon/D = 0.001$, we determine that the friction factor is $f = 0.0198$. To use the Fanno table (or equations), we need information on Mach numbers.

R = 53.3

μ = 3.8 × 10⁻⁷

g = 32.2

$$a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(32.2)(53.3)(530)]^{1/2} = 1128 \text{ ft/sec}$$

$$M_1 = \frac{V_1}{a_1} = \frac{406}{1128} = 0.36$$

From the Fanno table at $M_1 = 0.36$, we find that

$$\frac{p_1}{p^*} = 3.0042 \quad \frac{T_1}{T^*} = 1.1697 \quad \frac{fL_{1max}}{D} = 3.1801$$

The key to completing the problem is in establishing the Mach number at the outlet, and this is done through the *friction length*.

$$\frac{f\,\Delta x}{D} = \frac{(0.0198)(70)}{0.5} = 2.772$$

Looking at the physical sketch it is apparent (since f and D are constants) that

$$\frac{fL_{2max}}{D} = \frac{fL_{1max}}{D} - \frac{f\,\Delta x}{D} = 3.1801 - 2.772 = 0.408$$

We enter the Fanno table with this friction length and find that

$$M_2 = 0.623 \quad \frac{p_2}{p^*} = 1.6939 \quad \frac{T_2}{T^*} = 1.1136$$

Thus

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (1.6939)\left(\frac{1}{3.0042}\right)(20) = 11.28 \text{ psia}$$

and

$$T_2 = \frac{T_2}{T^*}\frac{T^*}{T_1}T_1 = (1.1136)\left(\frac{1}{1.1697}\right)(530) = 505^\circ\text{R}$$

In the example above, the friction factor was assumed constant. In fact, this assumption was made when equation (9.39) was integrated to obtain (9.40), and with the introduction of the * reference state, this became equation (9.46), which is listed in the Fanno table. Is this a reasonable assumption? Friction factors are functions of Reynolds numbers, which in turn depend on velocity and density—both of which can change quite rapidly in Fanno flow. Calculate the velocity at the outlet in Example 9.3 and compare it with that at the inlet. ($V_2 = 686$ ft/sec and $V_1 = 406$ ft/sec.)

But don't despair. From continuity we know that the product of ρV is always a constant, and thus the only variable in Reynolds number is the viscosity. Extremely large temperature variations are required to change the viscosity of a gas significantly, and thus variations in the Reynolds number are small for any given problem. We are also fortunate in that most engineering problems are well into the turbulent range where the friction factor is relatively insensitive to Reynolds number. A greater potential error is involved in the estimation of the duct roughness, which has a more significant effect on the friction factor.

Example 9.4 A converging-diverging nozzle with an area ratio of 5.42 connects to an 8-ft-long constant-area rectangular duct (see Figure E9.4). The duct is 8 × 4 in. in cross section and has a friction factor of $f = 0.02$. What is the minimum stagnation pressure feeding the nozzle if the flow is supersonic throughout the entire duct and it exhausts to 14.7 psia?

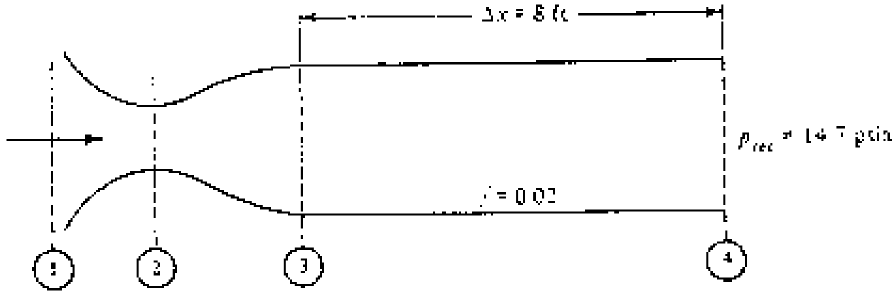


Figure E9.4

$$D_e = \frac{4A}{P} = \frac{(4)(32)}{24} = 5.334 \text{ in.}$$

$$\frac{f\,\Delta x}{D} = \frac{(0.02)(8)(12)}{5.334} = 0.36$$

To be supersonic with $A_3/A_2 = 5.42$, $M_3 = 3.26$, $p_3/p_{t3} = 0.0185$, $p_3/p^* = 0.1901$, and $fL_{3\max}/D = 0.5582$.

$$\frac{fL_{4\max}}{D} = \frac{fL_{3\max}}{D} - \frac{f\,\Delta x}{D} = 0.5582 - 0.36 = 0.1982$$

Thus

$$M_4 = 1.673 \quad \text{and} \quad \frac{p_4}{p^*} = 0.5243$$

and

$$p_{t1} = \frac{p_{t1}}{p_{t3}}\frac{p_{t3}}{p_3}\frac{p_3}{p^*}\frac{p^*}{p_4}p_4 = (1)\left(\frac{1}{0.0185}\right)(0.1901)\left(\frac{1}{0.5243}\right)(14.7) = 288 \text{ psia}$$

Any pressure above 288 psia will maintain the flow system as specified but with expansion waves outside the duct. (Recall an underexpanded nozzle.) Can you envision what would happen if the inlet stagnation pressure fell below 288 psia? (Recall the operation of an overexpanded nozzle.)

9.7 CORRELATION WITH SHOCKS

As you have progressed through this chapter you may have noticed some similarities between Fanno flow and normal shocks. Let us summarize some pertinent information.

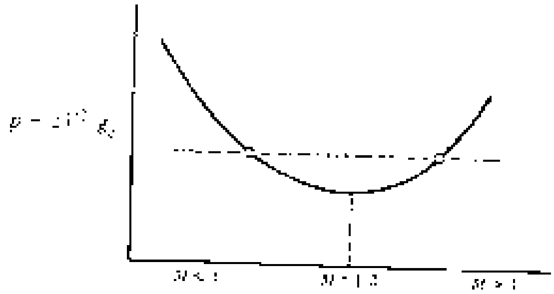


Figure 9.7 Variation of $p + \rho V^2/g_c$ in Fanno flow

The points just before and after a normal shock represent states with the same mass flow per unit area, the same value of $p + \rho V^2/g_c$, and the same stagnation enthalpy. These facts are the result of applying the basic concepts of continuity, momentum, and energy to any arbitrary fluid. This analysis resulted in equations (6.2), (6.3), and (6.9).

A Fanno line represents states with the same mass flow per unit area and the same stagnation enthalpy. This is confirmed by equations (9.2) and (9.5). To move *along* a Fanno line requires friction. At the end of Section 9.3 [see equation (9.17)] it was pointed out that it is this very friction which causes the value of $p + \rho V^2/g_c$ to change.

The variation of the quantity $p + \rho V^2/g_c$ along a Fanno line is quite interesting. Such a plot is shown in Figure 9.7. You will notice that for every point on the supersonic branch of the Fanno line there is a corresponding point on the subsonic branch with the same value of $p + \rho V^2/g_c$. Thus these two points satisfy all three conditions for the end points of a normal shock and could be connected by such a shock.

Now we can imagine a supersonic Fanno flow leading into a normal shock. If this is followed by additional duct, subsonic Fanno flow would occur. Such a situation is shown in Figure 9.8a. Note that the shock merely causes the flow to jump from the supersonic branch to the subsonic branch of the *same* Fanno line. [See Figure 9.8b.]

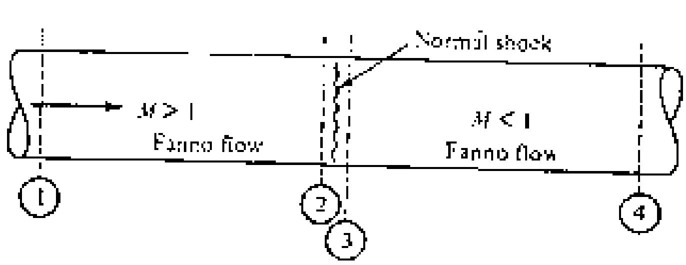


Figure 9.8a Combination of Fanno flow and normal shock (physical system).

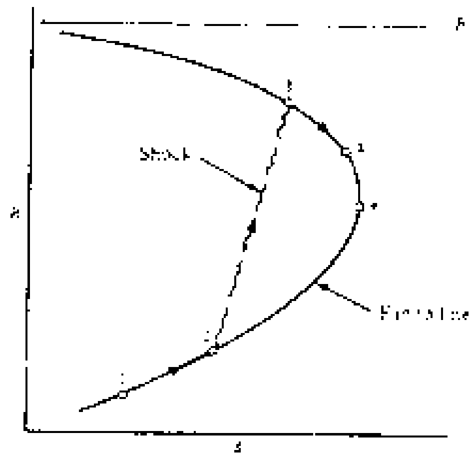


Figure 9.8b Combination of Fanno flow and normal shock

Example 9.5 A large chamber contains air at a temperature of 300 K and a pressure of 5 bar abs (Figure E9.5). The air enters a converging–diverging nozzle with an area ratio of 2.4. A constant-area duct is attached to the nozzle and a normal shock stands at the exit plane. Receiver pressure is 3 bar abs. Assume the entire system to be adiabatic and neglect friction in the nozzle. Compute the $f\,\Delta x/D$ for the duct.

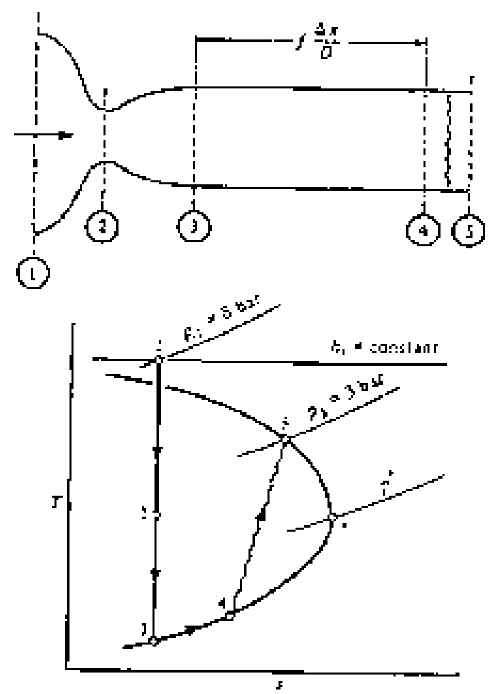


Figure E9.5

For a shock to occur as specified, the duct flow must be supersonic, which means that the nozzle is operating at its third critical point. The inlet conditions and nozzle area ratio fix conditions at location 3. We can then find p^* at the tip of the Fanno line. Then the ratio p_5/p^* can be computed and the Mach number after the shock is found from the Fanno table. This solution probably would not have occurred to us had we not drawn the T–s diagram and recognized that point 5 is on the same Fanno line as 3, 4, and *.

For $A_3/A_t = 2.4$, $M_3 = 2.4$ and $p_3/p_{t3} = 0.06840$. We proceed immediately to compute p_5/p^*:

$$\frac{p_5}{p^*} = \frac{p_5}{p_4}\frac{p_{t4}}{p_{t3}}\frac{p_{t3}}{p_3}\frac{p_3}{p^*} = \left(\frac{3}{8}\right)(1)\left(\frac{1}{0.2654}\right)(0.3111) = 1.7036$$

From the Fanno table we find that $M_5 = 0.619$, and then from the shock table $M_4 = 1.764$. Returning to the Fanno table, $fL_{3\,max}/D = 0.4099$ and $fL_{4\,max}/D = 0.2382$. Thus

$$\frac{f\,\Delta x}{D} = \frac{fL_{3\,max}}{D} - \frac{fL_{4\,max}}{D} = 0.4099 - 0.2382 = 0.172$$

9.8 FRICTION CHOKING

In Chapter 5 we discussed the operation of nozzles that were fed by constant stagnation inlet conditions (see Figures 5.6 and 5.8). We found that as the receiver pressure was lowered, the flow through the nozzle increased. When the *operating pressure ratio* reached a certain value, the section of minimum area developed a Mach number of unity. The nozzle was then said to be choked. Further reduction in the pressure ratio did not increase the flow rate. This was an example of *area choking*.

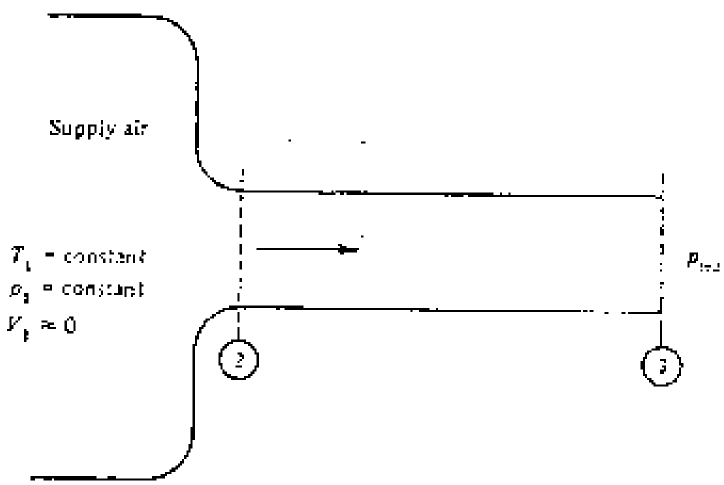


Figure 9.9 Converging nozzle and constant-area duct combination.

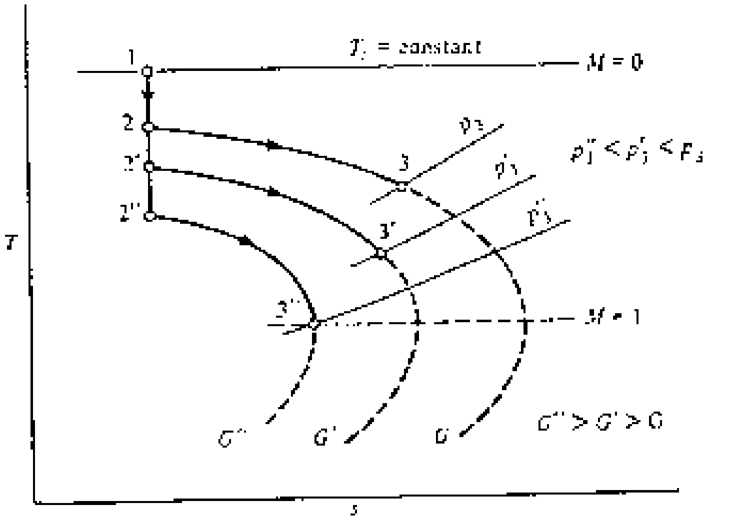


Figure 9.10 *T–s* diagram for nozzle-duct combination.

The subsonic Fanno flow situation is quite similar. Figure 9.9 shows a given length of duct fed by a large tank and converging nozzle. If the receiver pressure is below the tank pressure, flow will occur, producing a T–s diagram shown as path 1–2–3 in Figure 9.10. Note that we have isentropic flow at the entrance to the duct and then we move along a Fanno line. As the receiver pressure is lowered still more, the flow rate and exit Mach number continue to increase while the system moves to Fanno lines of higher mass velocities (shown as path 1–2′–3′). It is important to recognize that the receiver pressure (or more properly, the operating pressure ratio) is controlling the flow. This is because in subsonic flow the pressure at the duct exit must equal that of the receiver.

Eventually, when a certain pressure ratio is reached, the Mach number at the duct exit will be unity (shown as path 1–2″–3″). This is called *friction choking* and any further reduction in receiver pressure would not affect the flow conditions *inside* the system. What would occur as the flow leaves the duct and enters a region of reduced pressure?

Let us consider this last case of choked flow with the exit pressure equal to the receiver pressure. Now *suppose that the receiver pressure is maintained at this value* but more duct is added to the system. (Nothing can physically prevent us from doing this.) What happens? We know that we cannot move *around the Fanno line*, yet somehow we must reflect the added friction losses. This is done by moving to a new Fanno line at a *decreased* flow rate. The T–s diagram for this is shown as path 1–2‴–3‴–4 in Figure 9.11. Note that pressure equilibrium is still maintained at the exit but

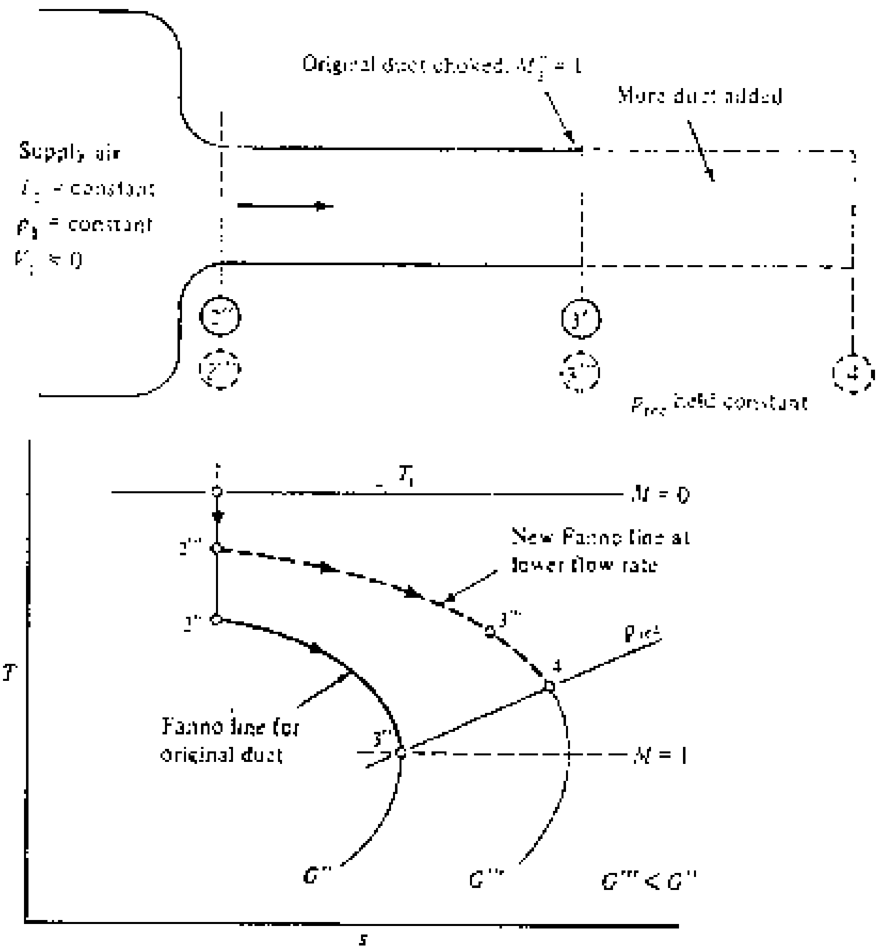


Figure 9.11 Addition of more duct when choked.

the system is no longer choked, although the flow rate has decreased. What would occur if the receiver pressure were now lowered?

In summary, when a *subsonic* Fanno flow has become *friction choked* and more duct is added to the system, the flow rate must decrease. Just how much it decreases and whether or not the exit velocity remains sonic depends on how much duct is added and the receiver pressure imposed on the system.

Now suppose that we are dealing with *supersonic* Fanno flow that is *friction choked*. In this case the addition of more duct causes a normal shock to form inside the duct. The resulting subsonic flow can accommodate the increased duct length at the same flow rate. For example, Figure 9.12 shows a Mach 2.18 flow that has an fL_{max}/D value of 0.356. If a normal shock were to occur at this point, the Mach number after the shock would be about 0.550, which corresponds to an fL_{max}/D

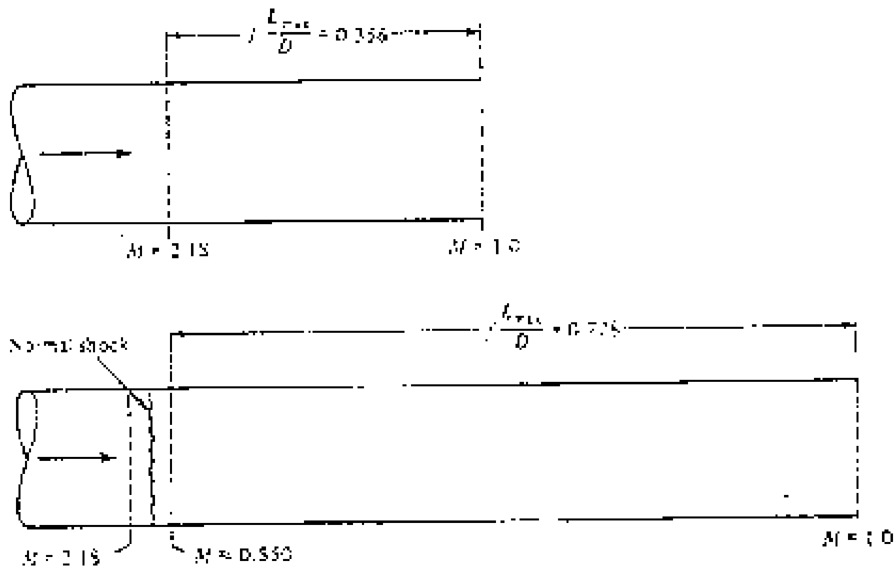


Figure 9.12 Influence of shock on maximum duct length.

value of 0.728. Thus, in this case, the appearance of the shock permits over twice the duct length to the choke point. This difference becomes even greater as higher Mach numbers are reached.

The shock location is determined by the amount of duct added. As more duct is added, the shock moves upstream and occurs at a higher Mach number. Eventually, the shock will move into that portion of the system that precedes the constant-area duct. (Most likely, a converging–diverging nozzle was used to produce the supersonic flow.) If sufficient friction length is added, the entire system will become subsonic and then the flow rate will decrease. Whether or not the exit velocity remains sonic will again depend on the receiver pressure.

9.9. WHEN γ IS NOT EQUAL TO 1.4

As indicated earlier, the Fanno flow table is for $\gamma = 1.4$. The behavior of fL_{max}/D, the friction function, is given in Figure 9.13 for $\gamma = 1.13$, 1.4, and 1.67 for Mach numbers up to $M = 5$. Here we can see that the dependence on γ is rather noticeable for $M \geq 1.4$. Thus, below this Mach number the tabulation in fanno table may be used with little error for any γ. This means that for subsonic flows, where most Fanno flow problems occur, there is little difference between the various gases. The desired accuracy of results will govern how far you want to carry this approximation into the supersonic region.

Strictly speaking, these curves are only representative for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of

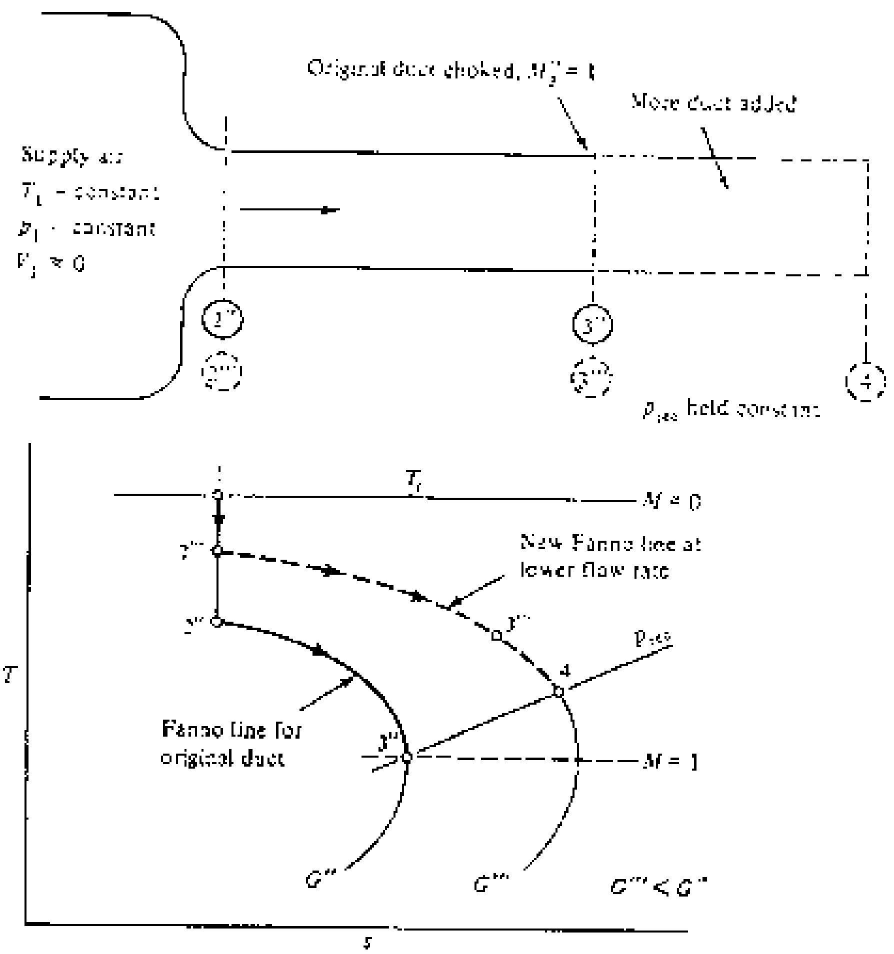


Figure 9.11 Addition of more duct when choked

the system is no longer choked, although the flow rate has decreased. What would occur if the receiver pressure were now lowered?

In summary, when a *subsonic* Fanno flow has become *friction choked* and more duct is added to the system, the flow rate must decrease. Just how much it decreases and whether or not the exit velocity remains sonic depends on how much duct is added and the receiver pressure imposed on the system.

Now suppose that we are dealing with *supersonic* Fanno flow that is *friction choked*. In this case the addition of more duct causes a normal shock to form inside the duct. The resulting subsonic flow can accommodate the increased duct length at the same flow rate. For example, Figure 9.12 shows a Mach 2.18 flow that has an fL_{max}/D value of 0.356. If a normal shock were to occur at this point, the Mach number after the shock would be about 0.550, which corresponds to an fL_{max}/D

Listed below are the precise inputs and program that you use in the computer

```
[ > g := 1.4;   X := 0.36;
[ > Y := ((g + 1)/(2*g))*ln(((g + 1)*(X^2)/2)/(1 +
    (g - 1)*(X^2)/2)) + (1/g)*((1/X^2) - 1);
                              Y := 3.180117523
```

We can proceed to find the Mach number at station 2. The new value of Y is $3.1801 - 2.772 = 0.408$. Now we use the same equation (9.46) but solve for M_2 as shown below. Note that since M is implicit in the equation, we are going to utilize "fsolve." Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)

$X \equiv$ the dependent variable (which in this case is M_2)

$Y \equiv$ the independent variable (which in this case is fL_{max}/D)

Listed below are the precise inputs and program that you use in the computer

```
[ > g2 := 1.4;   Y2 := 0.408;
[ > fsolve(Y2 = ((g2 + 1)/(2*g2))*log(((g2 + 1)*(X2^2)/2)/(1 +
    (g2 - 1)*(X2^2)/2)) + (1/g2)*((1/X2^2) - 1), X2, 0..1);
                              .6227097475
```

The answer of $M_2 = 0.6227$ is consistent with that obtained in Example 9.3. We can now proceed to calculate the required static properties, but this will be left as an exercise for the reader. ◆

9.11 SUMMARY

We have analyzed flow in a constant-area duct with friction but without heat transfer. The fluid properties change in a predictable manner dependent on the flow regime as shown in Table 9.3. The property variations in subsonic Fanno flow follow an intuitive pattern but we note that the supersonic flow behavior is completely different. The

Table 9.3 Fluid Property Variation for Fanno Flow

Property	Subsonic	Supersonic
Velocity	Increases	Decreases
Mach number	Increases	Decreases
Enthalpy*	Decreases	Increases
Stagnation enthalpy*	Constant	Constant
Pressure	Decreases	Increases
Density	Decreases	Increases
Stagnation pressure	Decreases	Decreases

* Also temperature if the fluid is a perfect gas.

only common occurrence is the decrease in stagnation pressure, which is indicative of the loss.

Perhaps the most significant equations are those that apply to all fluids:

$$\rho V = G = \text{constant} \quad (9.2)$$

$$h_t = h + \frac{G^2}{\rho^2 2g_c} = \text{constant} \quad (9.5)$$

Along with these equations you should keep in mind the appearance of Fanno lines in the h–s and T–s diagrams (see Figures 9.1 and 9.2). Remember that each Fanno line represents points with the same mass velocity (G) and stagnation enthalpy (h_t), and a normal shock can connect two points on opposite branches of a Fanno line which have the same value of $p + \rho V^2/g_c$. *Families* of Fanno lines could represent

1. Different values of G for the same h_t (such as those in Figure 9.10), or
2. The same G for different values of h_t (see Problem 10.17).

Detailed working equations were developed for perfect gases, and the introduction of a * reference point enabled the construction of a Fanno table which simplifies problem solution. The * condition for Fanno flow has no relation to the one used previously in isentropic flow (except in general definition). All Fanno flows proceed toward a limiting point of Mach 1. Friction choking of a flow passage is possible in Fanno flow just as area choking occurs in varying-area isentropic flow. An h–s (or T–s) diagram is of great help in the analysis of a complicated flow system. *Get into the habit of drawing these diagrams.*

PROBLEMS

In the problems that follow you may assume that all systems are completely adiabatic. Also, all ducts are of constant area unless otherwise indicated. You may neglect friction in the varying-area sections. You may also assume that the friction factor shown in charts applies to noncircular cross sections when the equivalent diameter concept is used and the flow is turbulent.

9.1. Conditions at the entrance to a duct are $M_1 = 3.0$ and $p_1 = 8 \times 10^4$ N/m². After a certain length the flow has reached $M_2 = 1.5$. Determine p_2 and $\bar{f}\,\Delta x/D$ if $\gamma = 1.4$.

9.2. A flow of nitrogen is discharged from a duct with $M_2 = 0.85$, $T_2 = 500$°R, and $p_2 = 28$ psia. The temperature at the inlet is 560°R. Compute the pressure at the inlet and the mass velocity (G).

9.3. Air enters a circular duct with a Mach number of 3.0. The friction factor is 0.01.

(a) How long a duct (measured in diameters) is required to reduce the Mach number to 2.0?

(b) What is the percentage change in temperature, pressure, and density?

(c) Determine the entropy increase of the air.

(d) Assume the same length of duct as computed in part (a), but the initial Mach number is 0.5. Compute the percentage change in temperature, pressure, density, and the entropy increase for this case. Compare the changes in the same length duct for subsonic and supersonic flow.

9.4. Oxygen enters a 6-in.-diameter duct with $T_1 = 600°R$, $p_1 = 50$ psia, and $V_1 = 600$ ft/sec. The friction factor is $f = 0.02$.

(a) What is the maximum length of duct permitted that will not change any of the conditions at the inlet?

(b) Determine T_2, p_2, and V_2 for the maximum duct length found in part (a).

9.5. Air flows in an 8-cm-inside diameter pipe that is 4 m long. The air enters with a Mach number of 0.45 and a temperature of 300 K.

(a) What friction factor would cause sonic velocity at the exit?

(b) If the pipe is made of cast iron, estimate the inlet pressure.

9.6. At one section in a constant-area duct the stagnation pressure is 66.8 psia and the Mach number is 0.80. At another section the pressure is 60 psia and the temperature is 120°F.

(a) Compute the temperature at the first section and the Mach number at the second section if the fluid is air.

(b) Which way is the air flowing?

(c) What is the friction length ($f\,\Delta x/D$) of the duct?

9.7. A 50 × 50 cm duct is 10 m in length. Nitrogen enters at $M_1 = 3.0$ and leaves at $M_2 = 1.7$, with $T_2 = 280$ K and $p_2 = 7 \times 10^4$ N/m^2.

(a) Find the static and stagnation conditions at the entrance.

(b) What is the friction factor of the duct?

9.8. A duct of 2 ft × 1 ft cross section is made of riveted steel and is 500 ft long. Air enters with a velocity of 174 ft/sec, $p_1 = 50$ psia, and $T_1 = 100°F$.

(a) Determine the temperature, pressure, and velocity at the exit.

(b) Compute the pressure drop assuming the flow to be incompressible. Use the entering conditions and equation (3.29). Note that equation (3.64) can easily be integrated to evaluate

$$\int T\,ds_i = f\,\frac{\Delta x}{D_e}\,\frac{V^2}{2g_c}$$

(c) How do the results of parts (a) and (b) compare? Did you expect this?

9.9. Air enters a duct with a mass flow rate of 35 lbm/sec at $T_1 = 520°R$ and $p_1 = 20$ psia. The duct is square and has an area of 0.64 ft^2. The outlet Mach number is unity.

(a) Compute the temperature and pressure at the outlet.

(b) Find the length of the duct if it is made of steel.

9.10. Consider the flow of a perfect gas along a Fanno line. Show that the pressure at the * reference state is given by the relation

$$p^* = \frac{\dot{m}}{A}\left[\frac{2RT_t}{\gamma g_c(\gamma + 1)}\right]^{1/2}$$

9.11. A 20-ft duct 12 in. in diameter contains oxygen flowing at the rate of 80 lbm/sec. Measurements at the inlet give $p_1 = 35$ psia and $T_1 = 800$°R. The pressure at the outlet is $p_2 = 13$ psia.

(a) Calculate M_1, M_2, V_2, T_{t2}, and p_{t2}.

(b) Determine the friction factor and estimate the absolute roughness of the duct material.

9.12. At the outlet of a 25-cm-diameter duct, air is traveling at sonic velocity with a temperature of 16°C and a pressure of 1 bar. The duct is very smooth and is 15 m long. There are two possible conditions that could exist at the entrance to the duct.

(a) Find the static and stagnation temperature and pressure for each entrance condition.

(b) Assuming the surrounding air to be at 1 bar pressure, how much horsepower is necessary to get ambient air into the duct for each case? (You may assume no losses in the work process.)

9.13. Ambient air at 60°F and 14.7 psia accelerates isentropically into a 12-in.-diameter duct. After 100 ft the duct transitions into an 8 × 8 in. square section where the Mach number is 0.50. Neglect all frictional effects except in the constant-area duct, where $f = 0.04$.

(a) Determine the Mach number at the duct entrance.

(b) What are the temperature and pressure in the square section?

(c) How much 8 × 8 in. square duct could be added before the flow chokes? (Assume that $f = 0.04$ in this duct also.)

9.14. Nitrogen with $p_t = 7 \times 10^5$ N/m² and $T_t = 340$ K enters a frictionless converging–diverging nozzle having an area ratio of 4.0. The nozzle discharges supersonically into a constant-area duct that has a friction length $f\,\Delta x/D = 0.355$. Determine the temperature and pressure at the exit of the duct.

9.15. Conditions before a normal shock are $M_1 = 2.5$, $p_{t1} = 67$ psia, and $T_{t1} = 700$°R. This is followed by a length of Fanno flow and a converging nozzle as shown in Figure P9.15. The area change is such that the system is choked. It is also known that $p_4 = p_{amb} = 14.7$ psia.

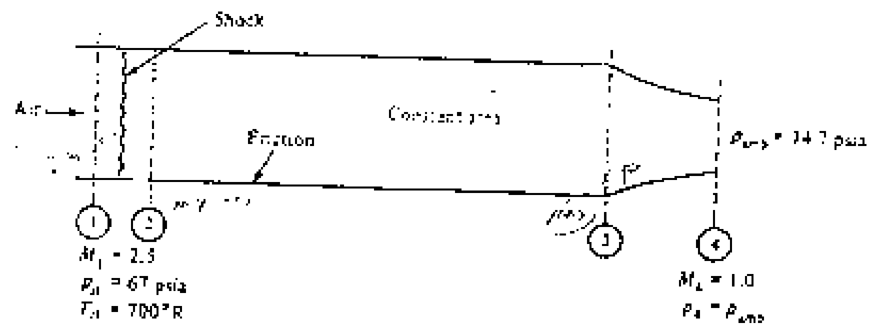


Figure P9.15

(a) Draw a T–s diagram for the system.

(b) Find M_2 and M_3.

(c) What is $f\,\Delta x/D$ for the duct?

9.16. A converging–diverging nozzle (Figure P9.16) has an area ratio of 3.0. The stagnation conditions of the inlet air are 150 psia and 550°R. A constant-area duct with a length of 12 diameters is attached to the nozzle outlet. The friction factor in the duct is 0.025.

(a) Compute the receiver pressure that would place a shock

(i) in the nozzle throat;

(ii) at the nozzle exit;

(iii) at the duct exit.

(b) What receiver pressure would cause supersonic flow throughout the duct with no shocks within the system (or after the duct exit)?

(c) Make a sketch similar to Figure 6.3 showing the pressure distribution for the various operating points of parts (a) and (b).

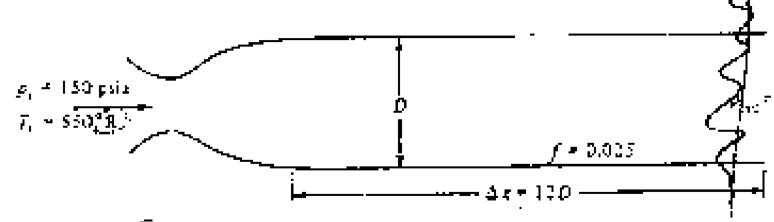


Figure P9.16

9.17. For a nozzle–duct system similar to that of Problem 9.16, the nozzle is designed to produce a Mach number of 2.8 with $\gamma = 1.4$. The inlet conditions are $p_{t1} = 10$ bar and $T_{t1} = 370$ K. The duct is 8 diameters in length, but the duct friction factor is unknown. The receiver pressure is fixed at 3 bar and a normal shock has formed at the duct exit.

(a) Sketch a T–s diagram for the system.

(b) Determine the friction factor of the duct.

(c) What is the total change in entropy for the system?

9.18. A large chamber contains air at 65 bar pressure and 400 K. The air passes through a converging-only nozzle and then into a constant-area duct. The friction length of the duct is $f\,\Delta x/D = 1.067$ and the Mach number at the duct exit is 0.96.

(a) Draw a T–s diagram for the system.

(b) Determine conditions at the duct entrance.

(c) What is the pressure in the receiver? (*Hint:* How is this related to the duct exit pressure?)

(d) If the length of the duct is doubled and the chamber and receiver conditions remain unchanged, what are the new Mach numbers at the entrance and exit of the duct?

9.19. A constant-area duct is fed by a converging-only nozzle as shown in Figure P9.19. The nozzle receives oxygen from a large chamber at $p_1 = 100$ psia and $T_1 = 1000$°R. The duct has a friction length of 5.3 and it is choked at the exit. The receiver pressure is exactly the same as the pressure at the duct exit.

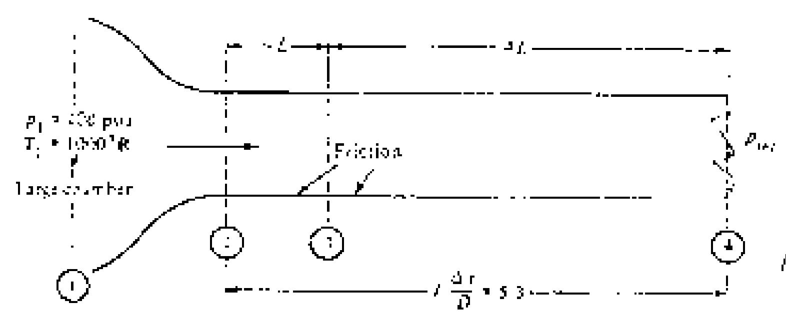


Figure P9.19

(a) What is the pressure at the end of the duct?

(b) Four-fifths of the duct is removed. (The end of the duct is now at 3.) The chamber pressure, receiver pressure, and friction factor remain unchanged. Now what is the pressure at the exit of the duct?

(c) Sketch both of the cases above on the same T–s diagram.

9.20. (a) Plot a Fanno line to scale in the T–s plane for air entering a duct with a Mach number of 0.20, a static pressure of 300 psia, and a static temperature of 540°R. Indicate the Mach number at various points along the curve.

(b) On the same diagram, plot another Fanno line for a flow with the same total enthalpy, the same entering entropy, but double the mass velocity.

9.21. Which, if any, of the ratios tabulated in the Fanno table (T/T^*, p/p^*, p_t/p_t^*, etc.) could also be listed in the Isentropic table with the same numerical values?

9.22. A contractor is to connect an air supply from a compressor to test apparatus 21 ft away. The exit diameter of the compressor is 2 in. and the entrance to the test equipment has a 1-in.-diameter pipe. The contractor has the choice of putting a reducer at the compressor followed by 1-in. tubing or using 2-in. tubing and putting the reducer at the entrance to the test equipment. Since smaller tubing is cheaper and less obtrusive, the contractor is leaning toward the first possibility, but just to be sure, he sends the problem to the engineering personnel. The air coming out of the compressor is at 520°R and the pressure is 40 psia. The flow rate is 0.7 lbm/sec. Consider that each size of tubing has an effective $f = 0.02$. What would be the conditions at the entrance to the test equipment for each tubing size? (You may assume isentropic flow everywhere but in the 21 ft of tubing.)

9.23. (*Optional*) (a) Introduce the * reference condition into equation (9.27) and develop an expression for $(s^* - s)/R$.

(b) Write a computer program for the expression developed in part (a) and compute a table of $(s^* - s)/R$ versus Mach number. Also include other entries of the Fanno table.

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

9.1. Sketch a Fanno line in the h–s plane. Include enough additional information as necessary to locate the sonic point and then identify the regions of subsonic and supersonic flow.

9.2. Fill in the blanks in Table CT9.2 to indicate whether the quantities *increase*, *decrease*, or *remain constant* in the case of Fanno flow.

Table CT9.2 Analysis of Fanno Flow

Property	Subsonic Regime	Supersonic Regime
Velocity		
Temperature		
Pressure		
Thrust function $(p + \rho V^2/g_c)$		

9.3. In the system shown in Figure CT9.3, the friction length of the duct is $f\,\Delta x/D = 12.40$ and the Mach number at the exit is 0.8. $A_3 = 1.5$ in² and $A_4 = 1.0$ in². What is the air pressure in the tank if the receiver is at 15 psia?

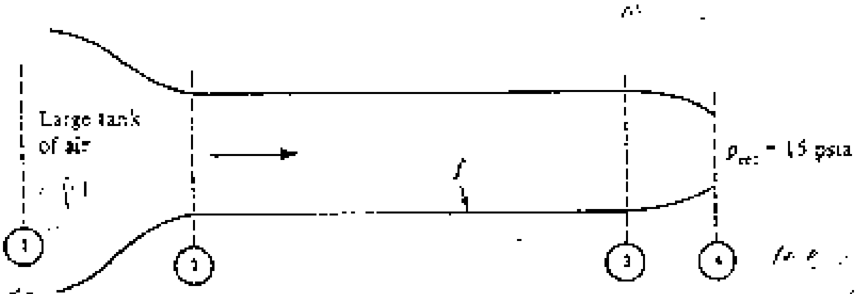


Figure CT9.3

9.4. Over what range of receiver pressures will normal shocks occur someplace within the system shown in Figure CT9.4? The area ratio of the nozzle is $A_3/A_2 = 2.403$ and the duct $f\,\Delta x/D = 0.30$.

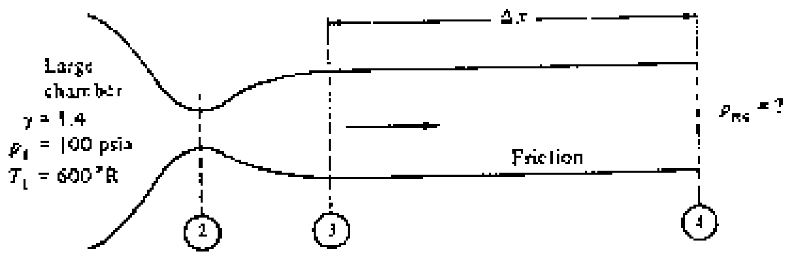


Figure CT9.4

9.5. There is no friction in the system shown in Figure CT9.5 except in the constant area ducts from 3 to 4 and from 6 to 7. Sketch the T–s diagram for the entire system.

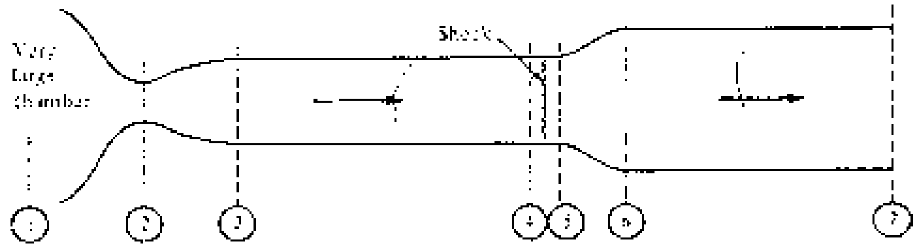


Figure CT9.5

9.6. Starting with the basic principles of continuity, energy, and so on, derive an expression for the property ratio p_2/p_1 in terms of Mach numbers and the specific heat ratio for Fanno flow with a perfect gas.

9.7. Work Problem 9.18.