Subject: Dynamics of Flight and Stability — **الموضوع : ديناميك طيران و استقرارية**

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Tutorial : 1 — **مناقشة : 1**

Experimental : — **عملي :**

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Flight Dynamics and Control
Dr.Emad N. Abdulwahab

1- Introduction to Aircraft Stability and Control

1.1 The Freedom of Motion of Aircraft-Basic Axes

Airplane *performance* is governed by *forces* along and perpendicular to the flight path. The translational motion of the airplane is a response to these forces. In contrast, airplane *stability and control* are governed by *moments* about the center of gravity, and the rotational motion of the plane is a response to these moments.

Figure 1 shows a rectangular right-handed coordinate system attached to the aircraft. The origin of the axes is at the aircraft's center of gravity. The x axis is along the fuselage, the y axis is along the wingspan, and the z axis points downward.

The aircraft's translational motion is given by the velocity components U, V, and W along the axes. Thus, the net velocity of the aircraft is the vector sum of these three velocity components. The rotational motion is given by the angular velocity components $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ about the x, y, and z axes.

In summary, the nomenclature associated with rotational motion is as follows:

- x axis: roll axis, L' = rolling moment, $\dot{\phi}$ = rolling velocity.
- y axis: pitch axis, M = pitching moment, $\dot{\theta}$ = pitching velocity.
- z axis: yaw axis, N = yawing moment, $\dot{\psi}$ = yawing velocity.

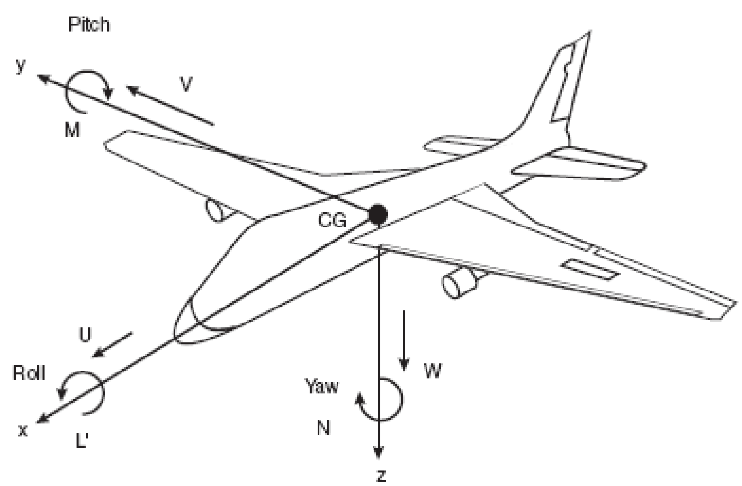
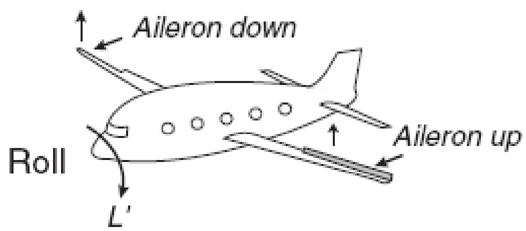


Fig.1 Definition of airplane's axes system

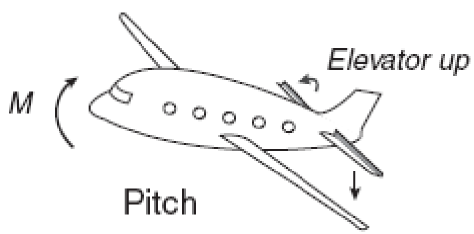
A classic airplane has three basic controls: *ailerons*, *elevator*, and *rudder*. They are designed to change and control the moments about the roll, pitch, and yaw axes. These control surfaces are flaplike surfaces that can be deflected back and forth at the command of the pilot.

A downward deflection of a control surface will increase the lift, since this makes the airfoil shape of the wing or tail "more bent downward" (in aeronautical jargon, it has a larger camber) and thus produces more lift. An increase or decrease of the deflection will change the moment and thus will result in a rotation about an axis.

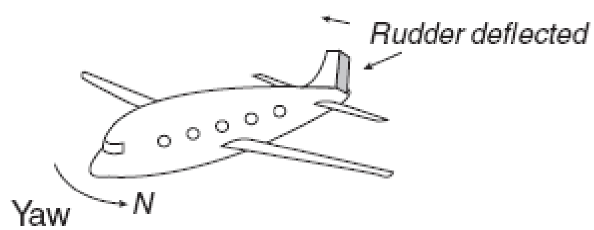
- *Rolling.* The ailerons control the roll or *lateral motion* and are therefore often called the lateral controls.



- *Pitching.* The elevator controls pitch or the *longitudinal motion* and thus is often called the longitudinal control.



- *Yawing.* The rudder controls yaw or the *directional motion* and thus is called the directional control.



2 | AIRPLANE STABILITY

Static Stability

Static stability can be visualized by a ball (or any object) on a surface. Initially the ball is in equilibrium. The ball is then displaced from the equilibrium position, and its initial behavior is observed.

- *Statically stable.* If the forces and moments on the body caused by a disturbance tend initially to return the body toward its equilibrium position, the body is *statically stable.*
- *Statically unstable.* If the forces and moments are such that the body continues to move away from its equilibrium position after being disturbed, the body is *statically unstable.*

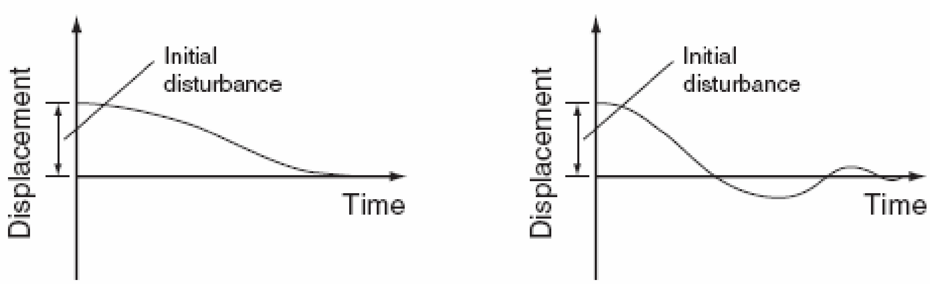
Dynamic Stability

Dynamic stability deals with the *time history* of the vehicle's motion after it initially responds to its static stability.

Consider an airplane flying at an angle of attack (AOA) α_e such that the moments about the center of gravity (cg) are zero. The aircraft is therefore in equilibrium at α_e and is said to be *trimmed,* and α_e is called the *trim angle of attack.*

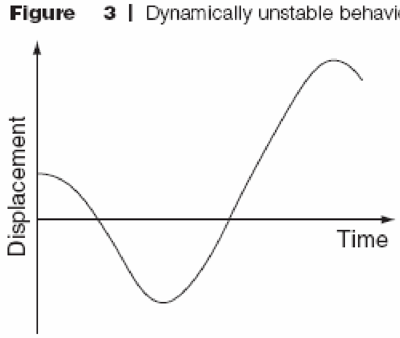
Now imagine that a wind gust disturbs the airplane and changes its angle of attack to some new value α. Hence, the plane was pitched through a displacement $\alpha - \alpha_e$. The plane's behavior could be as shown in Figure 2.

Figure 2 | Dynamically stable behavior.



It is important to note that static stability does not imply dynamic stability, as Figure 3 shows. The plane is dynamically unstable but still statically stable.

Figure 3 | Dynamically unstable behavior.

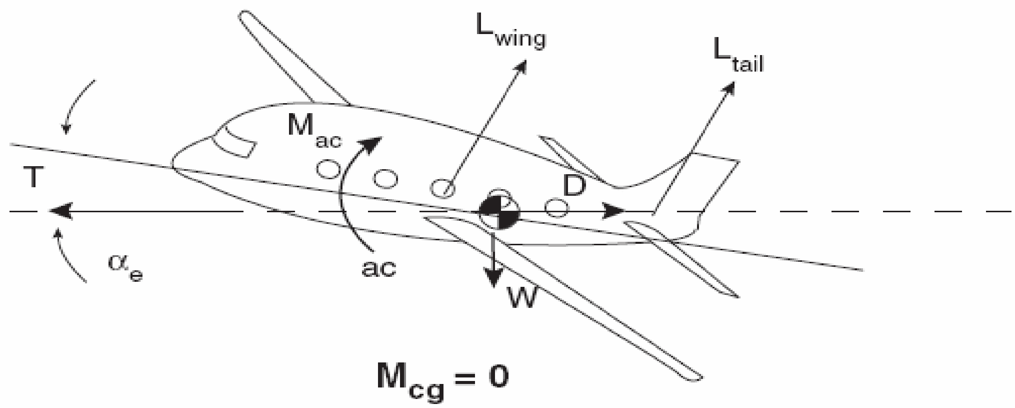


Moment on an Aircraft

Having looked at a wing only, we can now consider a complete airplane, as shown in Figure 4 In examining a whole aircraft, the pitching moment about the center of gravity (center of mass) is of interest. The moment coefficient about cg is defined analogous to the moment coefficient about the ac:

$$C_{M,cg} \equiv \frac{M_{cg}}{q_\infty S c}$$

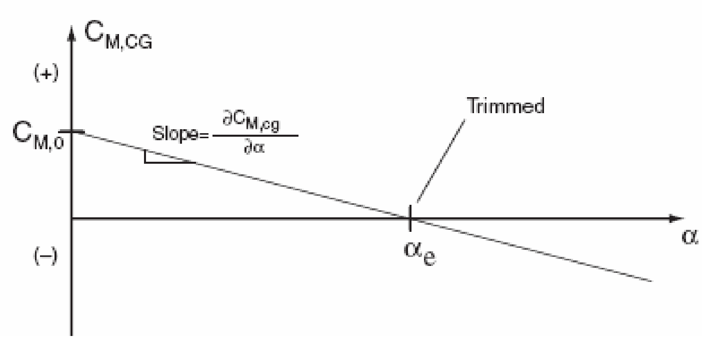
Figure 4 | Contributions to the moment acting about the center of gravity.



3- ATTAINING AIRCRAFT LONGITUDINAL STATIC STABILITY

Consider an airplane with fixed control surfaces. Wind tunnel testing may reveal the following behavior (see Figure 5 The plot is almost linear and shows the value of the $C_{M,\,cg}$ versus angle of attack α. The slope of the curve is $\partial C_{M,\,cg}/\partial\alpha$, and is sometimes denoted with the letter "a." (A partial derivative rather than a total derivative is used since the coefficient does not depend on α alone.) The value of $C_{M,\,cg}$ at an angle of attack equal to zero is denoted by $C_{M,\,0}$. The angle at which the moment coefficient is zero is, of course, the trim angle of attack.

Figure 5 | The moment coefficient about the center of gravity as a function of angle of attack for a longitudinally stable aircraft.



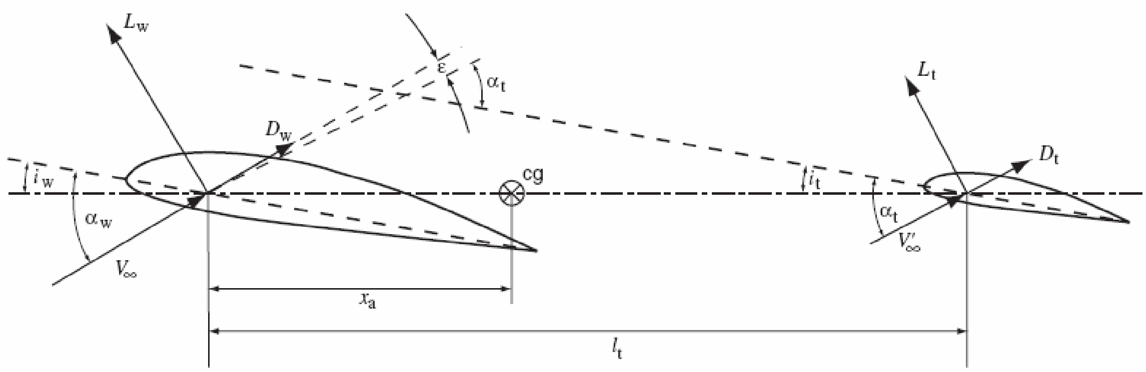
criteria for longitudinal static stability and balance as

$$\frac{\partial C_{M,\mathrm{cg}}}{\partial \alpha} < 0 \qquad \text{and} \qquad C_{M,0} > 0$$

Useful calculation and Example

Now consider an idealized wing-tail configuration in steady, level flight such as shown in Figure 6 . The wing and the tail are set at incidence angles i_w and i_t, respectively, with respect to the longitudinal aircraft axis. The relative wind V_∞ comes in at an angle α_w with respect to the wing. The relative wind V'_∞ comes in at an angle α_t with respect to the tail.

Figure 6 Geometry of a wing-tail combination



The tail angle of attack can be computed as follows:

$$\alpha_t = \alpha_w - \varepsilon + i_t - i_w$$

where ε is the downwash. Its value can be computed from the following equation:

$$\varepsilon = \varepsilon_\alpha \alpha_w$$

where $\varepsilon_\alpha \approx 0.3$ to 0.5. This allows us to rewrite the equation for α_t as

$$\alpha_t = (1 - \varepsilon_\alpha)\alpha_w + i_t - i_w$$

When the airplane is trimmed, the moments about the cg are zero. From this we can find the *trim condition.* The coefficients of lift for the wing and for the tail and the angle of attack can be defined as the product of the slope of the moment coefficient.

$$C_{L,w} = a_w \alpha_w$$

$$C_{L,t} = a_t[\alpha_w(1 - \varepsilon_\alpha) + i_t - i_w]$$

After several steps and a few simplifying assumptions, the trim condition can be

$$\frac{C_{M,\mathrm{cg}}}{a_w} = \left[\frac{x_a}{c} - \frac{A_t l_t a_t}{A_w c a_w}(1 - \varepsilon_\alpha)\right]\alpha_w + \left[\frac{C_{M,\mathrm{ac}}}{a_w} + \frac{A_t l_t a_t}{A_w c a_w}(i_w - i_t)\right] = 0$$

The first term in brackets is the sensitivity to the angle of attack. Consider again the situation where a wind gust disturbs a plane flying in trim and causes it to pitch up. For the plane to be stable, the moment coefficient $C_{M,\,cg}$ (which was 0) has to be negative in order for the plane to pitch down. For stability we can then write

$$\frac{\partial C_{M,cg}}{\partial \alpha} < 0 \qquad \text{or} \qquad \frac{x_a}{c} < \frac{A_t l_t a_t}{A_w c a_w}(1 - \varepsilon_\alpha)$$

In designing your LTA vehicle, you can place the cg (and thus set the value of x_a) such that the vehicle is stable, using the above inequality. In the limiting case, when the cg is as far back as possible,

$$\left(\frac{x_a}{c}\right)_{\max} = \frac{A_t l_t a_t}{A_w c a_w}(1 - \varepsilon_\alpha)$$

The cg is said to be at the *neutral point.*

The trim angle of attack can be written as

$$(\alpha_w)_{\text{trim}} = \frac{C_{M,\text{ac}}/a_w + [A_t l_t a_t/(A_w c a_w)](i_w - i_t)}{(x_a/c)_{\max} - x_a/c}$$

For lift generation, $\alpha_w < 0$, $C_{M,\,\text{ac}} < 0$, (usually), and $i_w - i_t > 0$. Also, L_t may have to be negative.

4 LATERAL STATIC STABILITY

Lateral static stability is concerned with the ability of the aircraft to maintain wings level equilibrium in the roll sense. Wing dihedral is the most visible parameter which confers lateral static stability on an aircraft although there are many other contributions, some of which are destabilising. Since all aircraft are required to fly with their wings level in the steady trim state lateral static stability is designed in from the outset. Dihedral is the easiest parameter to adjust in the design process in order to "tune" the degree of stability to an acceptable level. Remember that too much lateral static stability will result in an aircraft that is reluctant to manoeuvre laterally, so it is important to obtain the correct degree of stability.

The effect of dihedral as a means for providing lateral static stability is easily appreciated by considering the situation depicted in Fig. 7 Following a small lateral disturbance in roll ϕ the aircraft will commence to slide "downhill" sideways with a sideslip velocity v. Consider the resulting change in the aerodynamic conditions on the leading wing which has dihedral angle Γ. Since the wing has dihedral the sideslip velocity has a small component v' resolved perpendicular to the plane of the wing panel where

$$v' = v \sin \Gamma$$

The velocity component v' combines with the axial velocity component U_e to increase the angle of attack of the leading wing by α'. Since $v' << U_e$ the change in angle of attack α' is small and the total disturbed axial velocity component $U \cong U_e$. The

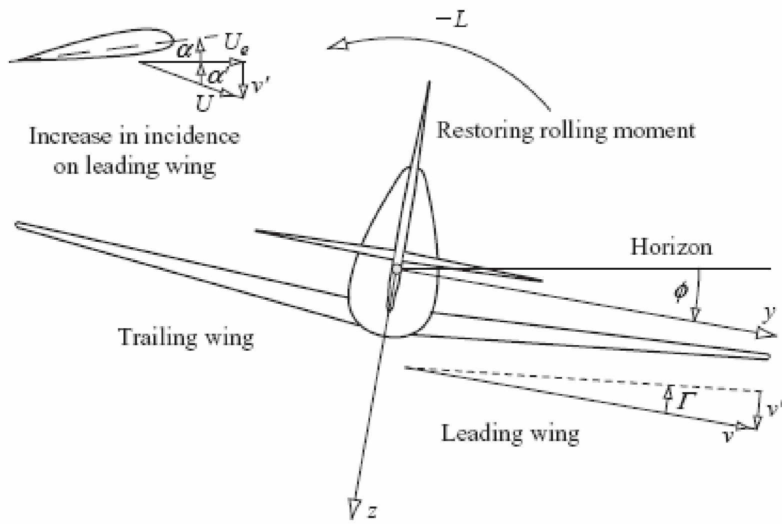


Figure 7 *Dihedral effect.*

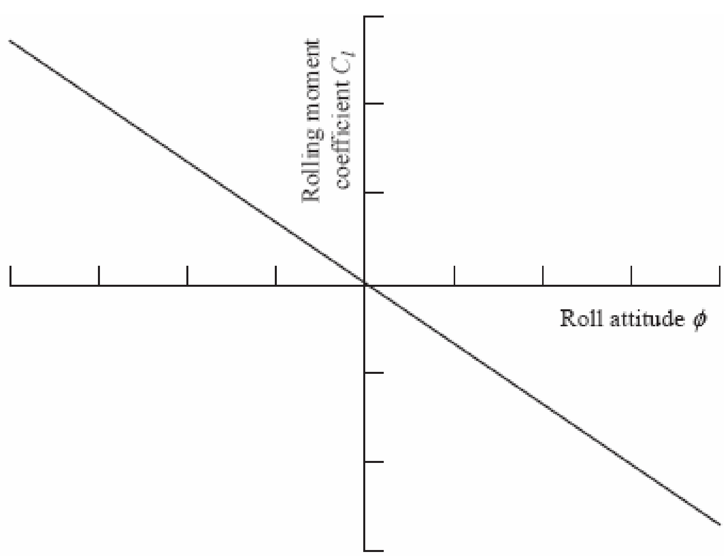


Figure 8 *C_l–ϕ plot for a stable aircraft.*

increase in angle of attack on the leading wing gives rise to an increase in lift which in turn gives rise to a restoring rolling moment $-L$. The corresponding aerodynamic change on the wing trailing into the sideslip results in a small decrease in lift which also produces a restoring rolling moment. The net effect therefore is to create a negative rolling moment which causes the aircraft to recover its zero sideslip wings level equilibrium. Thus, the condition for an aircraft to be laterally stable is that the rolling moment resulting from a positive disturbance in roll attitude must be negative, or in mathematical terms:

$$\frac{dC_l}{d\phi} < 0$$

where C_l is the rolling moment coefficient. This is shown graphically in Fig. 8

5 DIRECTIONAL STATIC STABILITY

Directional static stability is concerned with the ability of the aircraft to yaw or *weathercock* into wind in order to maintain directional equilibrium. Since all aircraft are required to fly with zero sideslip in the yaw sense, positive directional stability is designed in from the outset. The fin is the most visible contributor to directional static stability although, as in the case of lateral stability, there are many other contributions, some of which are destabilising.

The yawing moment is stabilising since it causes the aircraft to yaw to the right until the sideslip angle is reduced to zero. Thus, the condition for an aircraft to be directionally stable is readily established and is

$$\frac{dC_n}{d\psi} < 0 \quad \text{or, equivalently,} \quad \frac{dC_n}{d\beta} > 0$$

where C_n is the yawing moment coefficient.

6 Perturbation variables

The motion of the aircraft is described in terms of force, moment, linear and angular velocities and attitude resolved into components with respect to the chosen aircraft fixed axis system. For convenience it is preferable to assume a generalised *body axis* system in the first instance. Thus initially, the aircraft is assumed to be in steady rectilinear, but not necessarily level, flight when the body incidence is α_e and the steady velocity V_0 resolves into components U_e, V_e and W_e as indicated in Fig. 9 In steady non-accelerating flight the aircraft is in equilibrium and the forces and moments acting on the airframe are in balance and sum to zero. This initial condition is usually referred to as *trimmed equilibrium*.

Table .1 shows the summery of motion variables

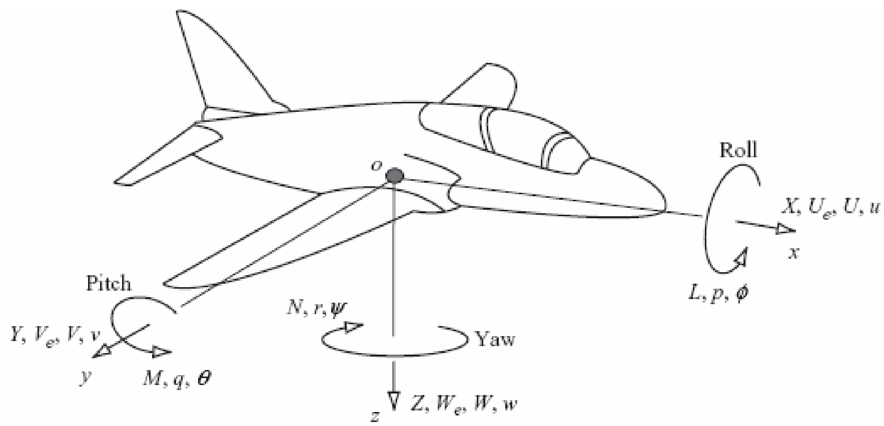


Figure 9 *Motion variables notation.*

Table 1 *Summary of motion variables*

	Trimmed equilibrium			*Perturbed*		
Aircraft axis	*ox*	*oy*	*oz*	*ox*	*oy*	*oz*
Force	0	0	0	X	Y	Z
Moment	0	0	0	L	M	N
Linear velocity	U_e	V_e	W_e	U	V	W
Angular velocity	0	0	0	p	q	r
Attitude	0	θ_e	0	ϕ	θ	ψ

7 THE EQUATIONS OF MOTION OF A RIGID SYMMETRIC AIRCRAFT

The object is to realise Newton's second law of motion for each of the six degrees of freedom which simply states that,

$$\text{mass} \times \text{acceleration} = \text{disturbing force} \qquad (1)$$

The components of velocity and force along the axes ox, oy and oz are denoted (U, V, W) and (X, Y, Z) respectively. The components of angular velocity and moment about the same axes are denoted (p, q, r) and (L, M, N) respectively. The point p is an arbitrarily chosen point within the body with coordinates (x, y, z). The local components of velocity and acceleration at p relative to the body axes are denoted (u, v, w) and (a_x, a_y, a_z) respectively.

The velocity components at $p(x, y, z)$ relative to o are given by

$$\begin{aligned} u &= \dot{x} - ry + qz \\ v &= \dot{y} - pz + rx \\ w &= \dot{z} - qx + py \end{aligned} \qquad (2)$$

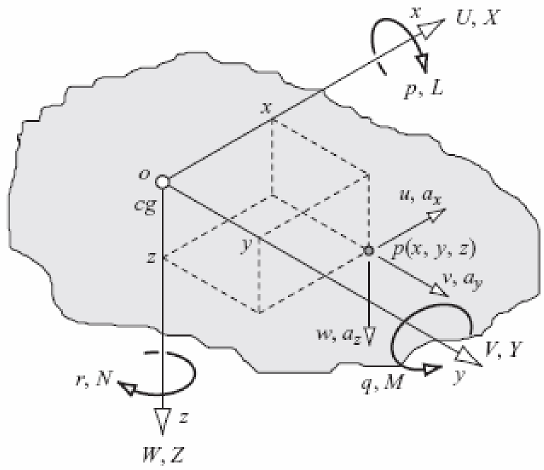


Figure 1 *Motion referred to generalised body axes.*

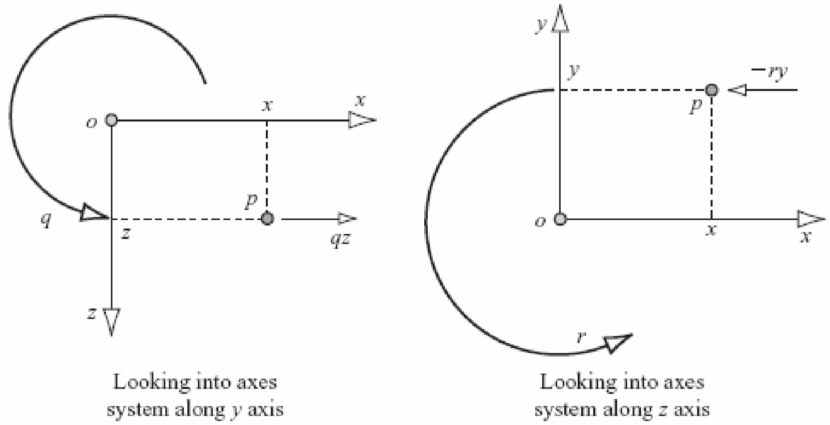


Figure 2 *Velocity terms due to rotary motion.*

It will be seen that the velocity components each comprise a linear term and two additional terms due to rotary motion. The origin of the terms due to rotary motion in the component u, for example, is illustrated in Fig. 2. Both $-ry$ and qz represent *tangential velocity* components acting along a line through $p(x, y, z)$ parallel to the ox axis. The rotary terms in the remaining two components of velocity are determined in a similar way. Now, since the generalised body shown in Fig. 1 represents the aircraft which is assumed to be rigid then

$$\dot{x} = \dot{y} = \dot{z} = 0 \tag{3}$$

and equations (2) reduce to

$$\begin{aligned} u &= qz - ry \\ v &= rx - pz \\ w &= py - qx \end{aligned} \tag{4}$$

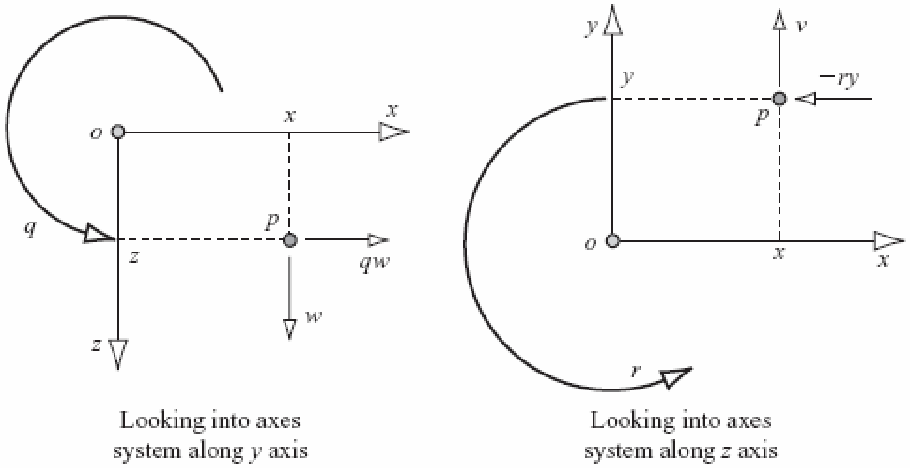


Figure 3 *Acceleration terms due to rotary motion*

The corresponding components of acceleration at $p(x, y, z)$ relative to o are given by

$$\begin{aligned} a_x &= \dot{u} - rv + qw \\ a_y &= \dot{v} - pw + ru \\ a_z &= \dot{w} - qu + pv \end{aligned} \quad (\cdot\ 5)$$

By superimposing the velocity components of the cg (U, V, W) on to the local velocity components (u, v, w) the absolute, or inertial, velocity components (u', v', w') of the point $p(x, y, z)$ are obtained. Thus

$$\begin{aligned} u' &= U + u = U - ry + qz \\ v' &= V + v = V - pz + rx \\ w' &= W + w = W - qx + py \end{aligned} \quad .6)$$

where the expressions for (u, v, w) are substituted from equations (4). Similarly, the components of inertial acceleration (a'_x, a'_y, a'_z) at the point $p(x, y, z)$ are obtained simply by substituting the expressions for (u', v', w'), equations (6), in place of (u, v, w) in equations (5). Whence

$$\begin{aligned} a'_x &= \dot{u}' - rv' + qw' \\ a'_y &= \dot{v}' - pw' + ru' \\ a'_z &= \dot{w}' - qu' + pv' \end{aligned} \quad 7)$$

Differentiate equations (6) with respect to time and note that since a rigid body is assumed equation (3) applies then

$$\begin{aligned} \dot{u}' &= \dot{U} - \dot{r}y + \dot{q}z \\ \dot{v}' &= \dot{V} - \dot{p}z + \dot{r}x \\ \dot{w}' &= \dot{W} - \dot{q}x + \dot{p}y \end{aligned} \qquad (\;8)$$

Thus, by substituting from equations (6) and (8) into equations (7) the inertial acceleration components of the point $p(x, y, z)$ in the rigid body are obtained which, after some rearrangement, may be written,

$$\begin{aligned} a'_x &= \dot{U} - rV + qW - x(q^2 + r^2) + y(pq - \dot{r}) + z(pr + \dot{q}) \\ a'_y &= \dot{V} - pW + rU + x(pq + \dot{r}) - y(p^2 + r^2) + z(qr - \dot{p}) \\ a'_z &= \dot{W} - qU + pV + x(pr - \dot{q}) + y(qr + \dot{p}) - z(p^2 + q^2) \end{aligned} \qquad 9)$$

Example

A pilot in an aerobatic aircraft performs a loop in 20 s at a steady velocity of 100 m/s. His seat is located 5 m ahead of, and 1 m above the *cg*. What total normal load factor does he experience at the top and at the bottom of the loop?

Assuming the motion is in the plane of symmetry only, then $V = \dot{p} = p = r = 0$ and since the pilot's seat is also in the plane of symmetry $y = 0$ and the expression for normal acceleration is, from equations (9):

$$a'_z = \dot{W} - qU + x\dot{q} - zq^2$$

Since the manoeuvre is steady, the further simplification can be made $\dot{W} = \dot{q} = 0$ and the expression for the normal acceleration at the pilots seat reduces to

$$a'_z = -qU - zq^2$$

Now,

$$q = \frac{2\pi}{20} = 0.314 \text{ rad/s}$$
$$U = 100 \text{ m/s}$$
$$x = 5 \text{ m}$$
$$z = -1 \text{ m (above } cg \text{ hence negative)}$$

whence $a'_z = -31.30 \text{ m/s}^2$. Now, by definition, the corresponding incremental normal load factor due to the manoeuvre is given by

$$n' = \frac{-a'_z}{g} = \frac{31.30}{9.81} = 3.19$$

The total normal load factor n comprises that due to the manoeuvre n' plus that due to gravity n_g. At the top of the loop $n_g = -1$, thus the total normal load factor is a given by

$$n = n' + n_g = 3.19 - 1 = 2.19$$

and at the bottom of the loop $n_g = 1$ and in this case the total normal load factor is given by

$$n = n' + n_g = 3.19 + 1 = 4.19$$

It is interesting to note that the normal acceleration measured by an accelerometer mounted at the pilots seat corresponds with the total normal load factor. The accelerometer would therefore give the following readings:

at the top of the loop $a_z = ng = 2.19 \times 9.81 = 21.48 \text{ m/s}^2$
at the bottom of the loop $a_z = ng = 4.19 \times 9.81 = 41.10 \text{ m/s}^2$

Therefore the resultant components of total force acting on the rigid body are given by

$$m(\dot{U} - rV + qW) = X$$
$$m(\dot{V} - pW + rU) = Y$$
$$m(\dot{W} - qU + pV) = Z$$

where m is the total mass of the body.

Thus the moment equations simplify to the following:

$$I_x\dot{p} - (I_y - I_z)qr - I_{xz}(pq + \dot{r}) = L$$
$$I_y\dot{q} + (I_x - I_z)pr + I_{xz}(p^2 - r^2) = M$$
$$I_z\dot{r} - (I_x - I_y)pq + I_{xz}(qr - \dot{p}) = N$$

Bringing together the total force and moment equations they may be written to include these contributions as follow

$$\begin{aligned} m(\dot{U} - rV + qW) &= X_a + X_g + X_c + X_p + X_d \\ m(\dot{V} - pW + rU) &= Y_a + Y_g + Y_c + Y_p + Y_d \\ m(\dot{W} - qU + pV) &= Z_a + Z_g + Z_c + Z_p + Z_d \end{aligned} \qquad 10$$

$$\begin{aligned} I_x\dot{p} - (I_y - I_z)qr - I_{xz}(pq + \dot{r}) &= L_a + L_g + L_c + L_p + L_d \\ I_y\dot{q} + (I_x - I_z)pr + I_{xz}(p^2 - r^2) &= M_a + M_g + M_c + M_p + M_d \\ I_z\dot{r} - (I_x - I_y)pq + I_{xz}(qr - \dot{p}) &= N_a + N_g + N_c + N_p + N_d \end{aligned} \qquad 11$$

After linearization the equation of motion in terms of stability derivatives can be written as follow

$$\begin{aligned} m\dot{u} - \mathring{X}_u u - \mathring{X}_{\dot{w}}\dot{w} - \mathring{X}_w w - \left(\mathring{X}_q - mW_e\right) q + mg\theta\cos\theta_e &= \mathring{X}_\eta \eta + \mathring{X}_\tau \tau \\ -\mathring{Z}_u u + \left(m - \mathring{Z}_{\dot{w}}\right)\dot{w} - \mathring{Z}_w w - \left(\mathring{Z}_q + mU_e\right) q + mg\theta\sin\theta_e &= \mathring{Z}_\eta \eta + \mathring{Z}_\tau \tau \\ -\mathring{M}_u u - \mathring{M}_{\dot{w}}\dot{w} - \mathring{M}_w w + I_y\dot{q} - \mathring{M}_q q &= \mathring{M}_\eta \eta + \mathring{M}_\tau \tau \end{aligned} \qquad 12$$

Stability derivatives used in equations of equations.

$$X_u = \frac{\overset{\circ}{X}_u}{\frac{1}{2}\rho V_0 S} = -2C_D - V_0\frac{\partial C_D}{\partial V} + \frac{1}{\frac{1}{2}\rho V_0 S}\frac{\partial \tau}{\partial V}$$

$$X_w = \frac{\overset{\circ}{X}_w}{\frac{1}{2}\rho V_0 S} = \left(C_L - \frac{\partial C_D}{\partial \alpha}\right)$$

$$Z_w = \frac{\overset{\circ}{Z}_w}{\frac{1}{2}\rho V_0 S} = -\left(\frac{\partial C_L}{\partial \alpha} + C_D\right)$$

$$Z_u = \frac{\overset{\circ}{Z}_u}{\frac{1}{2}\rho V_0 S} = -2C_L - V_0\frac{\partial C_L}{\partial V}$$

$$M_u = \frac{\overset{\circ}{M}_u}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = V_0\frac{\partial C_m}{\partial V}$$

$$M_w = \frac{\overset{\circ}{M}_w}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = \frac{\partial C_m}{\partial \alpha}$$

$$X_q = \frac{\overset{\circ}{X}_q}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = -\overline{V}_T\frac{\partial C_{D_T}}{\partial \alpha_T}$$

where the *tail volume ratio* is given by

$$\overline{V}_T = \frac{S_T l_T}{S\overline{\overline{c}}}$$

$$Z_q = \frac{\overset{\circ}{Z}_q}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}} = -\overline{V}_T a_1$$

$$M_q = \frac{\overset{\circ}{M}_q}{\frac{1}{2}\rho V_0 S\overline{\overline{c}}^2} = -\overline{V}_T\frac{l_T}{\overline{\overline{c}}}a_1 \equiv \frac{l_T}{\overline{\overline{c}}}Z_q$$

$$X_{\dot{w}} = \frac{\overset{\circ}{X}_{\dot{w}}}{\frac{1}{2}\rho S\overline{\overline{c}}} = -\overline{V}_T\frac{\partial C_{D_T}}{\partial \alpha_T}\frac{d\varepsilon}{d\alpha} \equiv X_q\frac{d\varepsilon}{d\alpha}$$

$$Z_{\dot{w}} = \frac{\overset{\circ}{Z}_{\dot{w}}}{\frac{1}{2}\rho S\overline{\overline{c}}} = -\overline{V}_T a_1 \frac{d\varepsilon}{d\alpha} \equiv Z_q \frac{d\varepsilon}{d\alpha}$$

$$M_{\dot{w}} = \frac{\overset{\circ}{M}_{\dot{w}}}{\frac{1}{2}\rho S\overline{\overline{c}}^2} = -\overline{V}_T \frac{l_T}{\overline{\overline{c}}} a_1 \frac{d\varepsilon}{d\alpha} \equiv M_q \frac{d\varepsilon}{d\alpha}$$

$$X_\eta = \frac{\overset{\circ}{X}_\eta}{\frac{1}{2}\rho V_0^2 S} = -2\frac{S_T}{S} k_T C_{L_T} a_2$$

$$Z_\eta = \frac{\overset{\circ}{Z}_\eta}{\frac{1}{2}\rho V_0^2 S} = -\frac{S_T}{S} a_2$$

$$M_\eta = \frac{\overset{\circ}{M}_\eta}{\frac{1}{2}\rho V_0^2 S\overline{\overline{c}}} = -\frac{S_T l_T}{S\overline{\overline{c}}} a_2 = -\overline{V}_T a_2$$

Example

Longitudinal derivative and other data for the McDonnell F-4C Phantom aeroplane was obtained from Heffley and Jewell (1972) for a flight condition of Mach 0.6 at an altitude of 35000 ft. The original data is presented in imperial units and in a format preferred in the USA. Normally, it is advisable to work with the equations of motion and the data in the format and units as given. Otherwise, conversion to another format can be tedious in the extreme and is easily subject to error. However, for the purposes of illustration, the derivative data has been converted to a form compatible with the equations developed above and the units have been changed to those of the more familiar SI system. The data is quite typical, it would normally be supplied in this, or similar, form by aerodynamicists and as such it represents the starting point in any flight dynamics analysis:

Flight path angle	$\gamma_e = 0°$	Air density	$\rho = 0.3809$ kg/m^3
Body incidence	$\alpha_e = 9.4°$	Wing area	$S = 49.239$ m^2
Velocity	$V_0 = 178$ m/s	Mean aerodynamic chord	$\overline{\overline{c}} = 4.889$ m
Mass	$m = 17642$ kg		
Pitch moment of inertia	$I_y = 165669$ kgm^2	Acceleration due to gravity	$g = 9.81$ m/s^2

Since the flight path angle $\gamma_e = 0$ and the body incidence α_e is non-zero it may be deduced that the following derivatives are referred to a body axes system and that $\theta_e \equiv \alpha_e$. The dimensionless longitudinal derivatives are given and any missing aerodynamic derivatives must be assumed insignificant, and hence zero. On the other hand, missing control derivatives may not be assumed insignificant although their absence will prohibit analysis of response to those controls:

$X_u = 0.0076$	$Z_u = -0.7273$	$M_u = 0.0340$
$X_w = 0.0483$	$Z_w = -3.1245$	$M_w = -0.2169$
$X_{\dot{w}} = 0$	$Z_{\dot{w}} = -0.3997$	$M_{\dot{w}} = -0.5910$
$X_q = 0$	$Z_q = -1.2109$	$M_q = -1.2732$
$X_\eta = 0.0618$	$Z_\eta = -0.3741$	$M_\eta = -0.5581$

Equations (12) are compatible with the data although the dimensional derivatives must first be calculated according to the definitions given . Thus the dimensional longitudinal equations of motion, referred to body axes, are obtained by substituting the appropriate values into equations 12 to give

$$17642\dot{u} - 12.67u - 80.62w + 512852.94q + 170744.06\theta = 18362.32\eta$$

$$1214.01u + 17660.33\dot{w} + 5215.44w - 3088229.7q + 28266.507\theta = -111154.41\eta$$

$$-277.47u + 132.47\dot{w} + 1770.07w + 165669\dot{q} + 50798.03q = -810886.19\eta$$

where $W_e = V_0 \sin\theta_e = 29.07$ m/s and $U_e = V_0 \cos\theta_e = 175.61$ m/s. Note that angular variables in the equations of motion have radian units. Clearly, when written like this the equations of motion are unwieldy. The equations can be simplified a little by dividing through by the mass or inertia as appropriate. Thus the first equation is divided by 17642, the second equation by 17660.33 and the third equation by 165669. After some rearrangement the following rather more convenient version is obtained:

$$\dot{u} = 0.0007u + 0.0046w - 29.0700q - 9.6783\theta + 1.0408\eta$$

$$\dot{w} = -0.0687u - 0.2953w + 174.8680q - 1.6000\theta - 6.2940\eta$$

$$\dot{q} + 0.0008\dot{w} = 0.0017u - 0.0107w - 0.3066q - 4.8946\eta$$