References

Automatic Control Engineering -*Francis H. Raven, McGraw-Hill 1978.*

 Modern Control Engineering -*Katsuhiko Ogata, Prentice Hall 2010.*

 Automatic Control Systems -*Farid Golnaraghi, Benjamin C. Kuo, Wiley 2010.*

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Lecture One

1-1 Introduction

control system is essential in any field of engineering and science and it is defined as a combination of components that act together in such a way that the overall system behaves automatically in a pre-specified desired manner.

Control system is means by which we can get the desired output. In other words, we can change the system functioning as per the requirements.

However, the basic components of any control system are (Figure 1): objectives of control or generally called the inputs; the results are also called outputs; and the control system which is responsible for controlling the outputs in some prescribed manner.

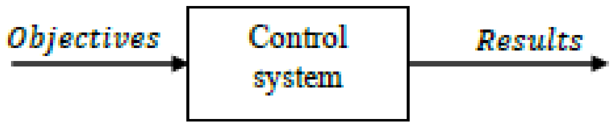


Figure 1: Basic components of a control system

In studying control engineering, we need to define additional terms that are necessary to describe control systems.

<u>Plants:</u> *a plant may be a piece of equipment, perhaps just a set of machine parts functioning together, the purpose of which is to perform a particular operation. In this lecture, we shall call any physical object to be controlled (such as a mechanical device, a heating furnace, a chemical reactor, or a spacecraft) a plant.*

<u>Process:</u> *a process to be a natural, progressively continuing operation or development marked by a series of gradual changes that succeed one another in a relatively fixed way and lead toward a particular result or end; or an artificial or voluntary, progressively continuing operation that consists of a series of controlled actions or movements systematically directed toward a particular result or end.*

<u>Systems:</u> *a system is a combination of components that act together and perform a certain objective. A system is not limited to physical ones. The concept of the system can be applied to abstract, dynamic phenomena such as those encountered in* *economics. The word system should, therefore, be interpreted to imply physical, biological, economic, and the like, systems.*

Disturbances: *a disturbance is a signal that tends to adversely affect the value of the output of a system. If a disturbance is generated within the system, it is called internal, while an external disturbance is generated outside the system and is an input.*

Feedback Control: *Feedback control refers to an operation that, in the presence of disturbances, tends to reduce the difference between the output of a system and some reference input and does so on the basis of this difference. Here only unpredictable disturbances are so specified, since predictable or known disturbances can always be compensated for within the system.*

Examples of Control Systems

Here we shall present several examples of control systems:

Speed Control System: *The basic principle of a Watt's speed governor for an engine is illustrated in the schematic diagram of Figure 1-2. The amount of fuel admitted to the engine is adjusted according to the difference between the desired and the actual engine speeds.*

The sequence of actions may be stated as follows: The speed governor is adjusted such that, at the desired speed, no pressured oil will flow into either side of the power cylinder. If the actual speed drops below the desired value due to disturbance, then the decrease in the centrifugal force of the speed governor causes the control valve to move downward, supplying more fuel, and the speed of the engine increases until the desired value is reached.

On the other hand, if the speed of the engine increases above the desired value, then the increase in the centrifugal force of the governor causes the control valve to move upward. This decreases the supply of fuel, and the speed of the engine decreases until the desired value is reached.

In this speed control system, the plant (controlled system) is the engine and the controlled variable is the speed of the engine. The difference between the desired

speed and the actual speed is the error signal. The control signal (the amount of fuel) to be applied to the plant (engine) is the actuating signal. The external input to disturb the controlled variable is the disturbance. An unexpected change in the load is a disturbance.

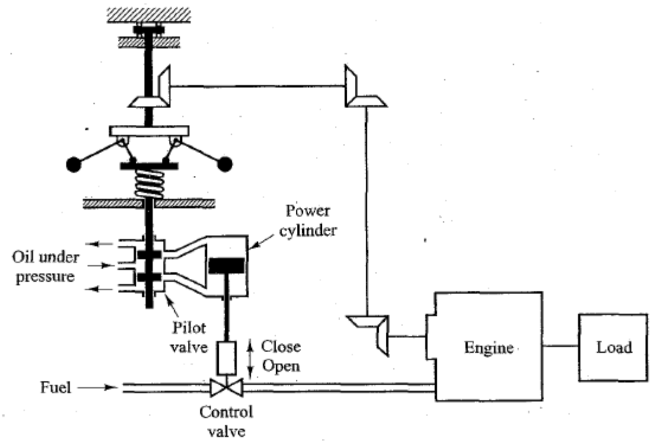


Figure 2: Speed Control System

Temperature Control System: *Figure 3 shows a schematic diagram of temperature control of an electric furnace. The temperature in the electric furnace is measured by a thermometer, which is an analog device. The analog temperature is converted to a digital temperature by an A/D converter. The digital temperature is fed to a controller through an interface. This digital temperature is compared with the programmed input temperature, and if there is any discrepancy (error), the controller sends out a signal to the heater, through an interface, amplifier, and relay, to bring the furnace temperature to a desired value.*

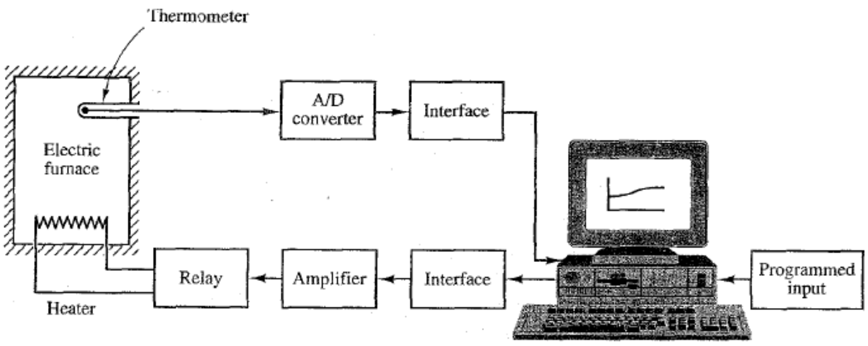
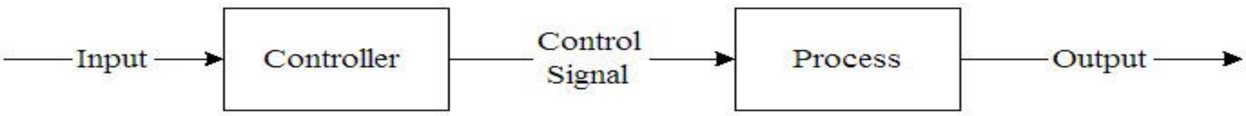


Figure 3: Temperature Control System

Types of Control System

a. Open loop control system

Is a system in which the output has no effect on the control action, also referred to as non-feedback system, is a type of continuous control system in which the output has no influence or effect on the control action of the input signal. In other words, in an open-loop control system the output is neither measured nor "fed back" for comparison with the input. Therefore, an open-loop system is expected to faithfully follow its input command or set point regardless of the final result. Also, an open-loop system has no knowledge of the output condition so cannot self-correct any errors it could make when the preset value drifts, even if this results in large deviations from the preset value.



Open Loop System

- **Advantages**

1. Simple and economic
2. No stability problem

- **Disadvantages**

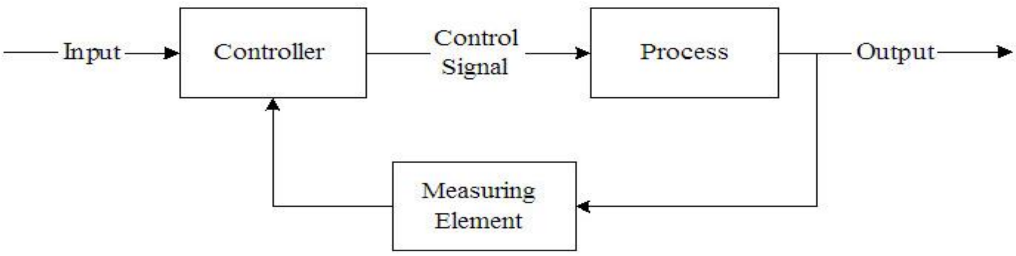
1. Inaccurate
2. Affected by system parameter variation and external noise it is insensitive to disturbances and unable to correct for these disturbances

Examples

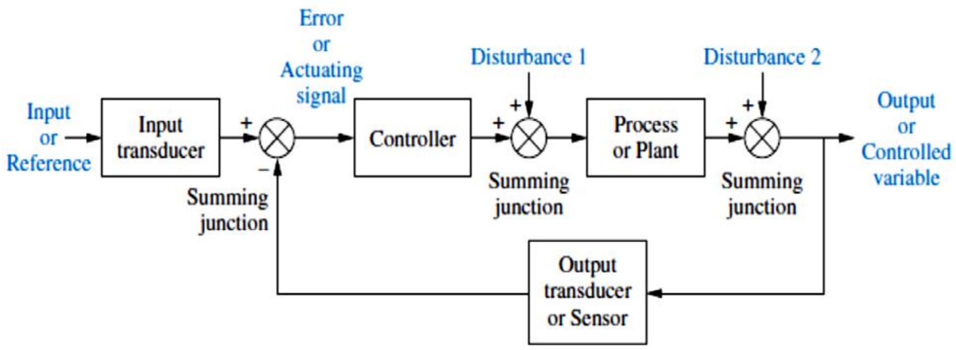
1. Traffic light controller
2. Electric washing machir
3. Bread toaster

b- Closed Loop (Feedback) Control System

a system that maintains a set relationship between the output and the reference input by comparing them and using the difference as a means of control. The main disadvantages of open-loop systems are insensitive to disturbances and unable to correct for these disturbances. Thus, to overcome these, closed-loop systems were introduced (Figure 5). Here, an **output transducer** (or **sensor**) is added to the system to measure the output response and convert it into the form that is utilized by the controller. An example would be a room temperature control system.



Closed Loop System

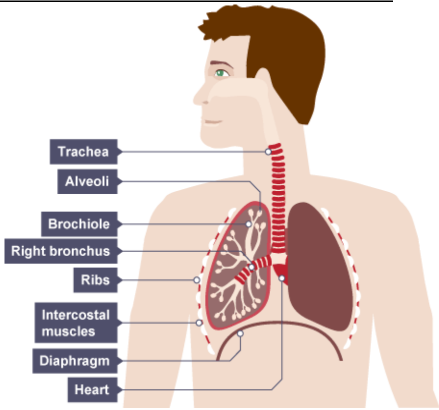


- **Advantages**
- Accurate
- Reduced effect of parameter variation

- **Disadvantages**
- The system is complex and costly
- Reduced the gain with negative feedback

Examples

1. Electric iron
2. DC motor speed control
3. Human respiratory system
4. Autopilot system



Example:

Temperature Control System, Figure 6 shows a schematic diagram of temperature control of an electric furnace.

A thermometer is used to measure the temperature in the electric furnace. The analogue temperature is converted to a digital temperature by an A/D (analogue to digital) converter. The digital temperature is fed to a controller through an interface. This digital temperature is compared with the programmed input temperature, and if there is any error, the controller sends out a signal to the heater, through an interface, amplifier, and relay, to bring the furnace temperature to a desired value.

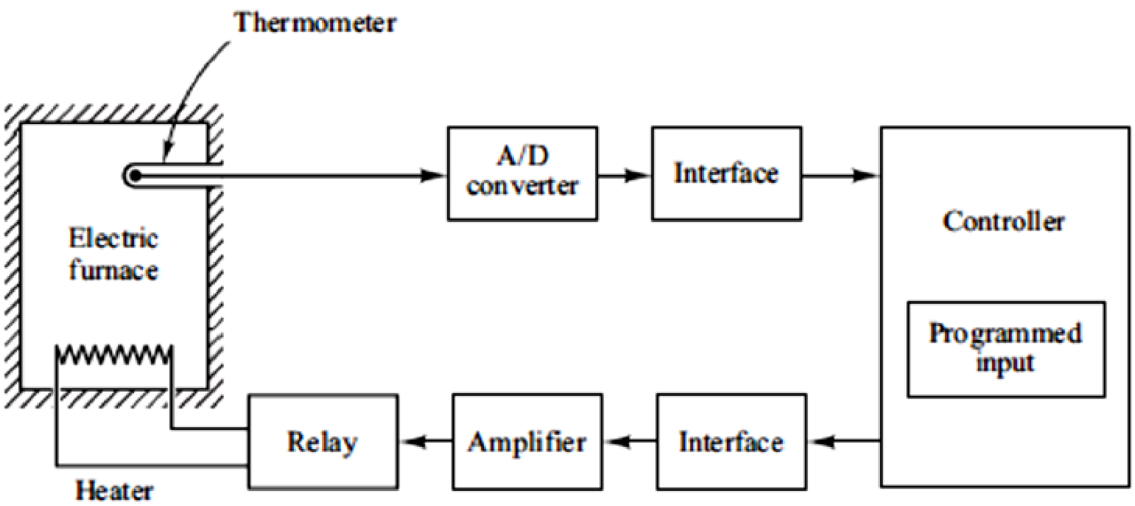


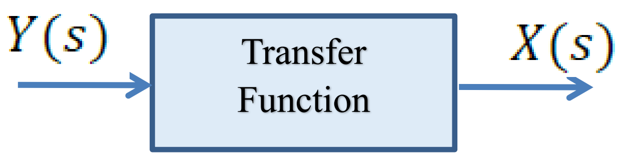
Figure 6: Temperature control system

1-2 Transfer Function

Generally, if the inputs and outputs of systems are considered as functions of time, then the relationship between the output and input is given by a differential equation. However, in order to make the control problem easy, a simpler relationship than a differential equation giving the relationship between input and output for a system is needed. To overcome this problem, the differential equations have to be transformed into a more convenient form by using the **Laplace transforms**. Transfer function, on the other hand, is used to relate the input Y(s) and output X(s) of a system. Thus, when we are working with inputs and outputs described as functions of S the transfer function is defined as

$$Transfer\ Function\ (T.F) = \frac{Output}{Input} = \frac{X(s)}{Y(s)}$$

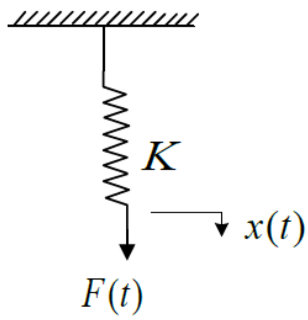
The transfer function can be represented by a block diagram with X(s) the input, Y(s) the output and the transfer function as the operator in the box that converts the input to the output, as shown in the figure below.

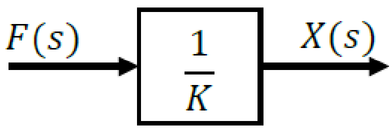


$$F(t) = K\ x(t)$$

$$F(s) = K\ X(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{K}$$





1-3 Laplace transform

The Laplace transform is a well-established mathematical technique for solving differential equations

$$X(s) = \mathscr{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt$$

Example: Find Laplace transform of x(t) =1

$$X(s) = \mathscr{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt$$

For $x(t) = \mathbf{1}$

$$X(s) = \mathscr{L}\ \mathbf{1} = \int_0^\infty e^{-st}dt$$

$$= -\frac{1}{s}\ e^{-st}\Big]_0^\infty$$

$$-\frac{1}{s}\ e^{-s(\infty)} + \frac{1}{s}\ e^{0}$$

$$= \frac{1}{s}$$

$$\mathscr{L}\ \boldsymbol{X}(\mathbf{t}) = \boldsymbol{X}(\boldsymbol{s})$$

$$\mathscr{L}\ \dot{\boldsymbol{X}}(\mathbf{t}) = \boldsymbol{S}\boldsymbol{X}(\boldsymbol{s}) - \boldsymbol{X}_{\boldsymbol{t}=\mathbf{0}}$$

$$\mathscr{L}\ \ddot{\boldsymbol{X}}(\mathbf{t}) = \boldsymbol{S}^2\boldsymbol{X}(\boldsymbol{s}) - \boldsymbol{S}\boldsymbol{X}_{\boldsymbol{t}=\mathbf{0}} - \dot{\boldsymbol{X}}_{\boldsymbol{t}=\mathbf{0}}$$

HW.1 Find the Laplace transform of an exponential function,

$$x(t) = ke^{-at} \qquad \text{where a \& k are constants}$$

$$X(s) = \int_0^\infty x(t)e^{-st}dt = \int_0^\infty ke^{-at}e^{-st}dt = k\int_0^\infty e^{-(a+s)t}dt = \frac{k}{s+a}$$

HW2. Find the Laplace transform of the following differential equation

$\ddot{X} + 3\dot{X} + 8X = 1$ **with initial condition at** ***t*****=0,** ***X*****= 4,** $\dot{X} = 0$

What Does the Laplace Transform Do?

The main idea behind the Laplace Transformation is that we can solve an equation (or system of equations) containing differential and integral terms by transforming the equation in "*t*-space" to one in "*s*-space". This makes the problem much easier to solve

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\frac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$\delta(t)$	1	(5)
$\delta(t-t_0)$	e^{-st_0}	(6)
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	(7)
$f'(t)$	$sF(s) - f(0)$	(8)
$f^n(t)$	$s^n F(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$	(9)
$\frac{ae^{at} - be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$	(19)
te^{at}	$\frac{1}{(s-a)^2}$	(20)
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	(21)
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$	(22)
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	(23)
$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$	(24)
$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$	(25)
$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$	(26)

Lecture Two

2-1 Modelling of mechanical systems

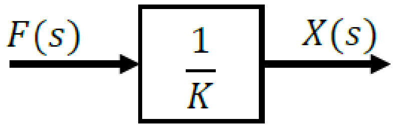
a- Spring:

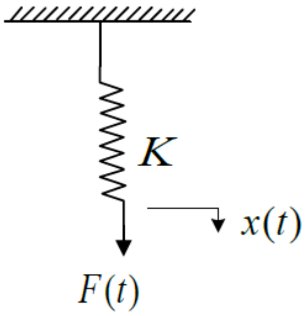
The stiffness of a system can be represented by a spring. For the spring in the figure below the extension x(t) is proportional to the applied extending force F(t):

$$F(t) = Kx(t)$$

$$F(s) = Kx(s)$$

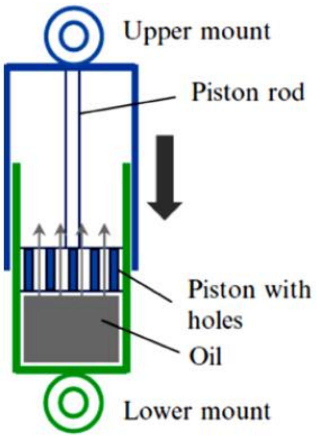
$$\frac{x(s)}{F(s)} = \frac{1}{k}$$

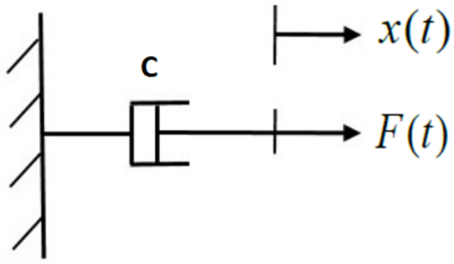




b- damping

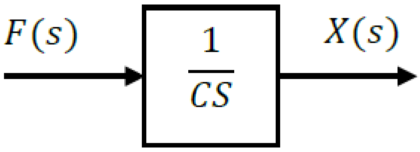
damping of a mechanical system can be represented by a dashpot. It typically contains a piston surrounded by viscous medium, such as oil (see the Figure below). The inward and outward movement of the piston will force the trapped oil to pass through the small holes in the piston from one side to the other. The faster the piston is displaced, the greater the resistance force, which means there is a proportional relation between the piston velocity and the resistance force. This can be mathematically represented as follow:





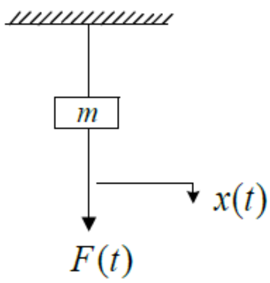
$$F(t) = C\,\dot{x} \longrightarrow F(s) = CS\,X(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{CS}$$



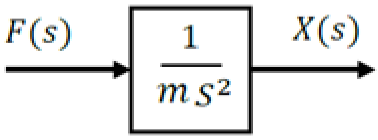
c- Mass:

Mass is the property of a body, which stores kinetic energy. If a force is applied on a body having mass M, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and friction are negligible.



$$F(t) = m\,\ddot{x} \longrightarrow F(s) = mS^2\,X(s)$$

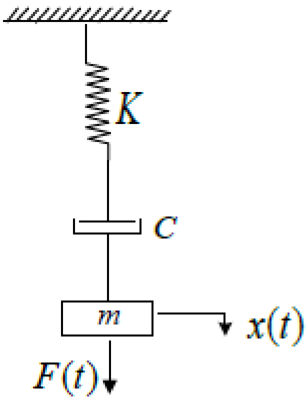
$$\frac{X(s)}{F(s)} = \frac{1}{mS^2}$$



2-2 Parallel and series elements connection in mechanical systems

a- Parallel element

For parallel elements, the same force *F* is transmitted through each element while the total deflection is seen to be the sum of the individual deflections of each element, as shown:

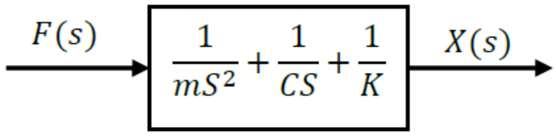


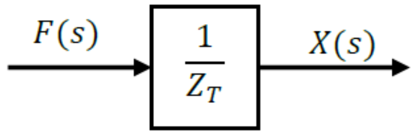
$$X(s) = \frac{F(s)}{MS^2} + \frac{F(s)}{CS} + \frac{F(s)}{K}$$

$$F(s) = \frac{1}{\frac{1}{MS^2} + \frac{1}{CS} + \frac{1}{K}} X(s)$$

$$F(s) = Z_T X(s)$$

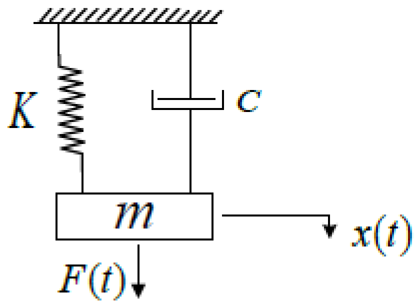
$$\frac{X(s)}{F(s)} = \frac{1}{Z_T}$$

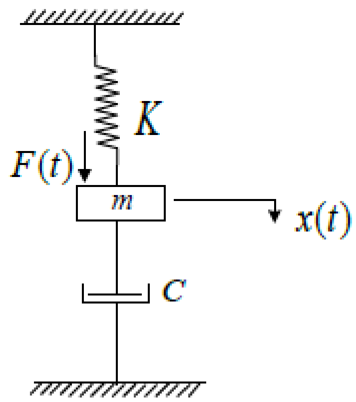




b- Series element

For series elements, the force F is equal to the summation of the forces acting on each individual component, and each element experiences the same displacement, as shown:

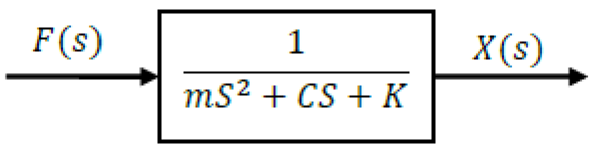


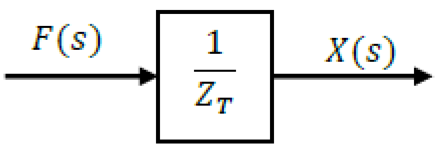


$$F(s) = (ms^2 + Cs + K)\, X(s)$$

$$F(s) = Z_T\, X(s)$$

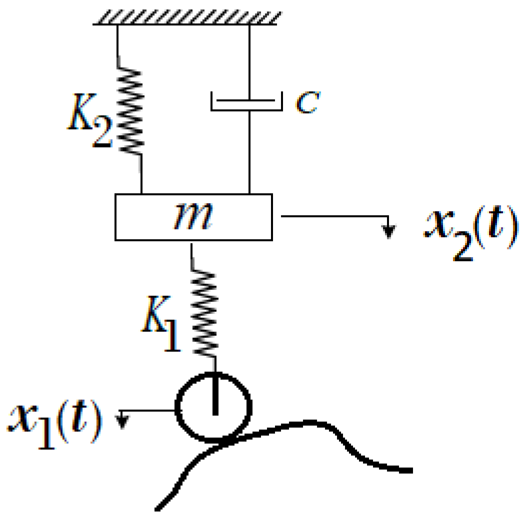
$$\frac{X(s)}{F(s)} = \frac{1}{Z_T}$$





Example 2-1:

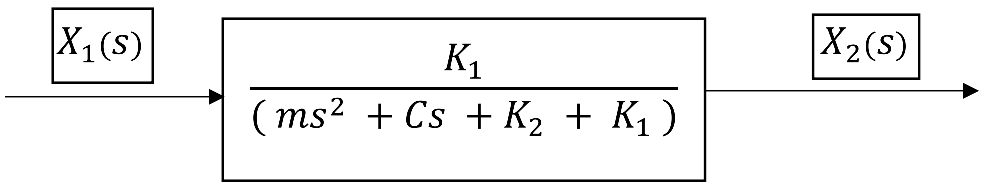
For the mass-spring-damper combination shown in the Figure below, determine the equation relating x_1 and x_2.



$$K_1 (X_1(s) - X_2(s)) = (ms^2 + Cs + K_2) X_2(s)$$

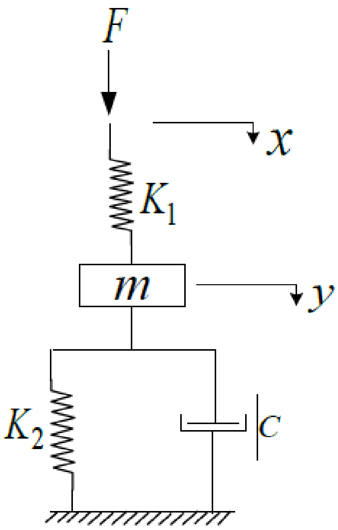
$$K_1 X_1(s) = (ms^2 + Cs + K_2 + K_1) X_2(s)$$

$$\frac{X_2(s)}{X_1(s)} = \frac{K_1}{(ms^2 + Cs + K_2 + K_1)}$$



Example 2-2:

For the mass-spring-damper combination shown in the Figure below, determine the equation relating *F* and *x*, the equation relating *F* and *y*, and the equation relating *x* and *y*.



(a)

Solution:

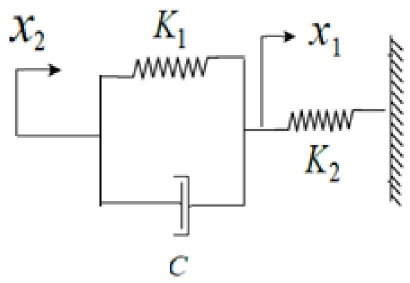
b) Equation relating *F* and *y*

F

c) Equation relating *x* and *y*

Example 2-3 :

Find T.F for the figure shown.

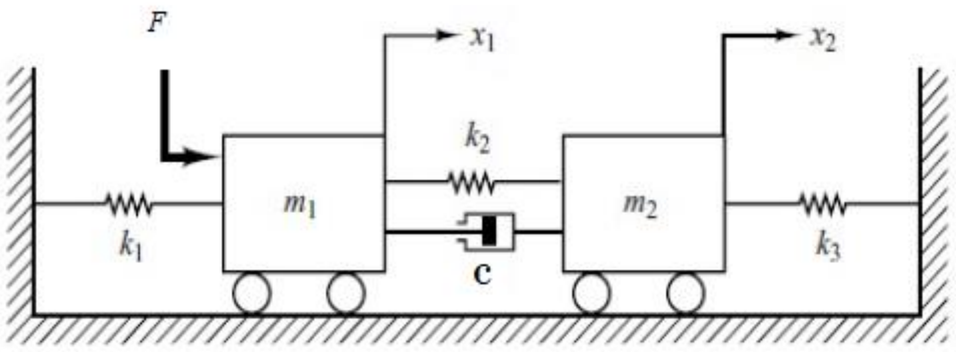


$$CS\,X_2(s) - CS\,X_1(s) + K_1\,X_2(s) - K_1\,X_1(s) = K_2\,X_1(s)$$

$$(CS + K_1)\,X_2(s) = (CS + K_1 + K_2)\,X_1(s)$$

$$\frac{X_1(s)}{X_2(s)} = \frac{(CS + K_1)}{(CS + K_1 + K_2)}$$

Example 2-4 : Find the transfer function ($\frac{X_1(s)}{F\ (s)}$) for the system shown in the figure below.



Solution:

1- using Grounded-Chair representation.

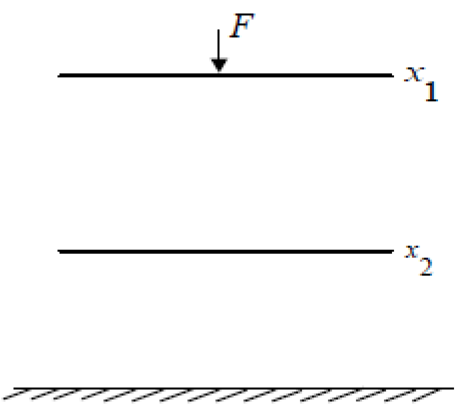
Steps of solution

1- Draw the ground coordinates at the bottom of the drawing.

2- Draw the coordinate at which the force acts at the top of the drawing.

3- Put all the other coordinates between the ground and force coordinates.

4- Insert each element in its correct orientation with respect to these coordinates, as follow.

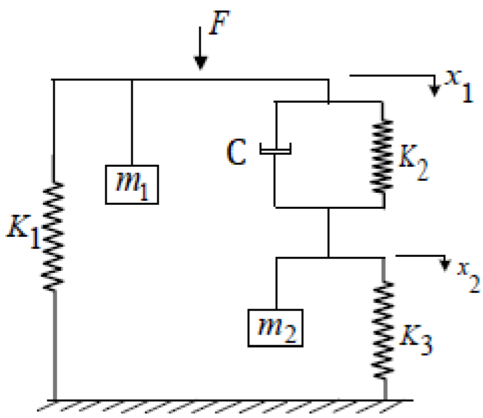


$$Z_1 = m_1 S^2 + k_1$$

$$Z_2 = CS + k_2$$

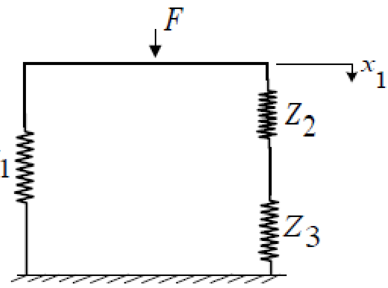
$$Z_3 = m_2 S^2 + k_3$$

$$\frac{X_1(s)}{F(s)} = \frac{1}{Z_T}$$



$$\frac{1}{Z_{eq}} = \frac{1}{Z_2} + \frac{1}{Z_3} = \frac{Z_3 + Z_2}{Z_2 * Z_3}$$

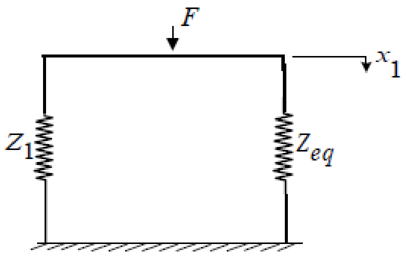
$$Z_{eq} = \frac{Z_2 * Z_3}{Z_3 + Z_2}$$

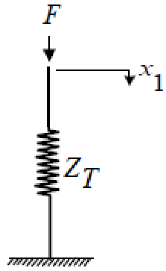


$$Z_T = Z_1 + Z_{eq}$$

$$Z_T = Z_1 + \frac{Z_2 * Z_3}{Z_3 + Z_2}$$

$$Z_T = \frac{Z_1 Z_3 + Z_1 Z_{2+} Z_2 Z_3}{Z_3 + Z_2}$$



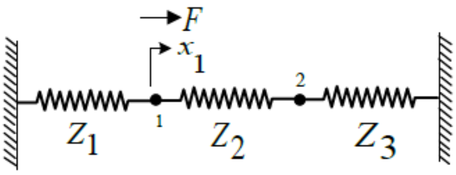


2- using the normal series and parallel connection

$$Z_1 = m_1 S^2 + k_1$$

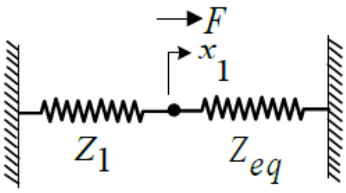
$$Z_2 = bS + k_2$$

$$Z_3 = m_2 S^2 + k_3$$



$$\frac{1}{Z_{eq}} = \frac{1}{Z_2} + \frac{1}{Z_3} = \frac{Z_3 + Z_2}{Z_2 * Z_3}$$

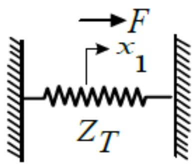
$$Z_{eq} = \frac{Z_2 * Z_3}{Z_3 + Z_2}$$



$$Z_T = Z_1 + Z_{eq}$$

$$Z_T = Z_1 + \frac{Z_2 * Z_3}{Z_3 + Z_2}$$

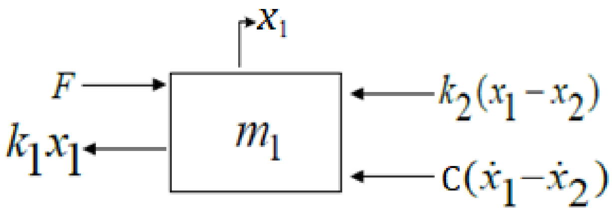
$$Z_T = \frac{Z_1 Z_3 + Z_1 Z_{2+} Z_2 Z_3}{Z_3 + Z_2}$$



$$\frac{X(s)}{F(s)} = \frac{1}{Z_T}$$

3- using force equilibrium based on Newton's second law of motion

Free body diagram of *m1*



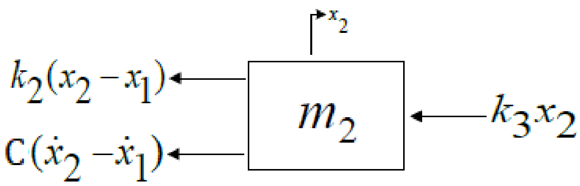
$$\sum F = m\ddot{x}_1$$

$$m_1\ddot{x}_1 = F - k_1x_1 - k_2(x_1 - x_2) - c(\dot{x}_1 - \dot{x}_2)$$

$$m_1\ddot{x}_1 = F - k_1x_1 - k_2x_1 + k_2x_2 - c\dot{x}_1 + c\dot{x}_2$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 + c\dot{x}_1 = F + k_2x_2 + c\dot{x}_2 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Free body diagram of *m2*



$$\sum F = m\ddot{x}_2$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1) - k_3x_2$$

$$m_2\ddot{x}_2 = -k_2x_2 + k_2x_1 - c\dot{x}_2 + c\dot{x}_1 - k_3x_2$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 + c\dot{x}_2 = k_2x_1 + c\dot{x}_1 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Taking the Laplace transforms for Equation (1) & (2), we obtain:

$$[m_1S^2 + CS + (k_1 + k_2)]X_1(s) = (k_2 + CS)X_2(s) + F(s) \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$[m_2S^2 + CS + (k_2 + k_3)]X_2(s) = (k_2 + CS)X_1(s) \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Solving Equation (4) for *X2(s)*:

$$X_2(s) = \frac{(k_2 + CS)}{[m_2S^2 + CS + (k_2 + k_3)]}X_1(s) \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Substituting Equation (5) in Equation (3) we get:

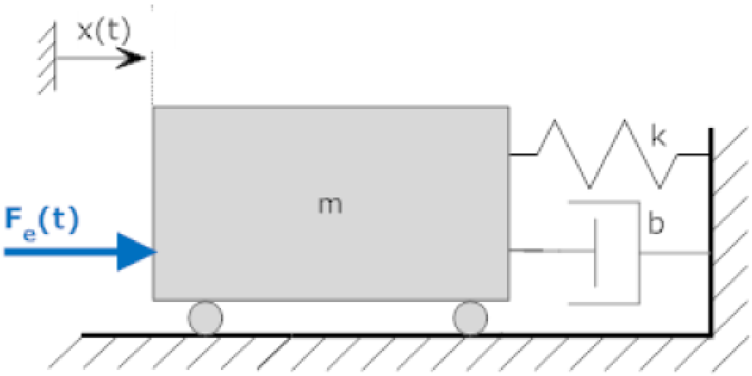
$$[m_1S^2 + CS + (k_1 + k_2)]X_1(s) = (k_2 + CS)\frac{(k_2 + CS)}{[m_2S^2 + CS + (k_2 + k_3)]}X_1(s) + F(s) \ldots\ldots (6)$$

$$\left[[m_1S^2 + CS + (k_1 + k_2)] - \frac{(k_2 + CS)^2}{[m_2S^2 + CS + (k_2 + k_3)]}\right]X_1(s) = F(s)$$

$$\left[\frac{[m_1S^2 + CS + (k_1 + k_2)][m_2S^2 + CS + (k_2 + k_3)] - (k_2 + CS)^2}{[m_2S^2 + CS + (k_2 + k_3)]}\right]X_1(s) = F(s)$$

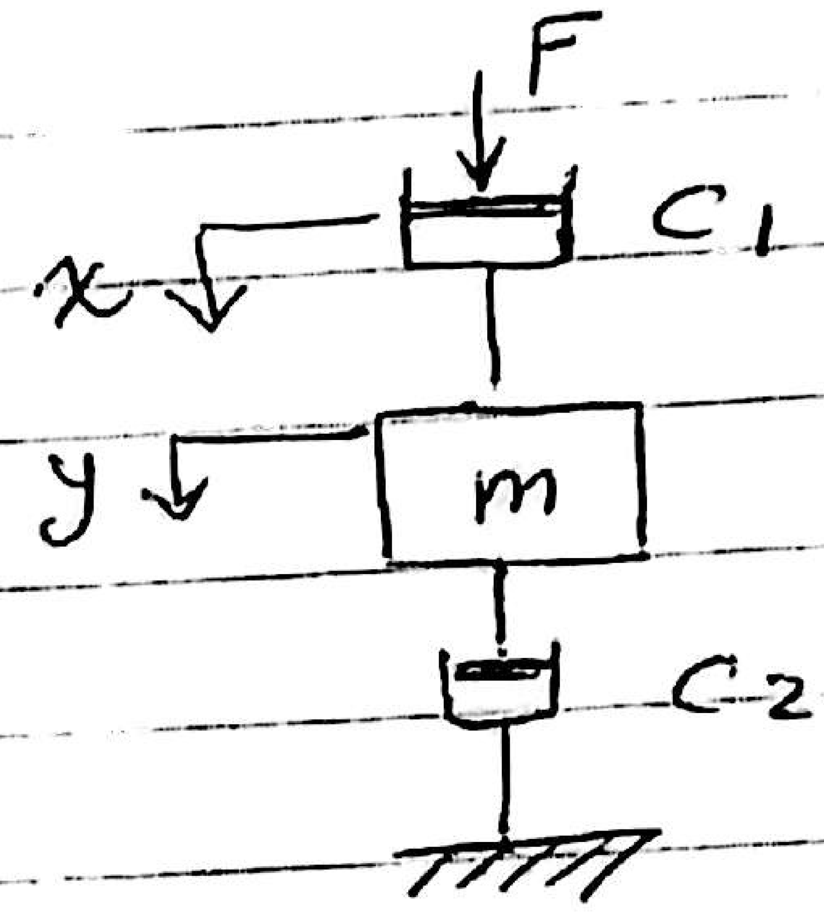
$$\boxed{\frac{X_1(s)}{F(s)} = \frac{[m_2S^2 + CS + (k_2 + k_3)]}{[m_1S^2 + CS + (k_1 + k_2)][m_2S^2 + CS + (k_2 + k_3)] - (k_2 + CS)^2}}$$

HW :



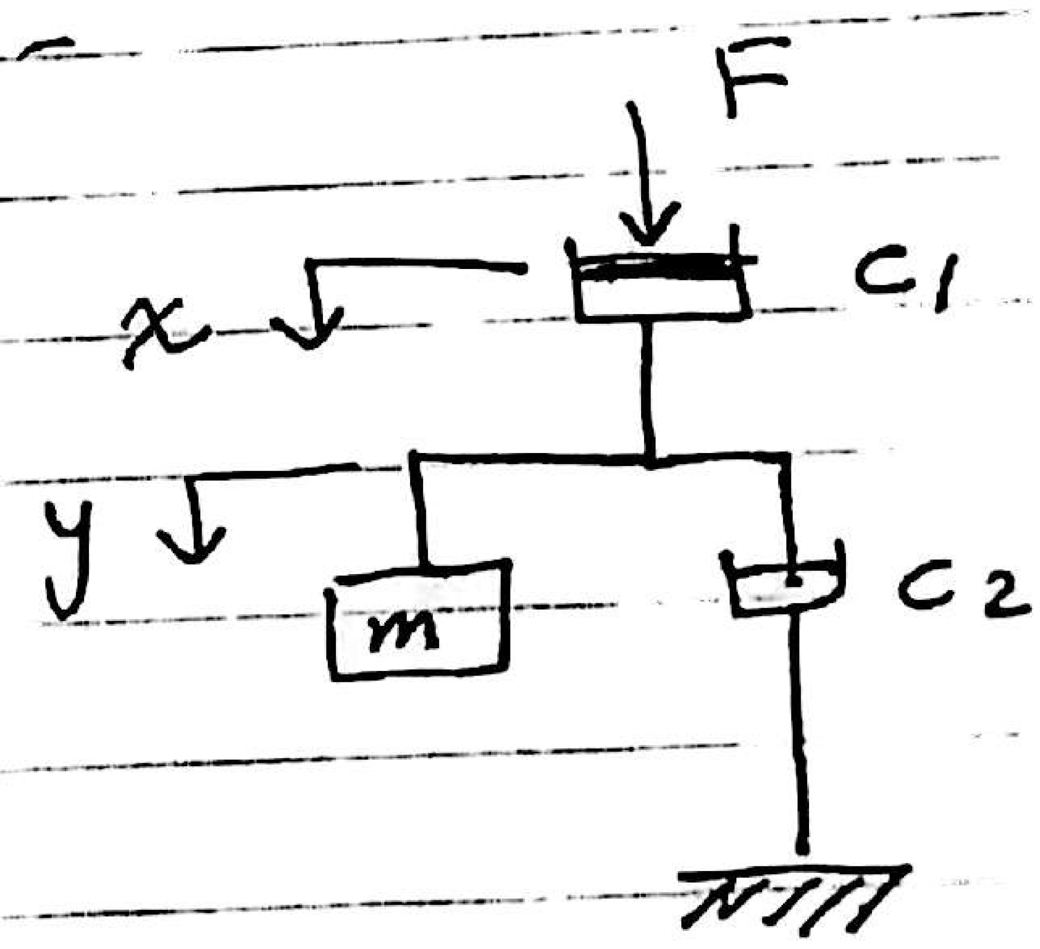
Example 2-5

Determine the equation relating F and x. The equation relating F and y and the equation relating x and y for the system shown in the figure.



Solution

Ground chair :-

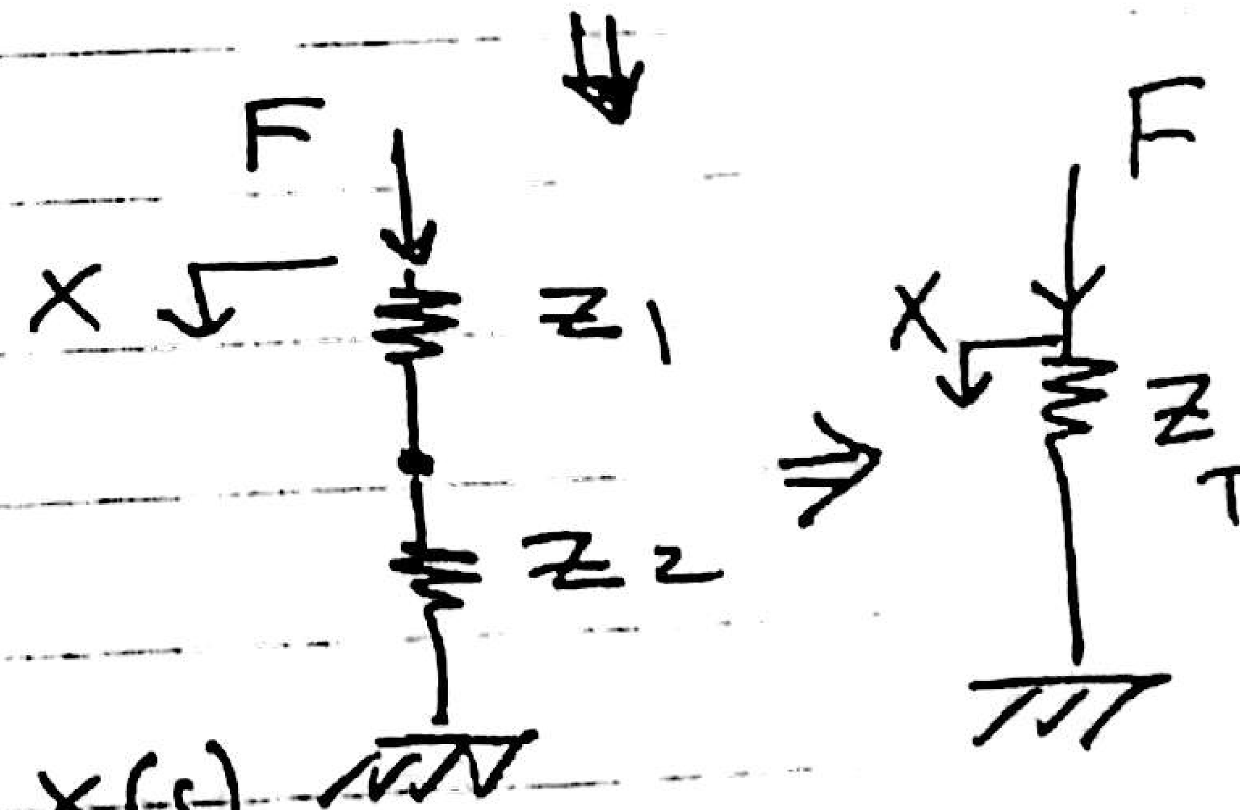


$Z_1 = C_1 s$

$Z_2 = ms^2 + C_2 s$

$$\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

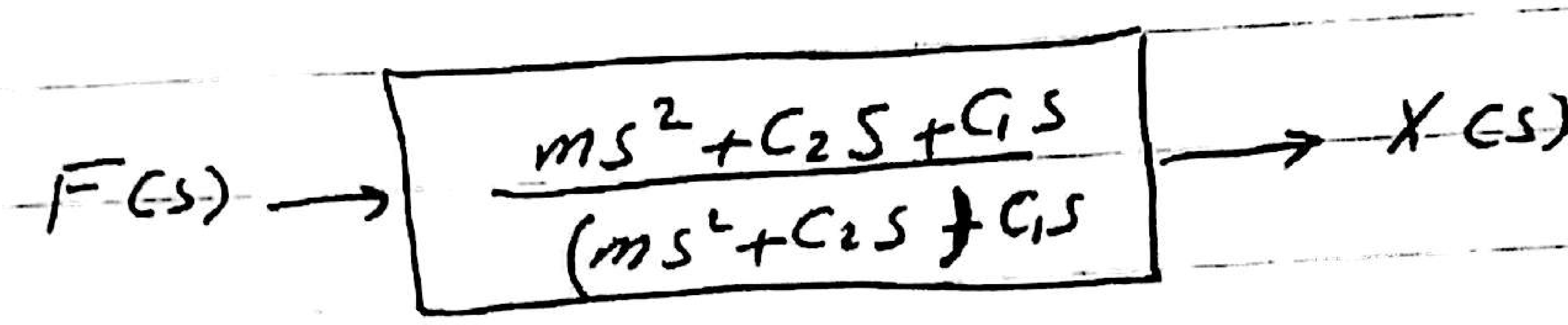
$$Z_T = \frac{(ms^2 + C_2 s)\,C_1 s}{ms^2 + C_2 s + C_1 s}$$



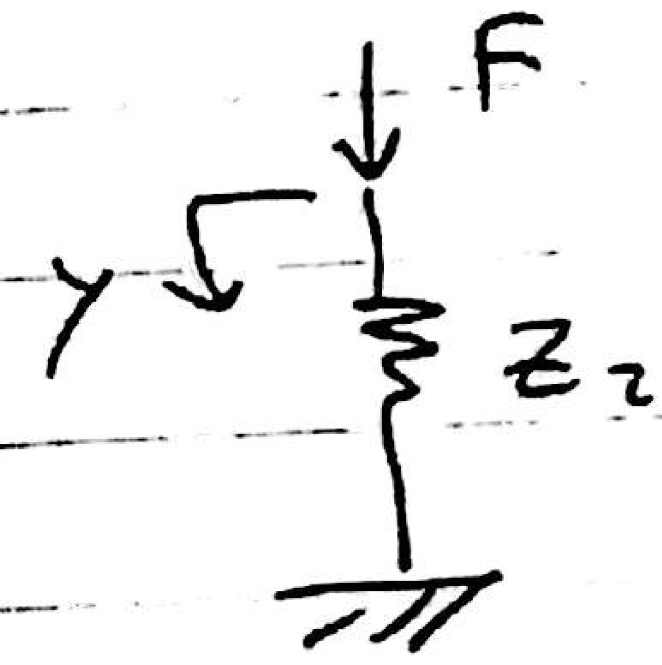
(1) $F(s) = Z_T \; X(s)$

$$F(s) = \frac{(ms^2 + C_2 s)\,C_1 s}{ms^2 + C_2 s + C_1 s}\; X(s)$$

$$\frac{X(s)}{F(s)} = \frac{ms^2 + C_2 S + C_1 s}{(ms^2 + C_2 S)\, C_1 S}$$

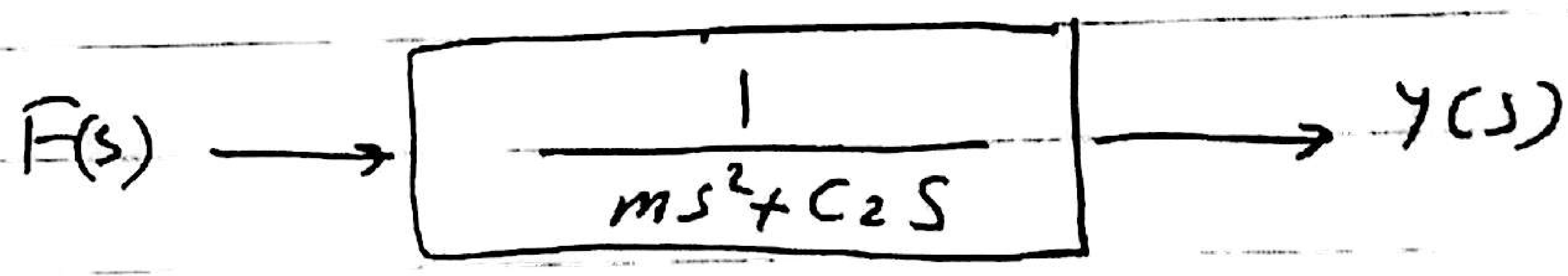


(2) $F(s) = Z_2 \, Y(s)$



$$F(s) = (ms^2 + C_2 S)\, Y(s)$$

$$\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + C_2 S}$$

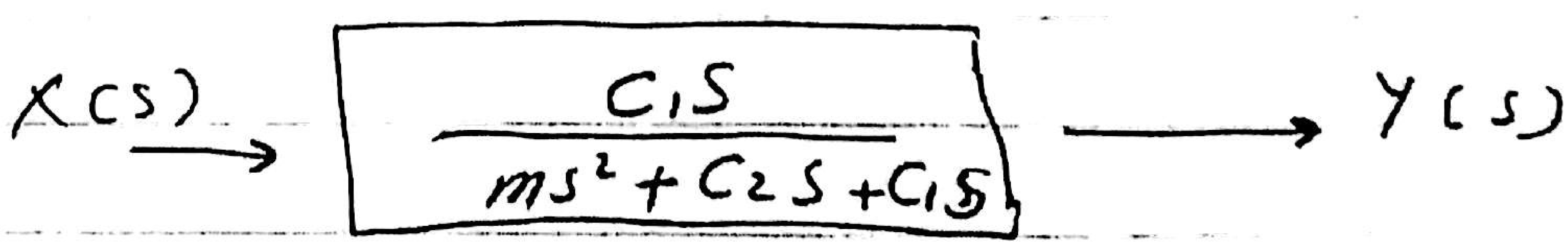


(3)

$$C_1 S\,(X(s) - Y(s)) = (ms^2 + C_2 S)\; Y(s)$$

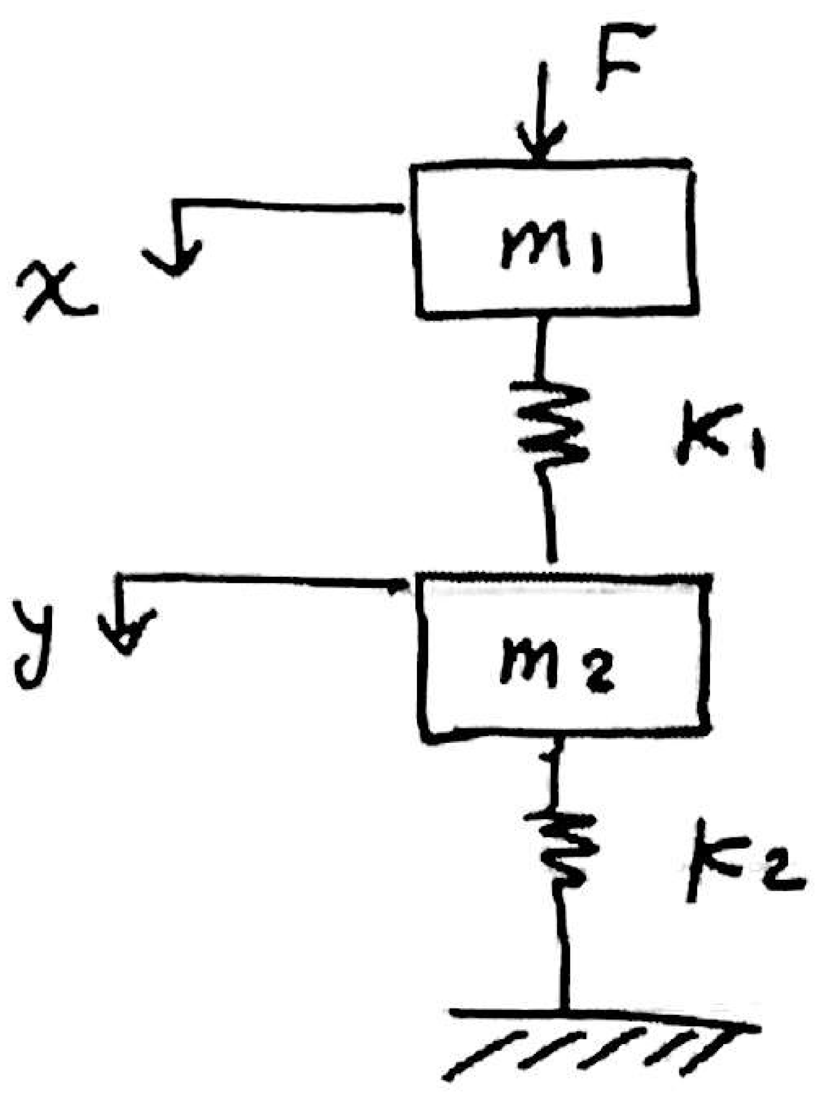
$$C_1 S \; X(s) = (ms^2 + C_2 S + C_1 S)\; Y(s)$$

$$\frac{Y(s)}{X(s)} = \frac{C_1 S}{ms^2 + C_2 S + C_1 S}$$



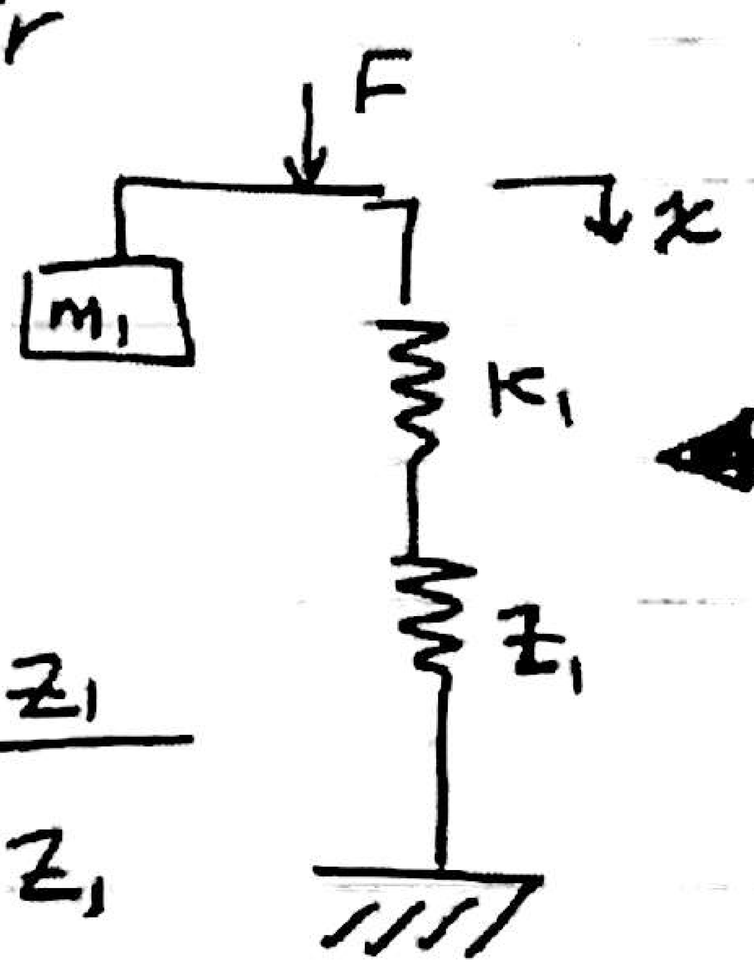
Example 2-6

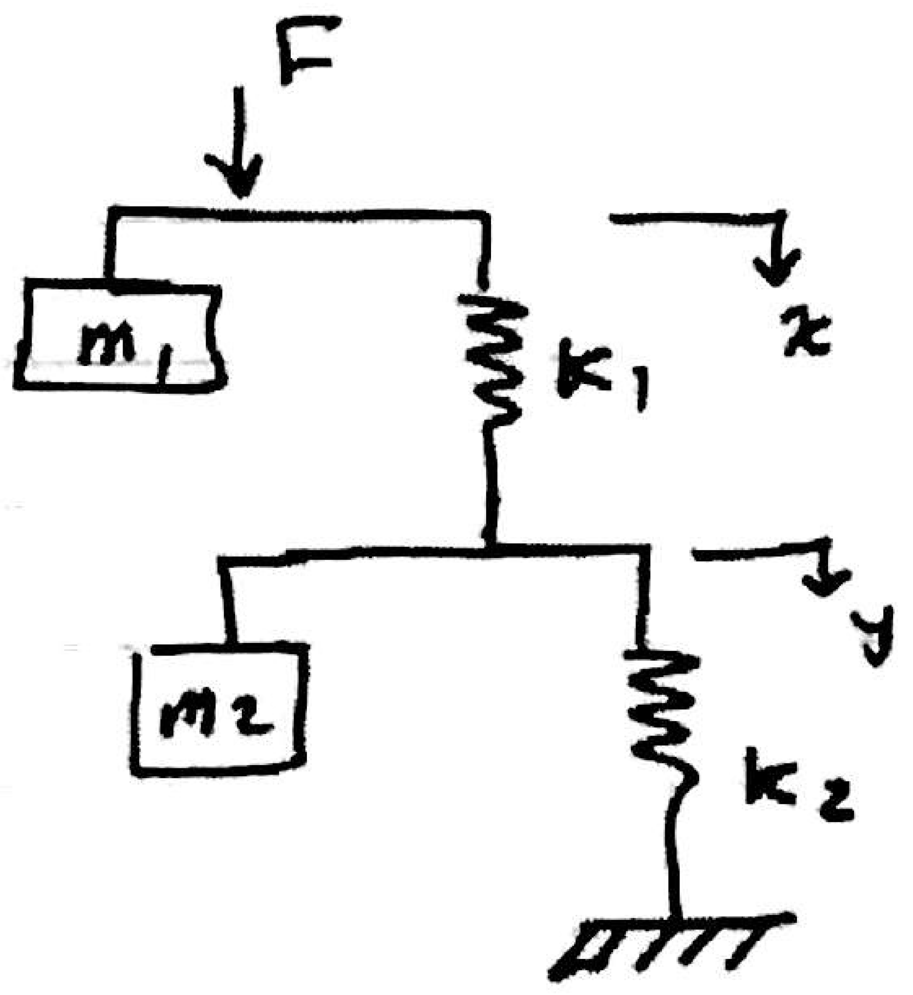
Find Transfer Function for the system shown.



Solution

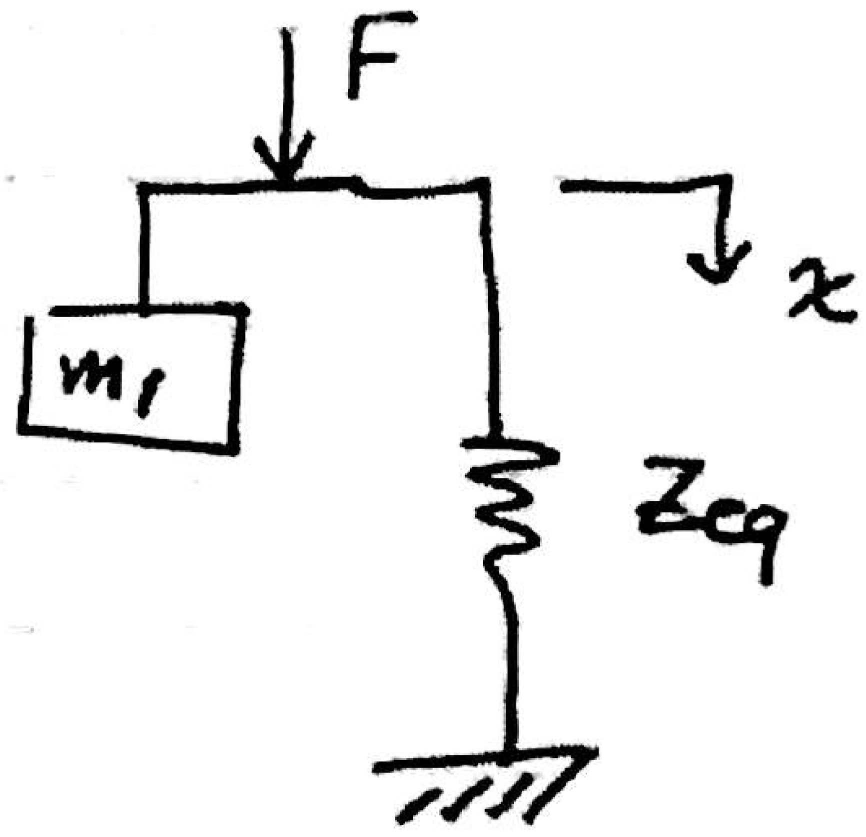
1- Graund chair





$$Z_1 = m_2 s^2 + K_2$$

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{K_1} = \frac{K_1 + Z_1}{K_1 Z_1}$$



$$Z_{eq} = \frac{(m_2 s^2 + K_2) K_1}{m_2 s^2 + K_2 + K_1}$$

$$Z_T = \frac{(m_2 s^2 + K_2) K_1}{m_2 s^2 + K_2 + K_1} + m_1 s^2$$

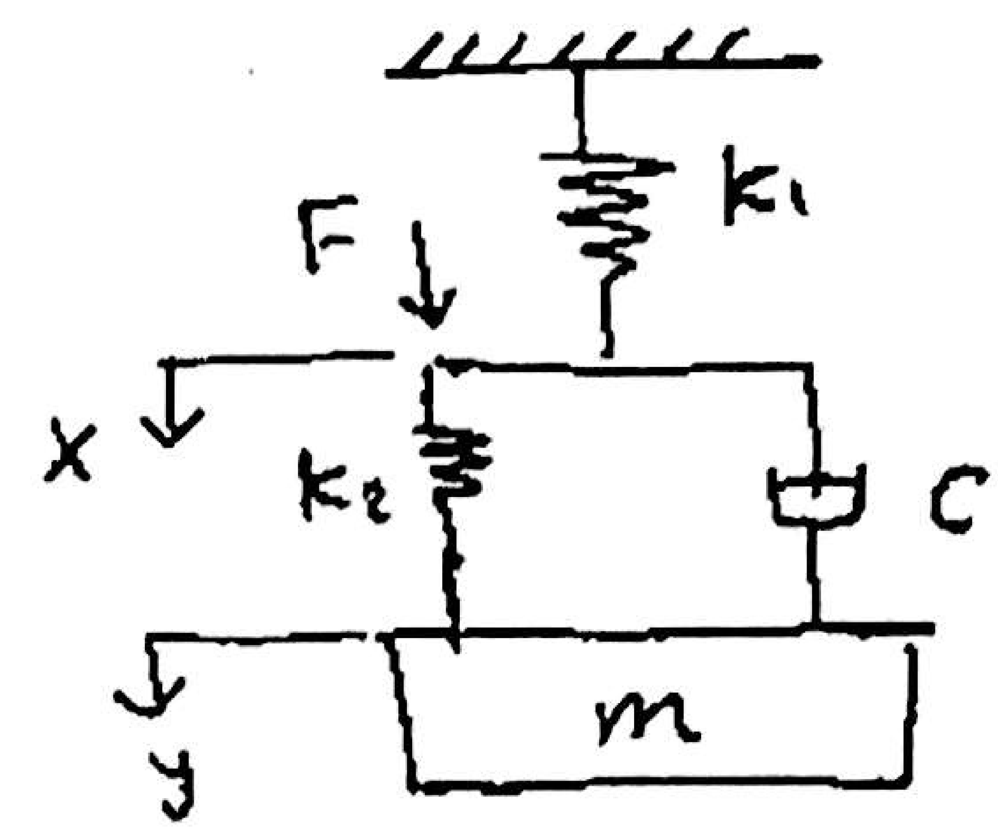
$$F(s) = \left[\frac{(m_2 s^2 + K_2) K_1}{m_2 s^2 + K_2 + K_1} + m_1 s^2 \right] X(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{\left[\frac{(m_2 s^2 + K_2) K_1}{m_2 s^2 + K_2 + K_1} + m_1 s^2 \right]}$$

Example 2-7

For the system shown in the Figure, determine the equation relating F and (X)

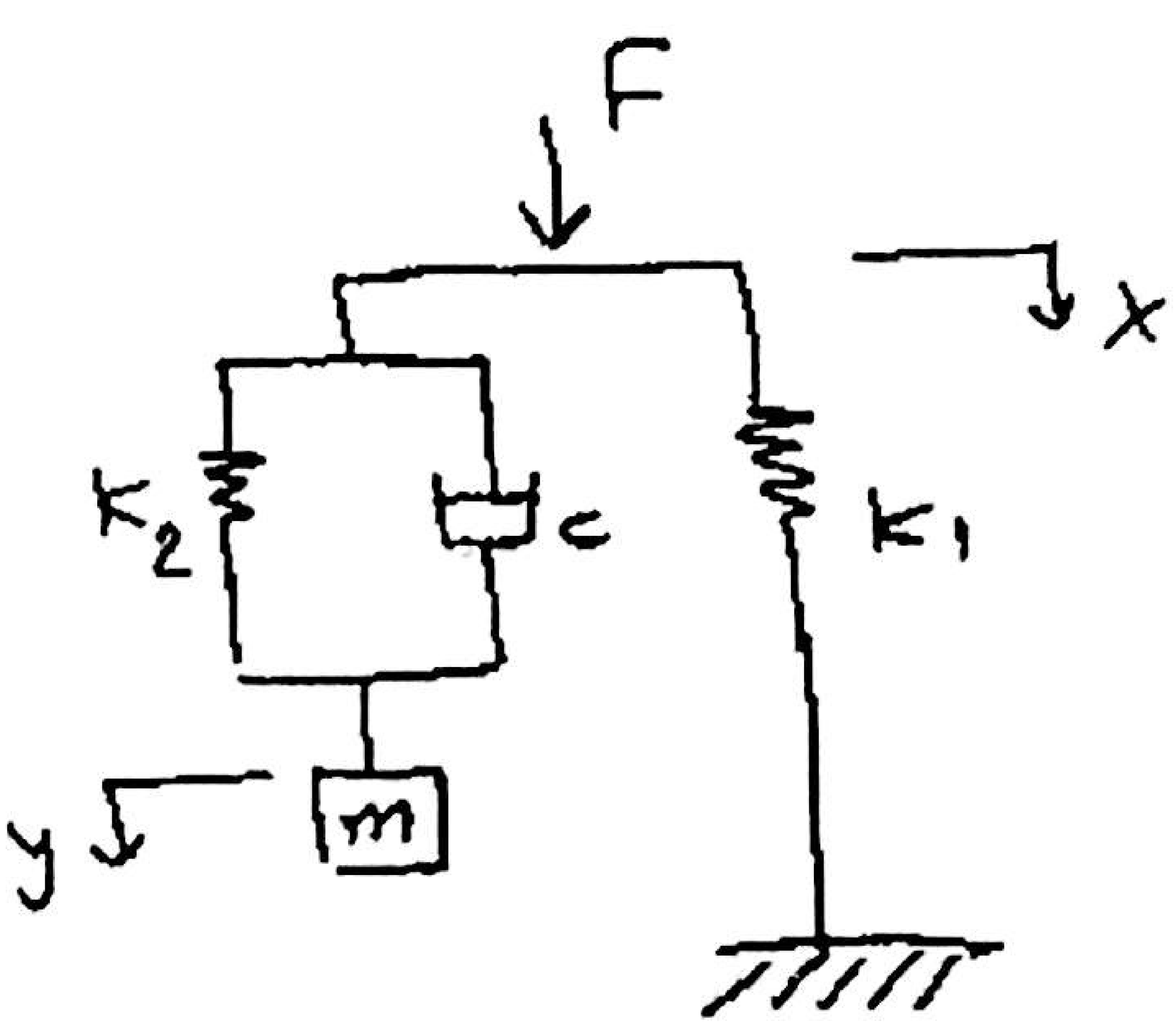
Solution



$$Z_1 = cs + k_2$$

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{ms^2}$$

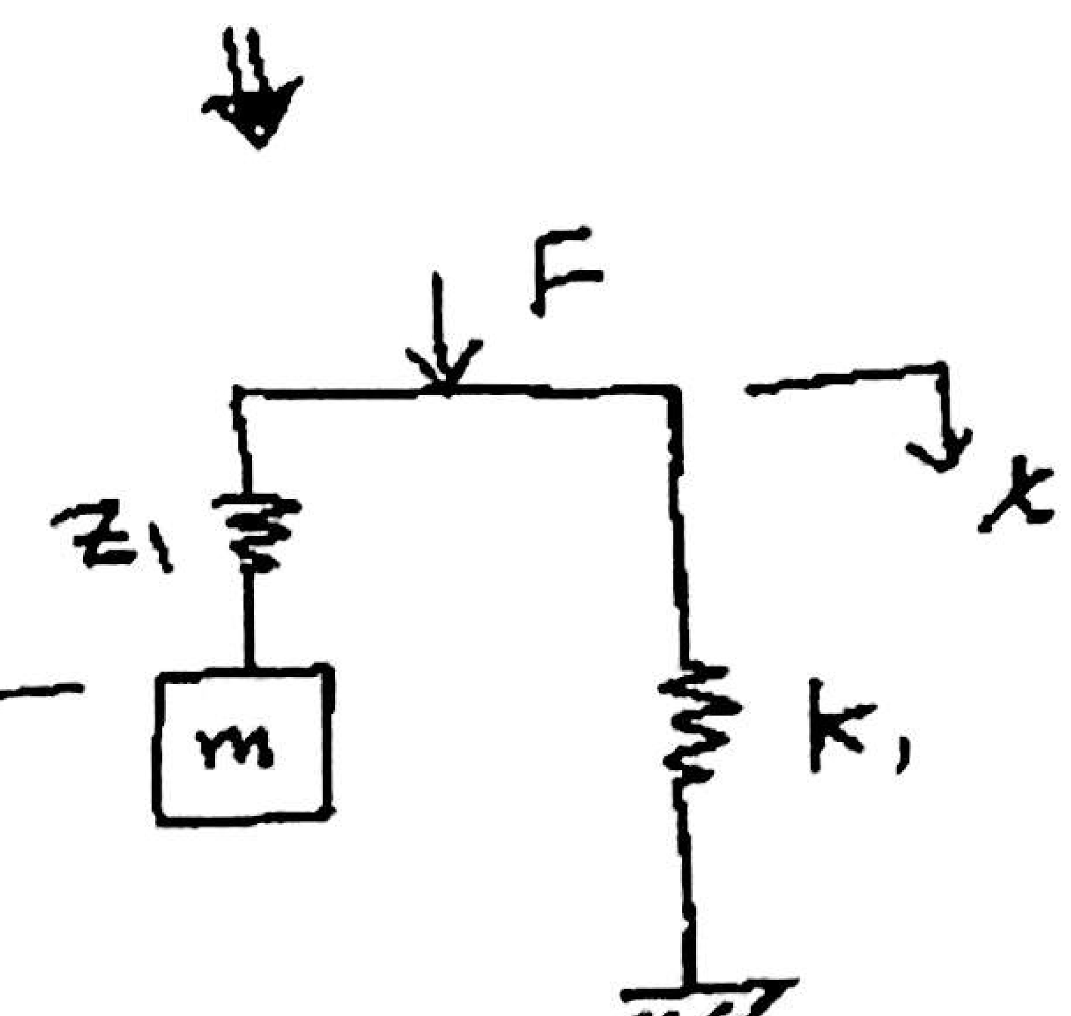
$$Z_{eq} = \frac{1}{\frac{1}{cs+k_2} + \frac{1}{ms^2}}$$

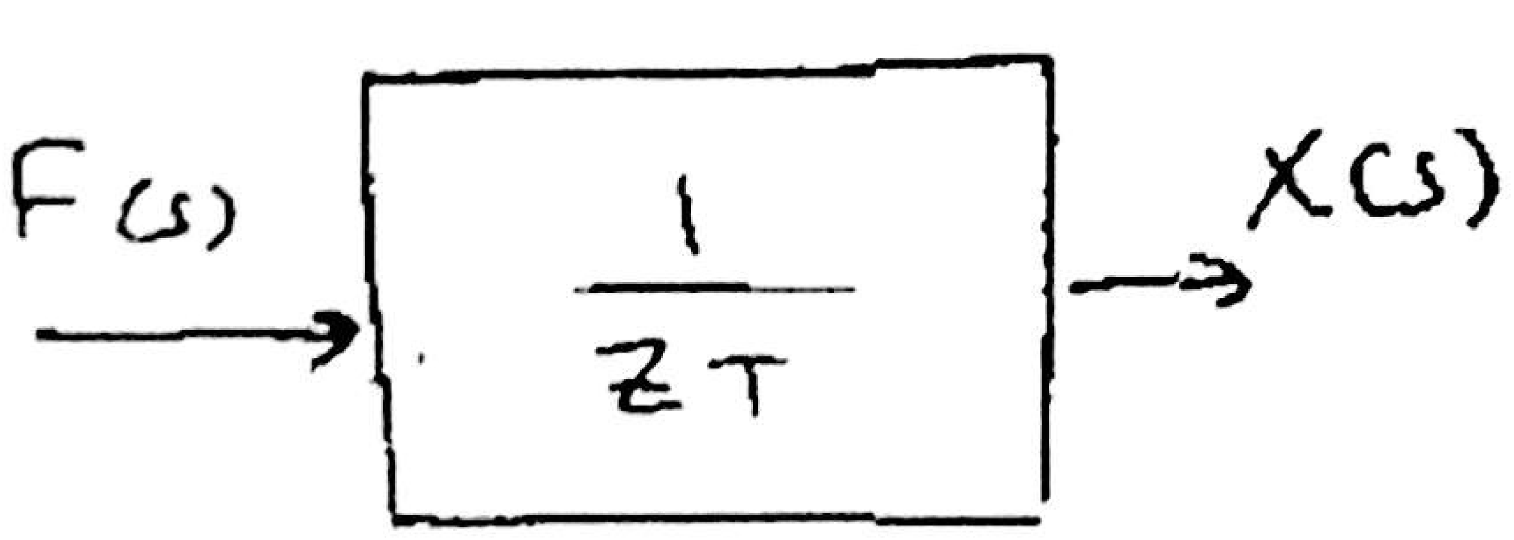


$$Z_T = Z_{eq} + k_1$$

$$F(s) = Z_T \; X(s)$$

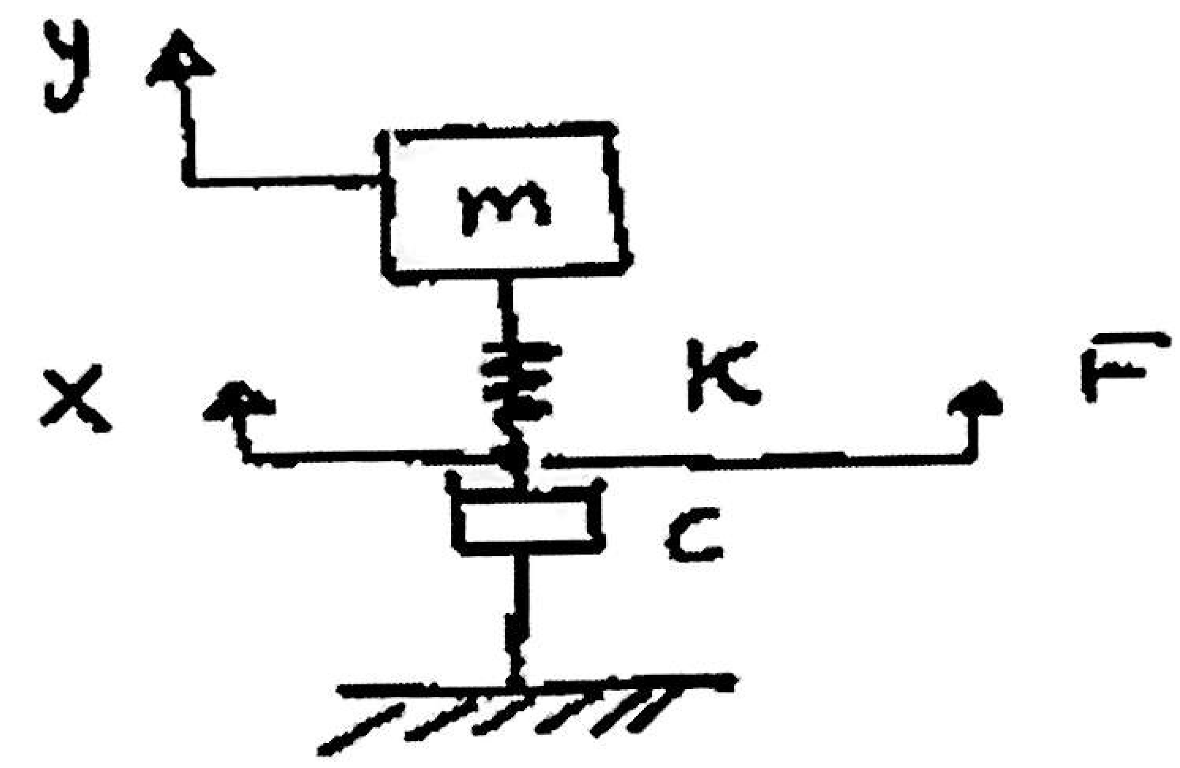
$$F(s) = \left[\frac{1}{\frac{1}{cs+k_2} + \frac{1}{ms^2}} + k_1\right] X(s)$$



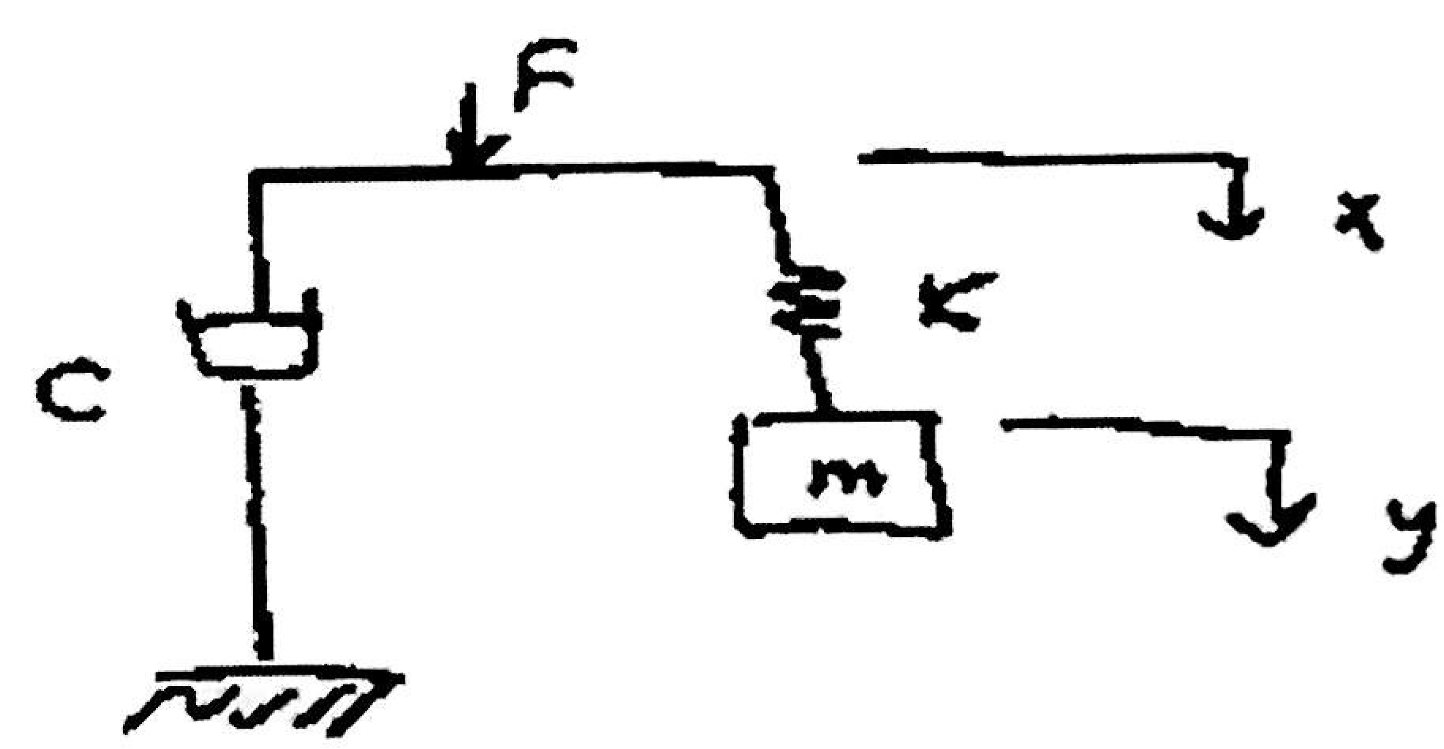


Example 2-8

Determine the equation relating F and x shown in the figure.



Solution



$$\frac{1}{Z_{eq}} = \frac{1}{ms^2} + \frac{1}{k}$$

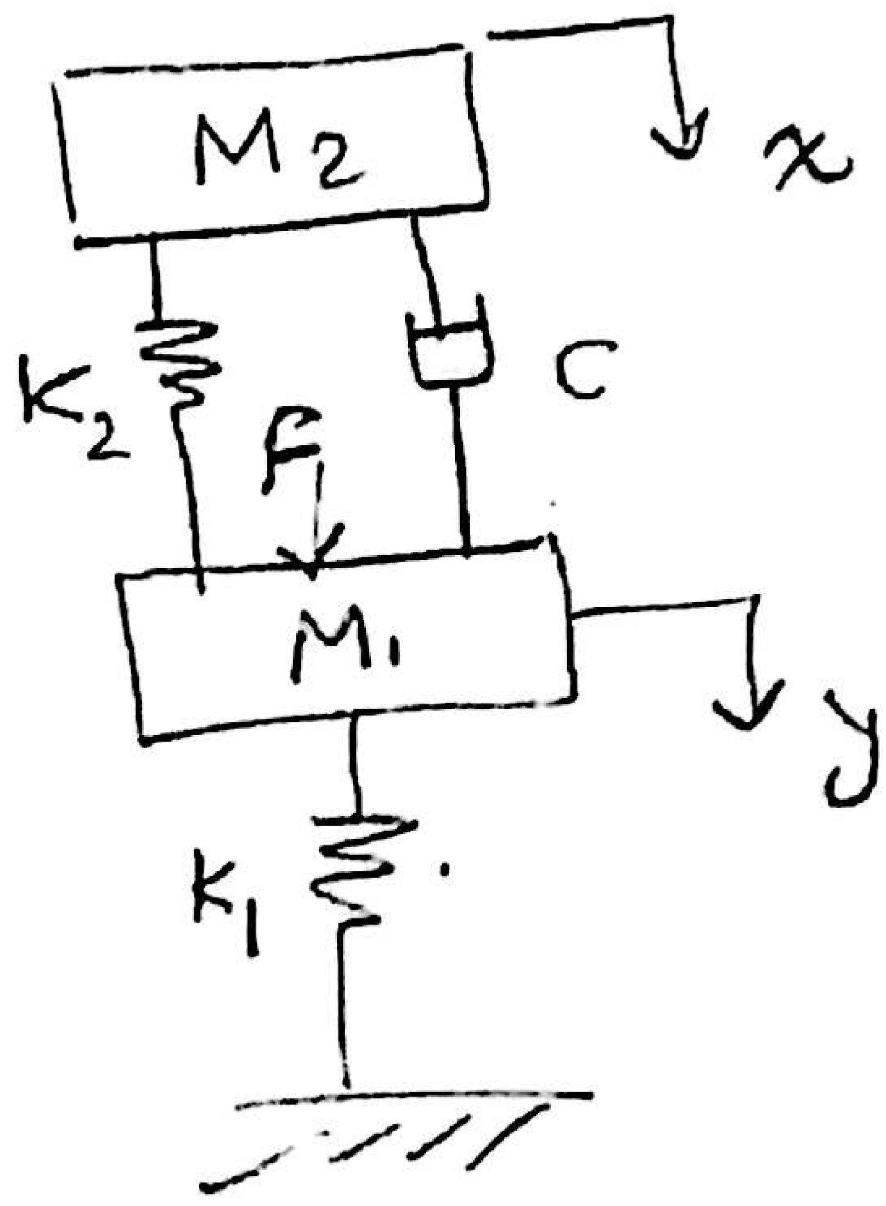
$$Z_{eq} = \frac{1}{\frac{1}{ms^2} + \frac{1}{k}}$$

$$Z_T = Cs + Z_{eq}$$

$$F(s) = \left[Cs + \frac{1}{\frac{1}{ms^2} + \frac{1}{k}} \right] X(s)$$

$$\frac{X(s)}{F(s)} = \frac{ms^2 + k}{cs(ms^2 + k) + ms^2 k}$$

Example 2-9

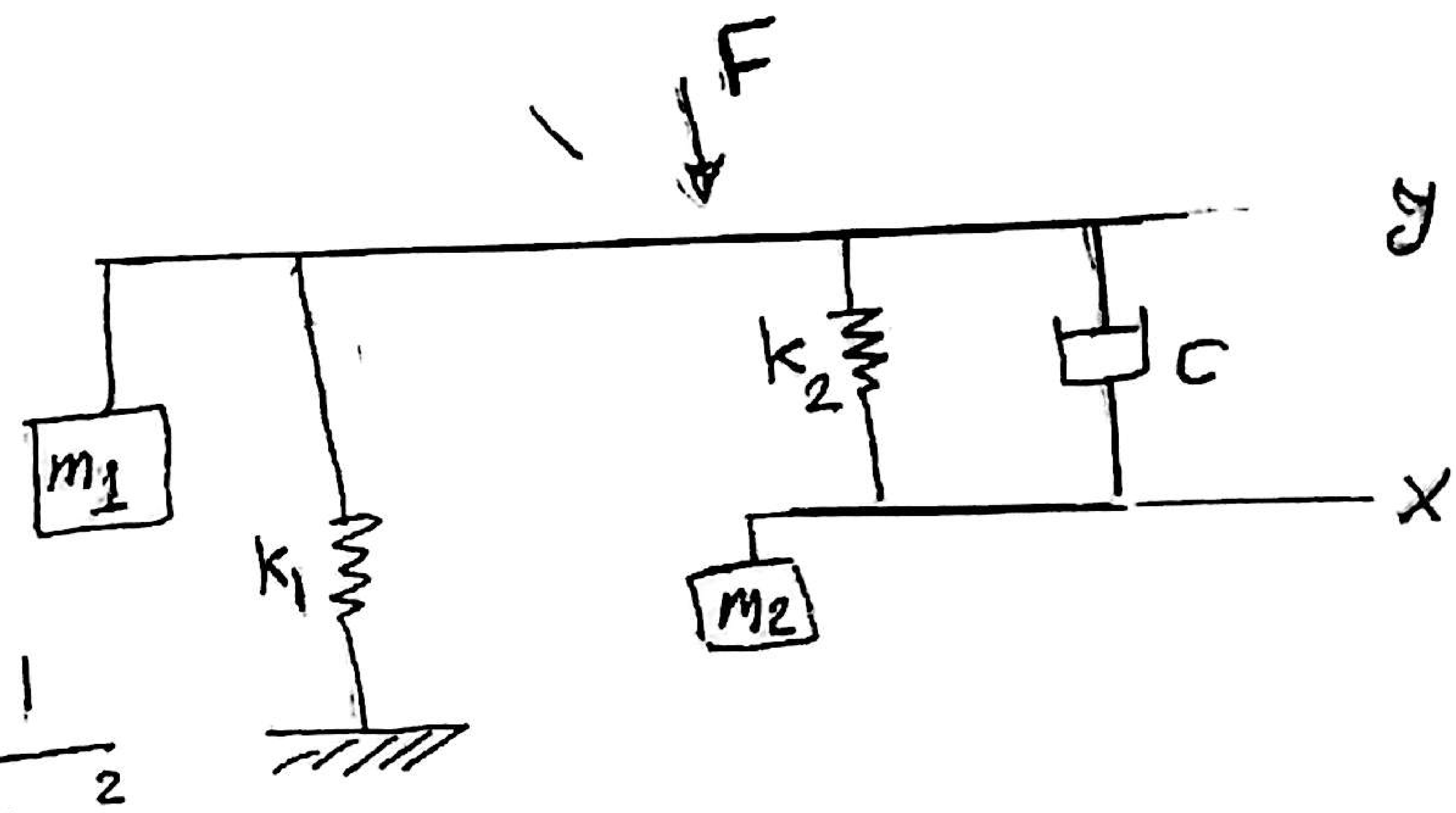


Solution

$$Z_1 = k_2 + Cs$$

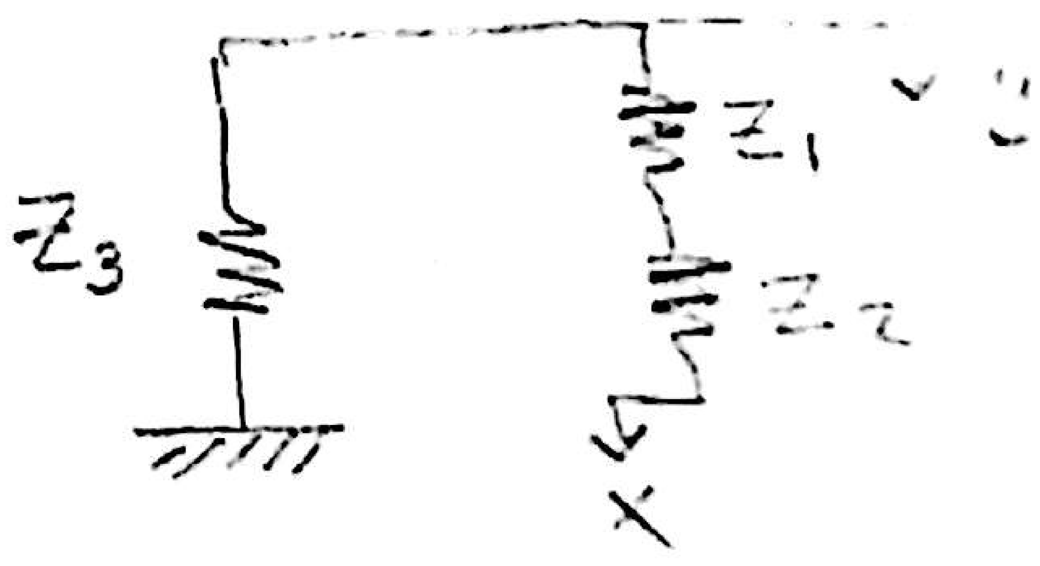
$$Z_2 = m_2 s^2$$

$$Z_3 = K_1 + m_1 s^2$$



$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

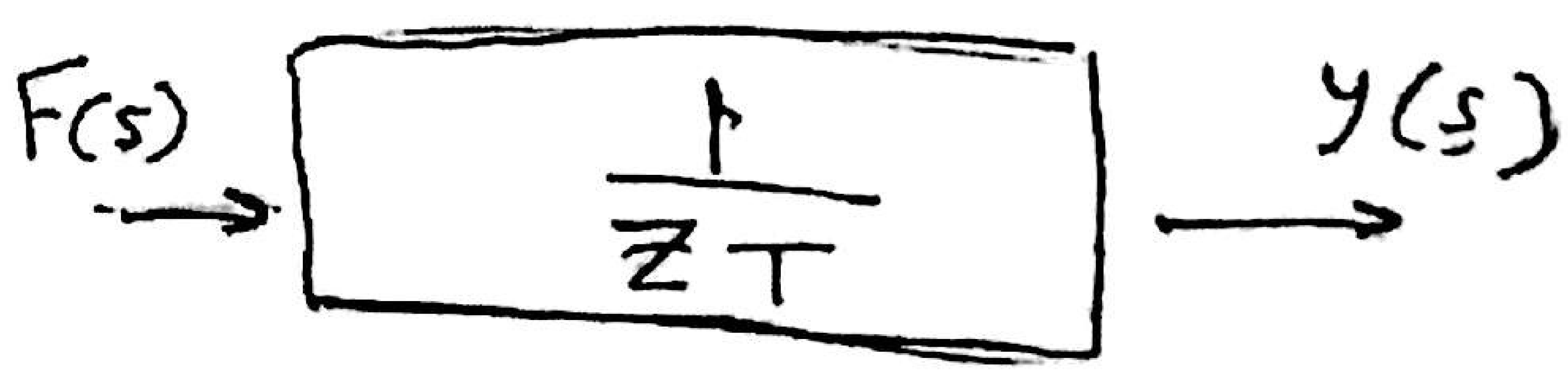
$$\frac{1}{Z_{eq}} = \frac{1}{k_2 + Cs} + \frac{1}{m_2 s^2}$$



$$Z_{eq} = \frac{1}{\frac{1}{k_2 + Cs} + \frac{1}{m_2 s^2}}$$

$$Z_T = \left(\frac{1}{\frac{1}{k_2 + Cs} + \frac{1}{m_2 s^2}} \right) + \left(K_1 + m_1 s^2 \right)$$

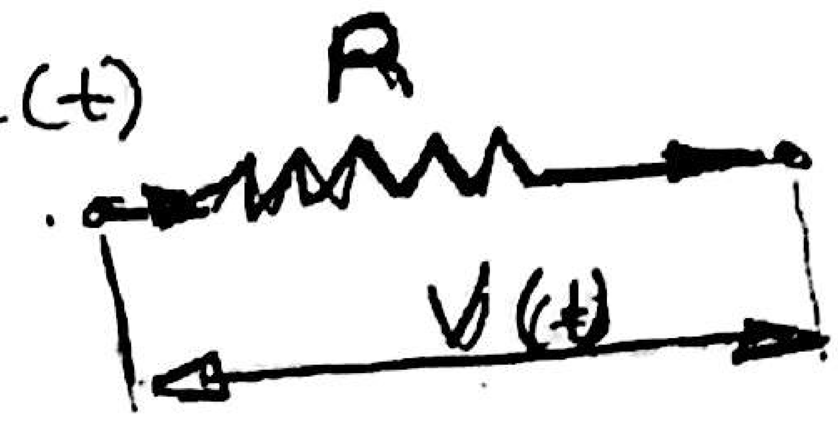
$$F = Z_T \; y(s)$$



Lecture Three

3-1 Modeling of Electrical systems

a- Resistor



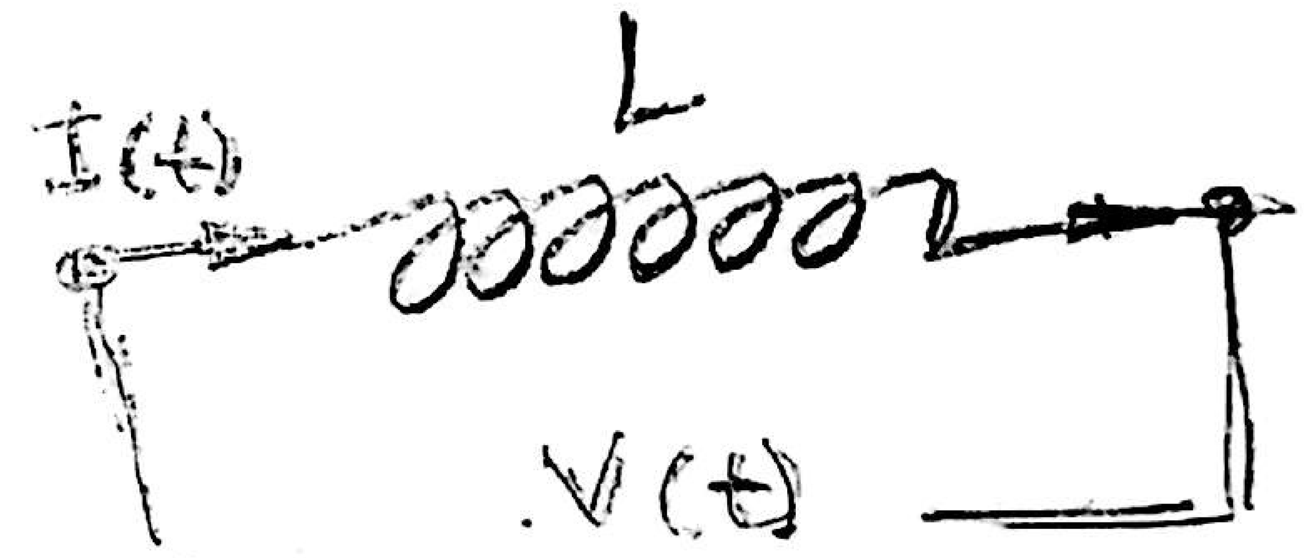
$$V(t) = R\,I(t)$$

$$V(s) = R\,I(s)$$

$$\frac{I(s)}{V(s)} = \frac{1}{R}$$

$$Z = R$$

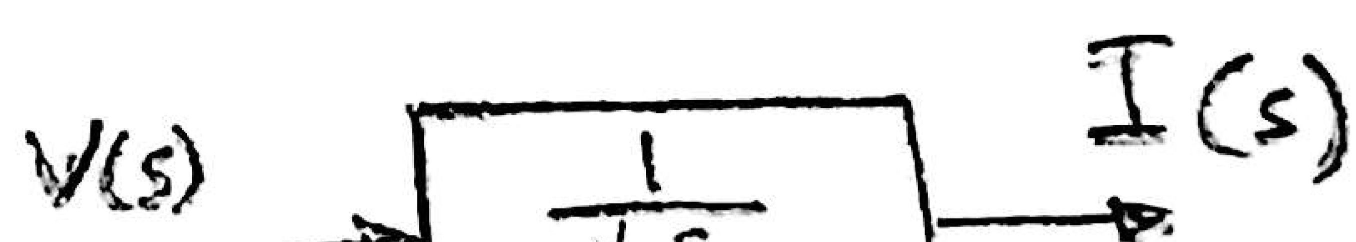
b- Inductor



$$V(t) = L\frac{dI}{d(t)} = L\dot{I}$$

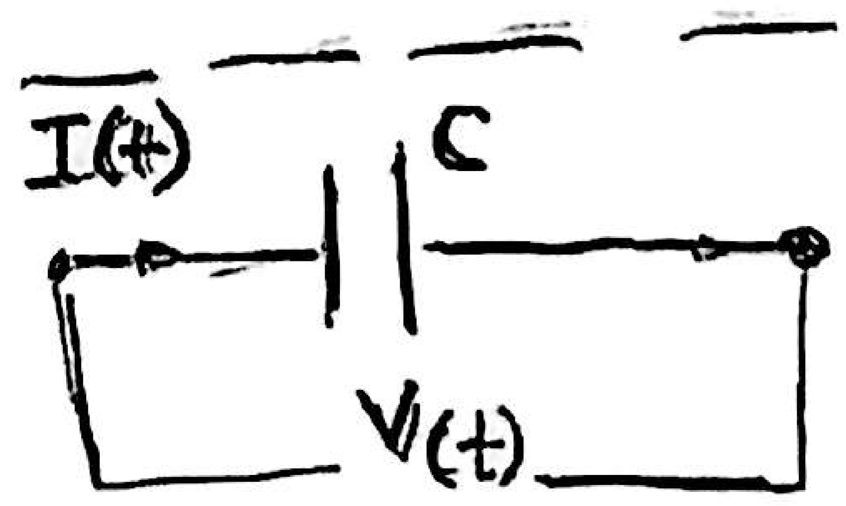
$$V(s) = Ls\,I(s)$$

$$\frac{I(s)}{V(s)} = \frac{1}{Ls}$$



$$Z = Ls$$

Capacitor



$$v(t) = \frac{1}{C}\int I\,d(t)$$

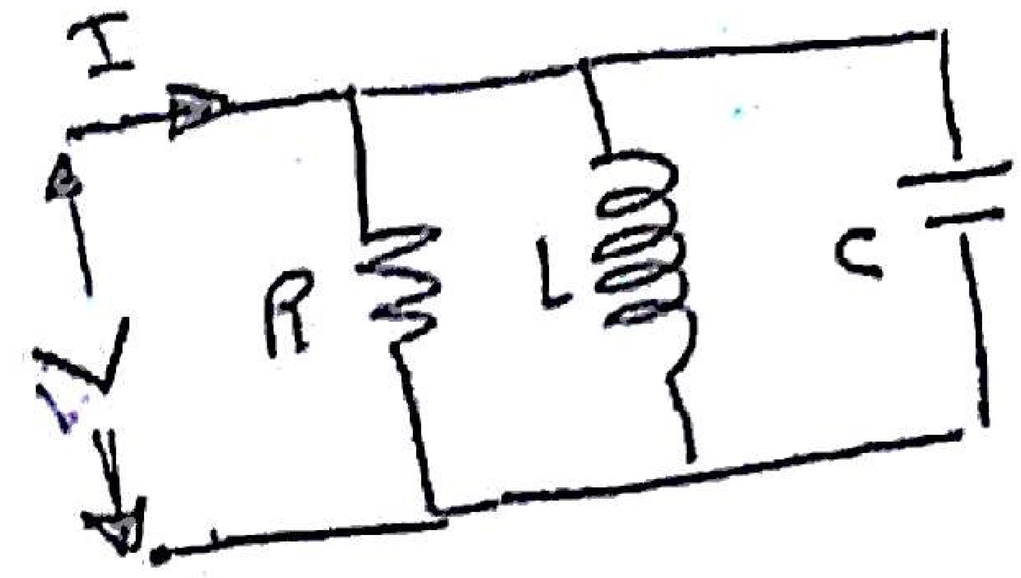
$$V(s) = \frac{1}{C}\frac{I(s)}{s} \Rightarrow$$

$$V(s) = \frac{1}{Cs}I(s) \Rightarrow \frac{I(s)}{V(s)} = Cs$$

$$Z = \frac{1}{Cs}$$

3-2 Parallel and series element Connection

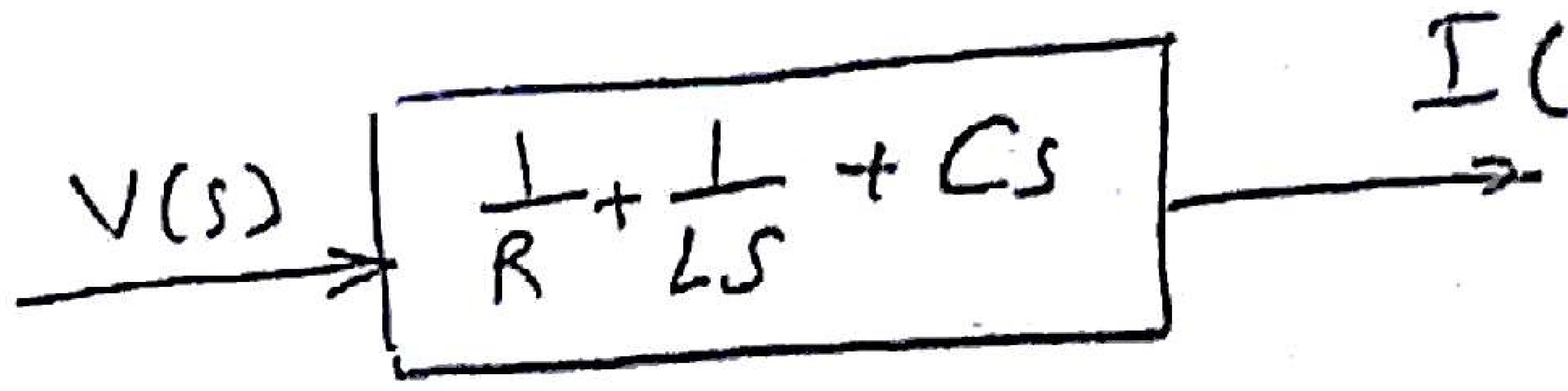
a- Parallel Connection



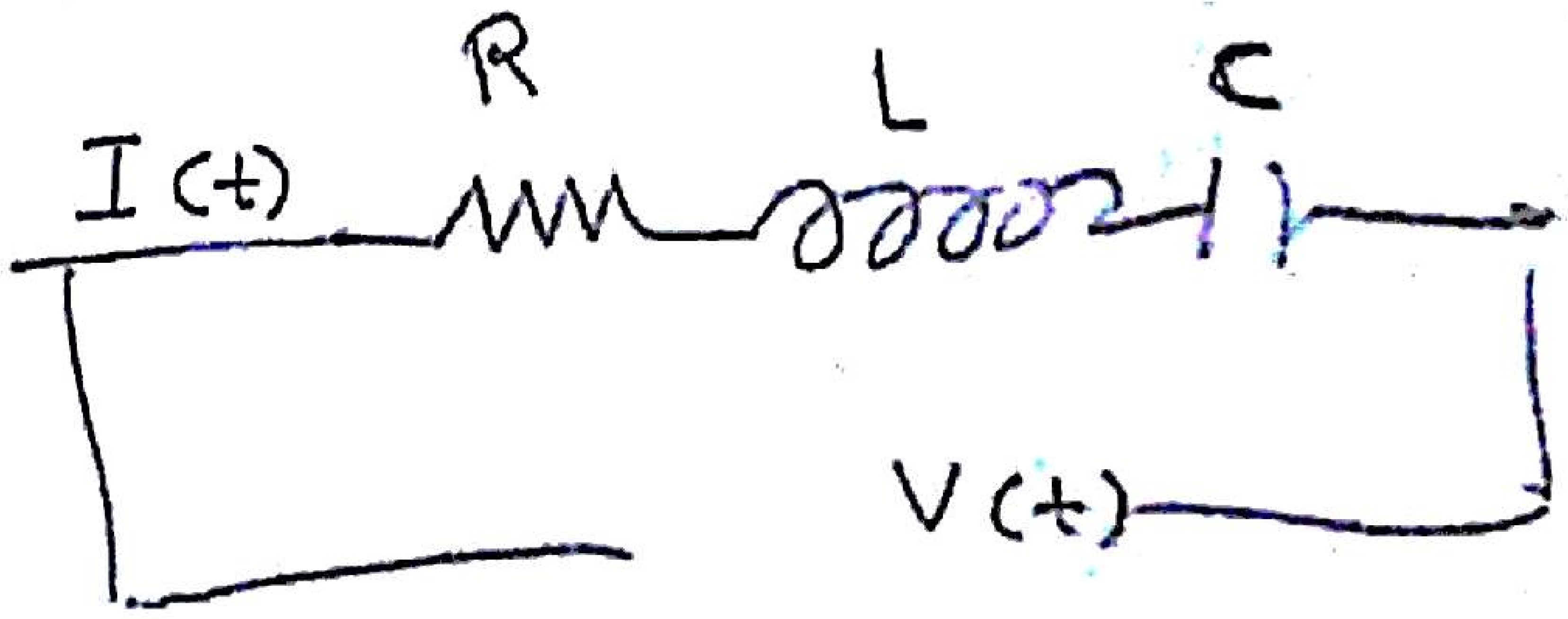
$$\frac{1}{Z_T}=\frac{1}{Z_R}+\frac{1}{Z_L}+\frac{1}{Z_C}$$

$$\frac{1}{Z_T}=\frac{1}{R}+\frac{1}{LS}+CS$$

$$TF=\frac{1}{Z_T}$$



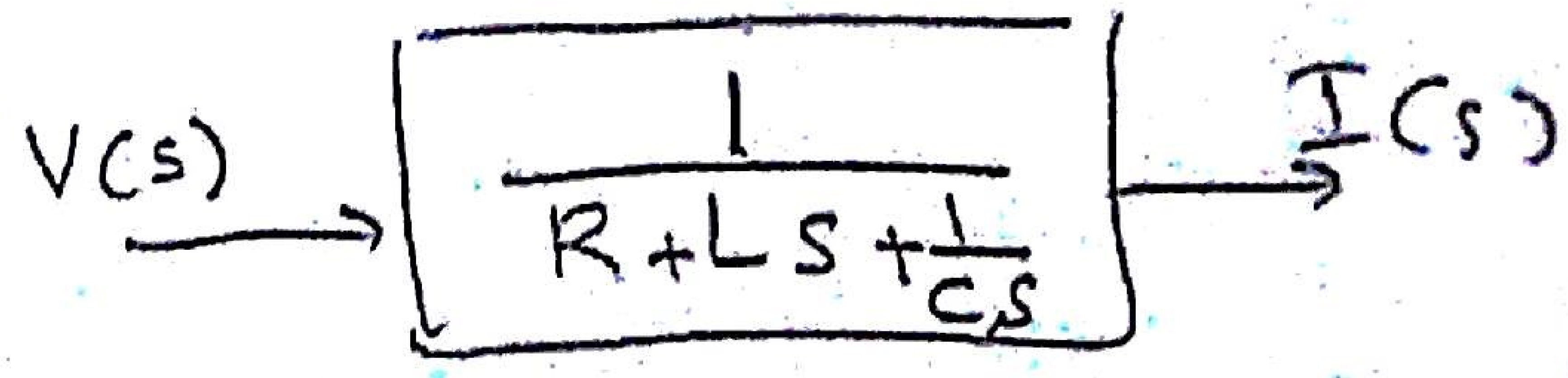
b- Series Connection



$$Z_T=Z_R+Z_L+Z_C$$

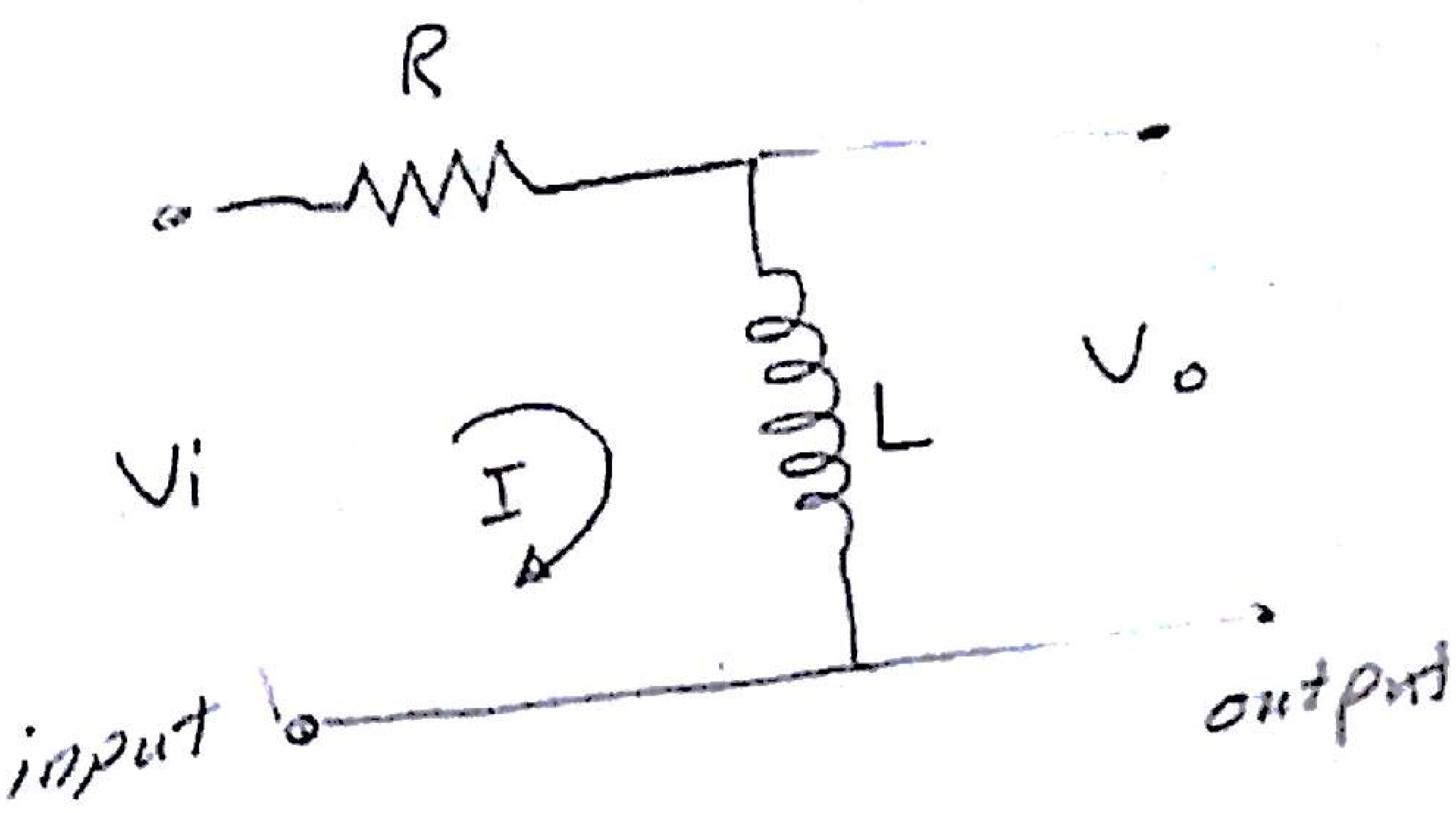
$$Z_T=R+LS+\frac{1}{CS}$$

$$TF=\frac{1}{Z_T}$$



Example 3-1

Find T.F of the given circuit



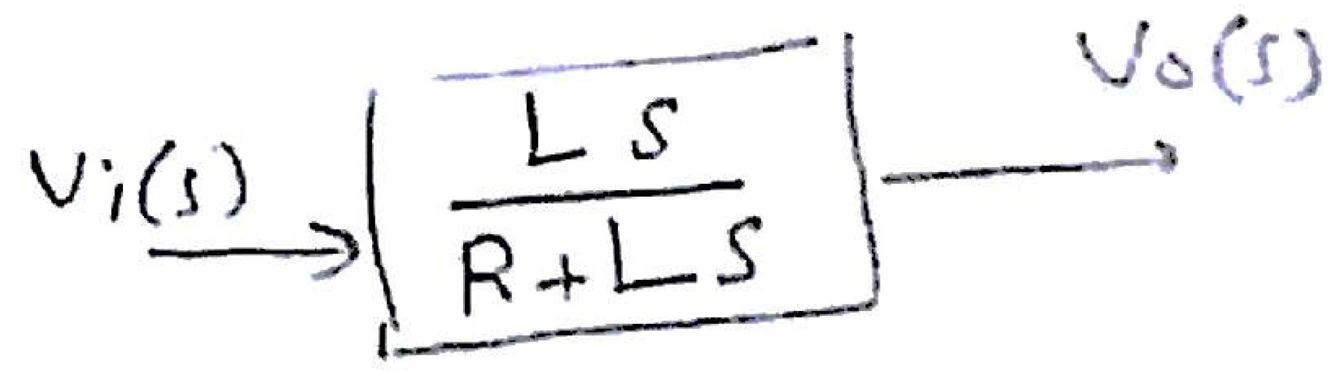
Solution

$$V_i(s) = R\,I(s) + Ls\,I(s) \quad \text{--(1)}$$

$$V_o(s) = Ls\,I(s)$$

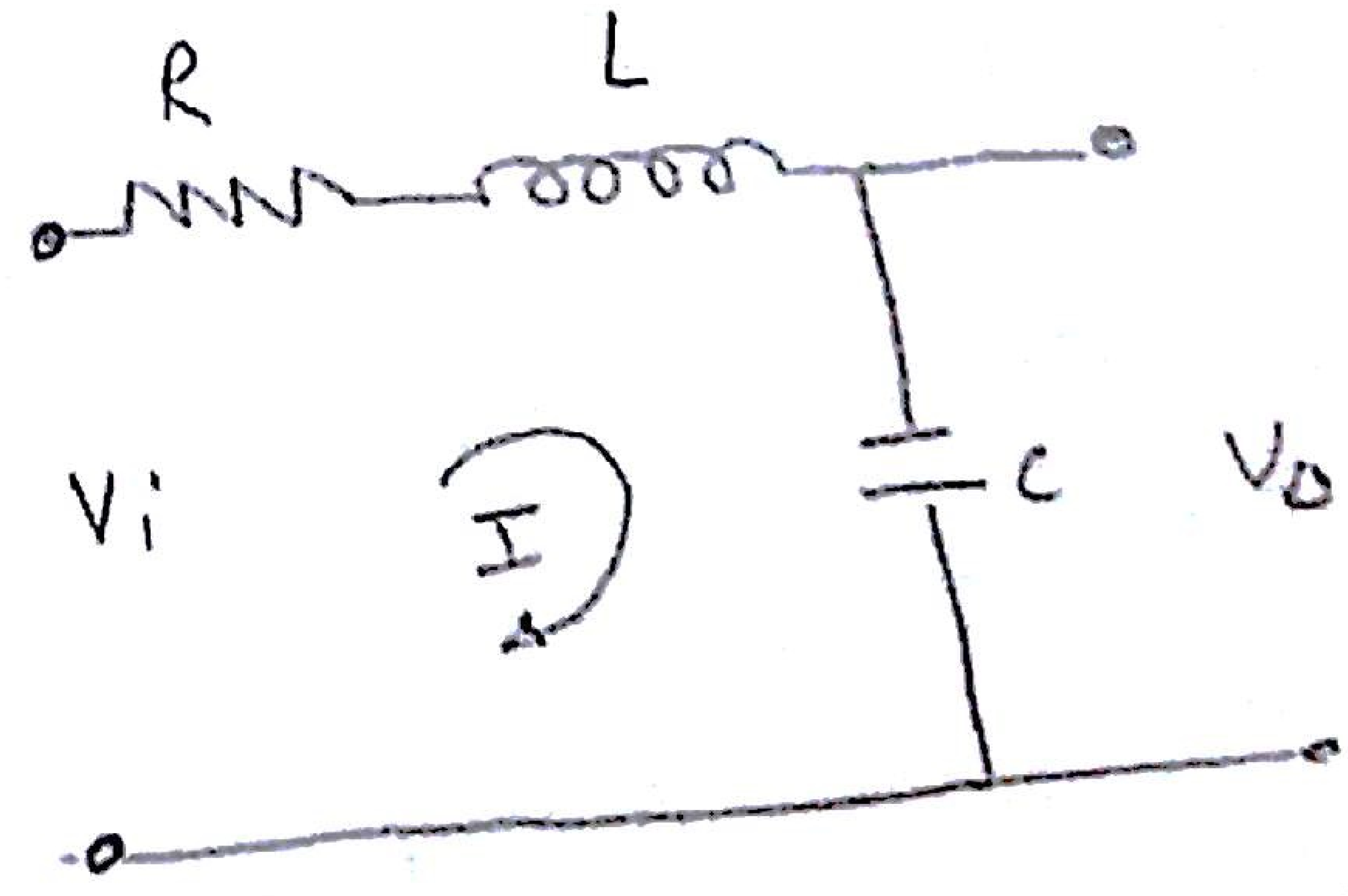
$$\frac{V_o(s)}{V_i(s)} = \frac{Ls\,I(s)}{(R + Ls)\,I(s)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{Ls}{R + Ls}$$



Example 3-2

Determine the T.F of the electrical circuit



Solution

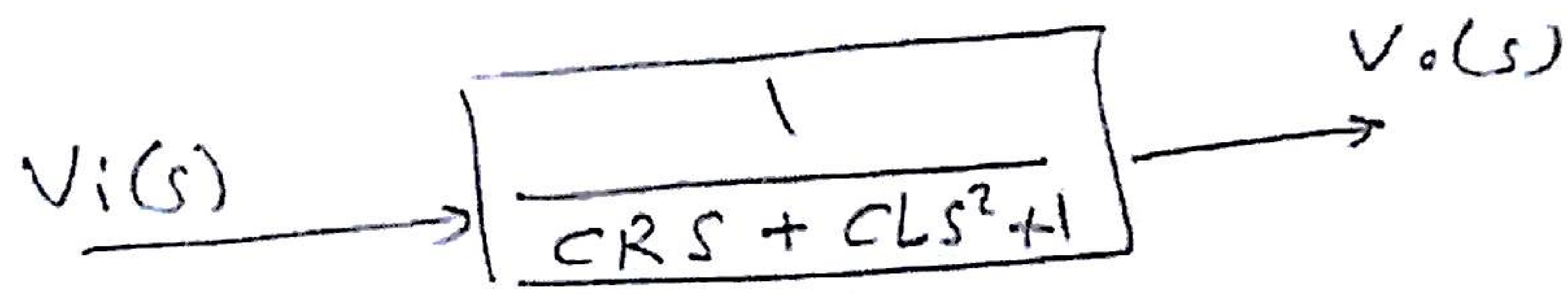
$$V_i(s) = R\,I(s) + Ls\,I(s) + \frac{1}{Cs} I(s)$$

$$V_o(s) = \frac{1}{Cs} I(s)$$

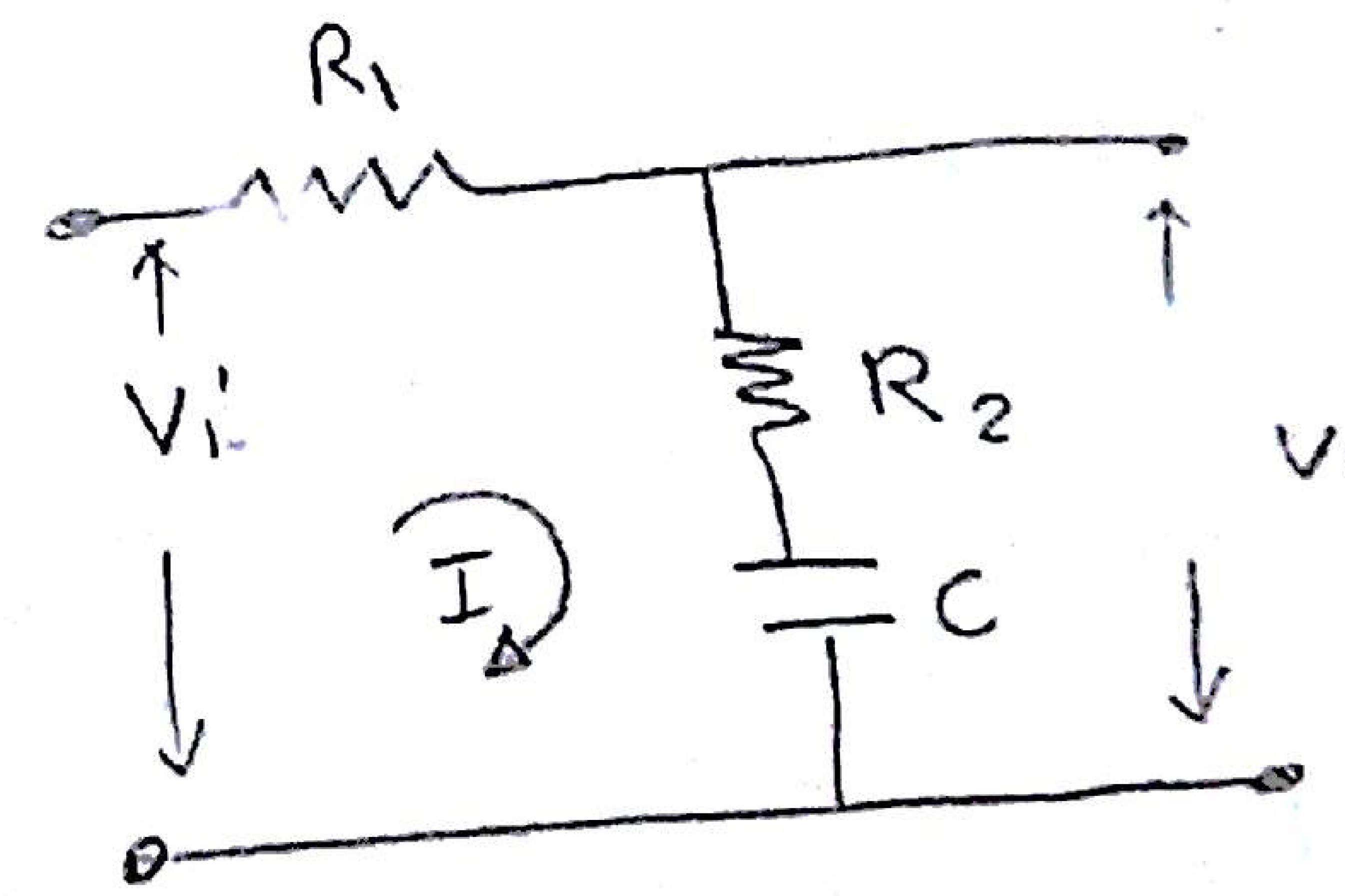
$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{Cs} I(s)}{\left(R + Ls + \frac{1}{Cs}\right) I(s)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{cs\left(R + Ls + \frac{1}{cs}\right)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{CRs + CLs^2 + 1}$$



Example 3-3   Find T.F



$$Z_T = Z_R + Z_C$$

$$Z_T = R_1 + R_2 + \frac{1}{CS}$$

$$V_i(s) = R_1 I(s) + R_2 I(s) + \frac{1}{CS} I(s)$$

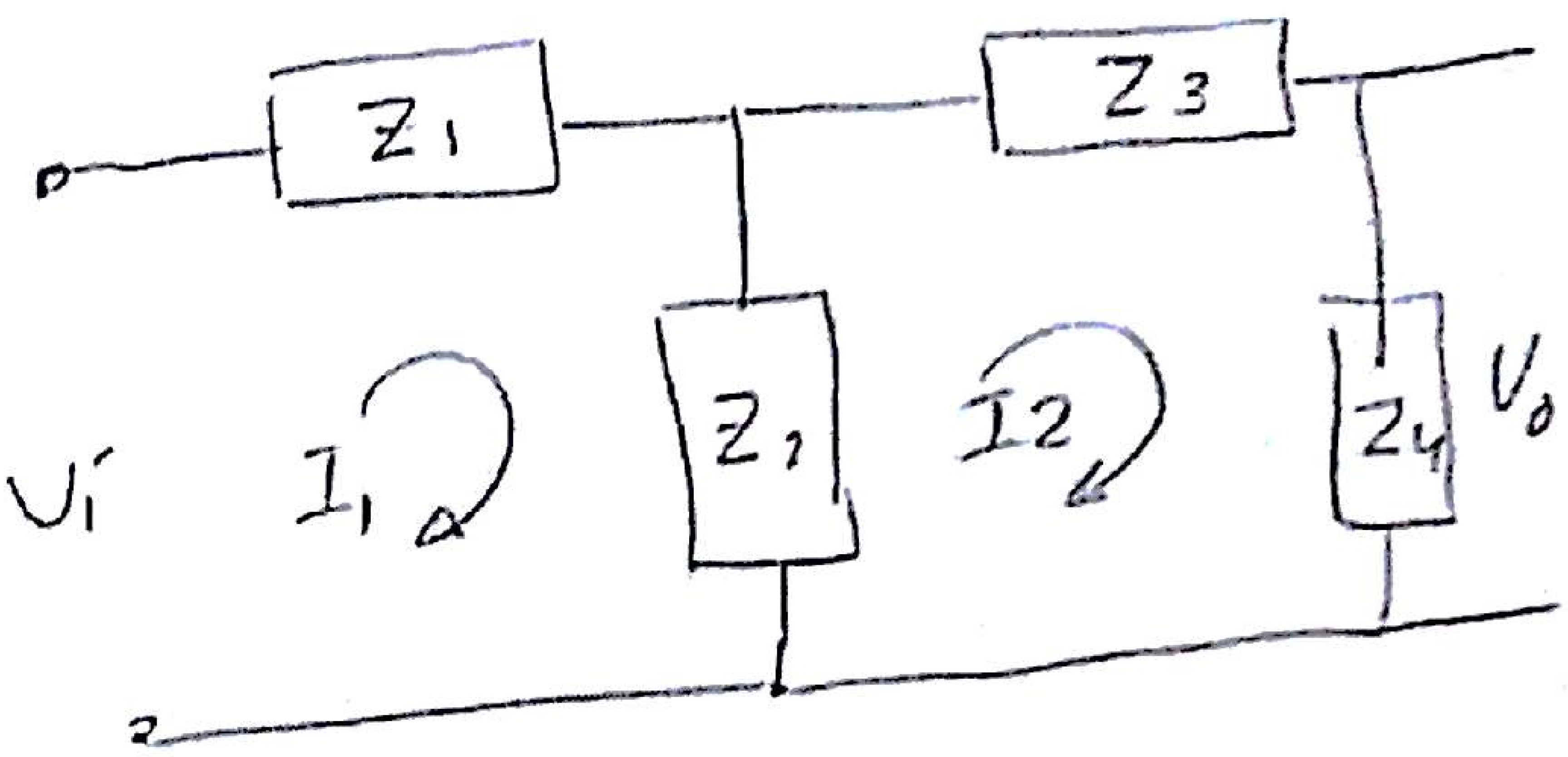
$$V_i(s) = \left(R_1 + R_2 + \frac{1}{CS}\right) I(s)$$

$$V_o(s) = \left(R_2 + \frac{1}{CS}\right) I(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\left(R_2 + \frac{1}{CS}\right) I(s)}{\left(R_1 + R_2 + \frac{1}{CS}\right) I(s)}$$

Example 3-4

Find T.F



Solution

$$V_i(s) = Z_1 I_1 + Z_2 I_1 - Z_2 I_2 \quad \text{——} (1)$$

Voltag droop around closed loop = 0

$$Z_2 I_2 + Z_3 I_2 + Z_4 I_2 - Z_2 I_1 = 0 \quad \text{——} (2)$$

Then

$$I_1 = \frac{(Z_2+Z_3+Z_4) I_2}{Z_2} \quad \text{——} (3)$$

$$V_o = Z_4 I_2 \quad (4)$$

sub (3) in (1)

$$V_i(s) = (Z_1+Z_2)\frac{(Z_2+Z_3+Z_4) I_2}{Z_2} - Z_2 I_2$$

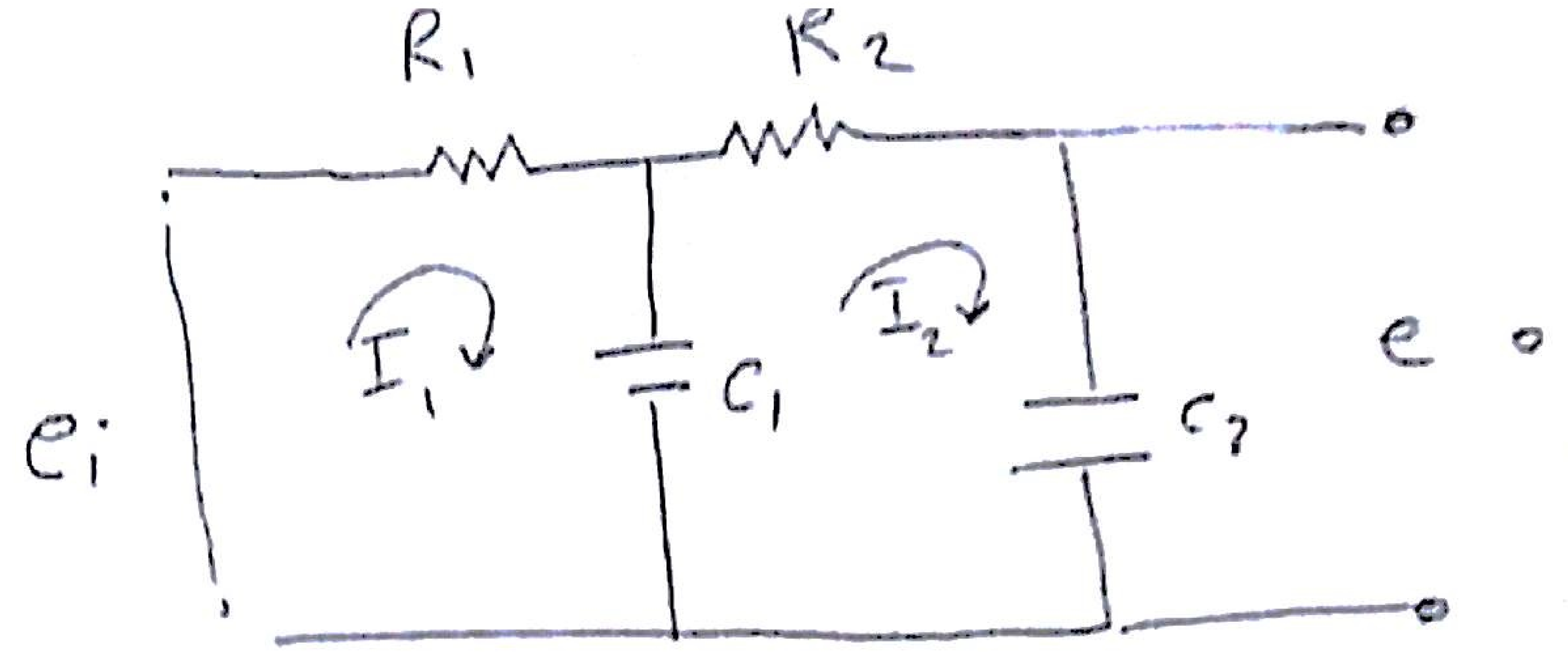
$$V_i(s) = \frac{\left[(Z_1+Z_2)(Z_2+Z_3+Z_4) - Z_2^2\right] I_2}{Z_2}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_4 Z_2 \, \cancel{I_2}}{\left[(Z_1+Z_2)(Z_2+Z_3+Z_4) - Z_2^2\right] \cancel{I_2}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_2 Z_4}{Z_1 Z_2 + Z_1 Z_3 + Z_1 Z_4 + Z_2 Z_3 + Z_2 Z_4}$$

H.W

Find T.F



Ans

$$\frac{e_o(s)}{e_i(s)} = \frac{1}{R_2C_2S + R_1C_2S + R_1R_2C_1C_2S^2 + R_1C_1S + 1}$$

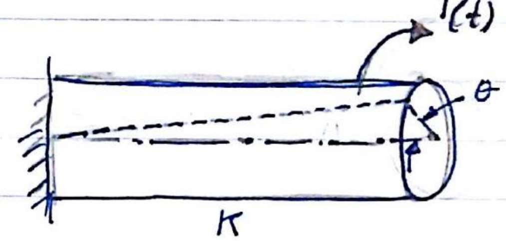
Lecture four

4-1 Modeling of Rotational systems

a- Torsional spring :-

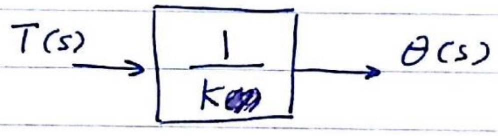
The restoring torque of a spring is proportinal to angular displacement θ and is given by

$$T(t) = K\,\theta(t)$$



$$T(s) = K\,\theta(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{K}$$



Where :- K is Torsional stiffness of the spring
θ is the angular displacement.

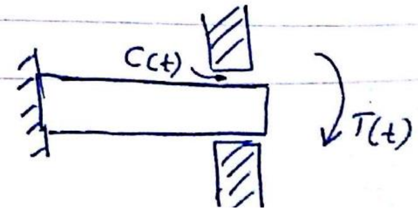
---

b- Torsional Damping

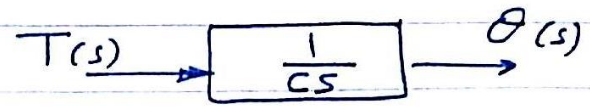
Frictional Torque it is opposes to rotation motion.

$$T(t) = C_t\,\dot{\theta}(t)$$

$$T(s) = C s\,\theta(s)$$



$$\frac{\theta(s)}{T(s)} = \frac{1}{Cs}$$



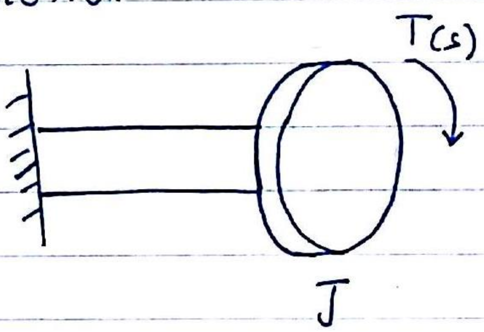
where : C: is the rotational frictional
$\dot{\theta}$ Is the angular velocity

(C) Moment of inertia

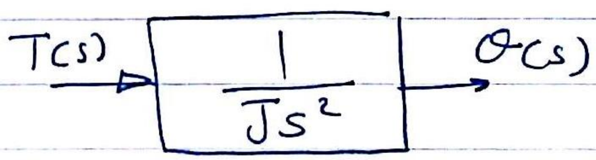
opposes to rotational motion.

$$T(t) = J\ddot{\theta}(t)$$

$$T(s) = Js^2\,\theta(s)$$



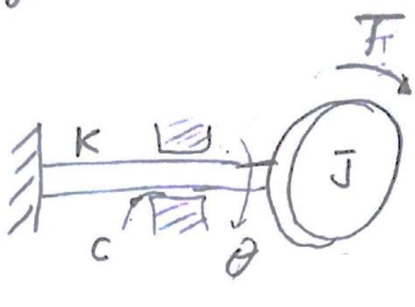
$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2}$$



Where J: moment of inertia of the body.

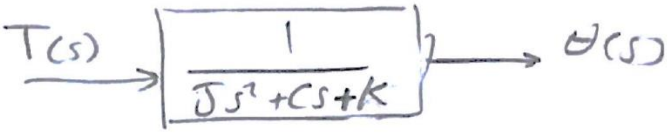
$\ddot{\theta}$: angular acceleration.

Example 4:1 Find the T.F of the system shown in the Figure.

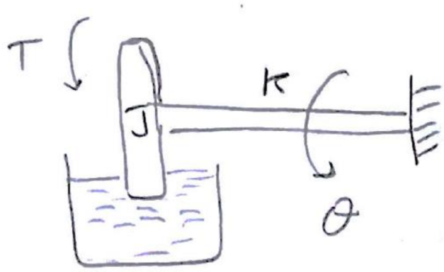


$$T(s) = Js^2\theta(s) + Cs\,\theta(s) + K\theta(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Cs + k}$$

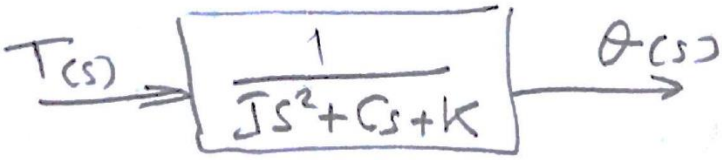


Example 4:2 Find T.F



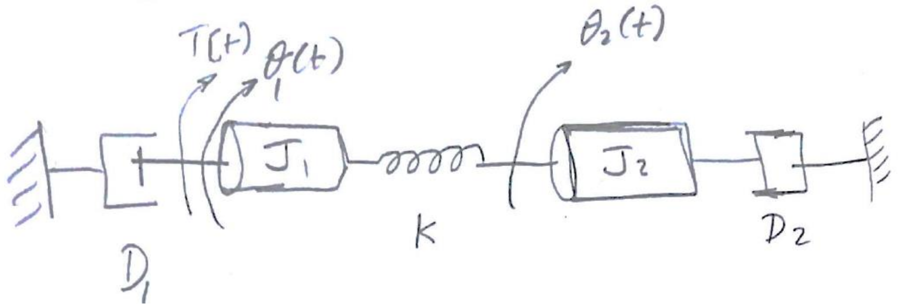
$$T(s) = (Js^2 + Cs + k)\,\theta(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Cs + k}$$



Example 4-3

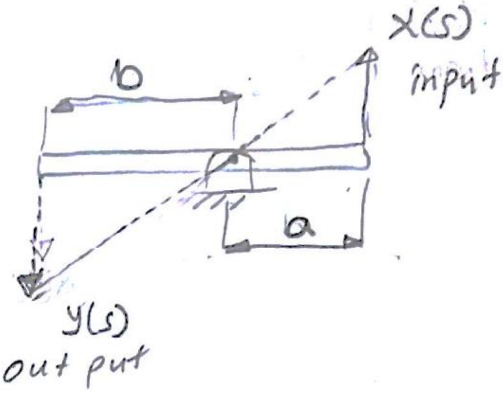
Find the T.F $\frac{\theta_2(s)}{T(s)}$ for the rotational system shown

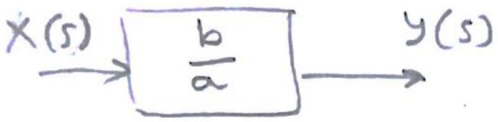


4-2 Leverage Systems

a- Middle - Fixed lever

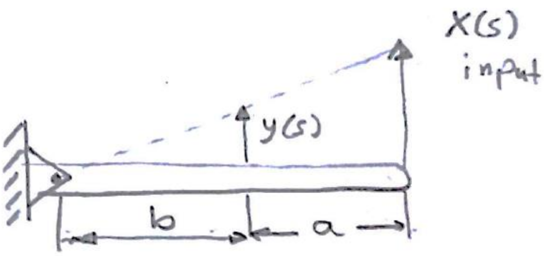
$$\frac{Y(s)}{X(s)} = \frac{b}{a}$$

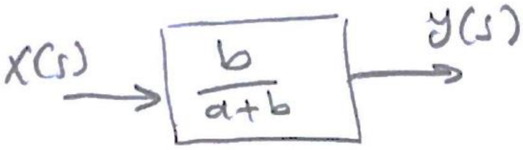




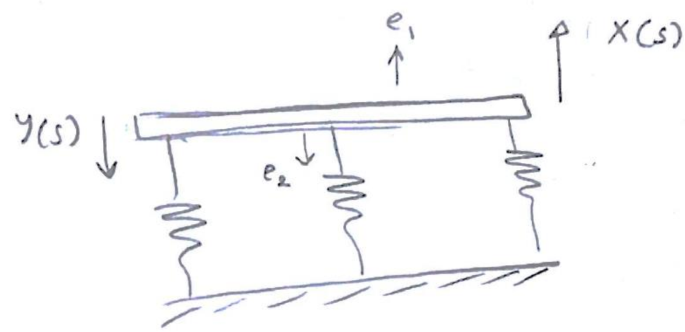
b- Fixed-end lever

$$\frac{Y(s)}{X(s)} = \frac{b}{a+b}$$

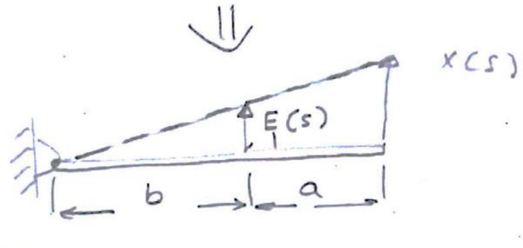




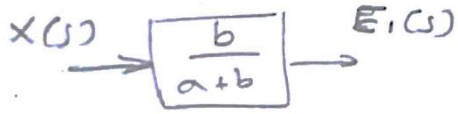
c. Free-end lever



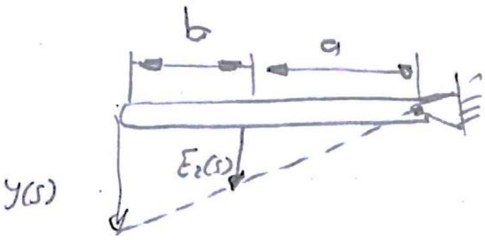
$$\frac{E(s)}{X(s)} = \frac{b}{a+b}$$



$$E_1(s) = \frac{b}{a+b} X(s)$$

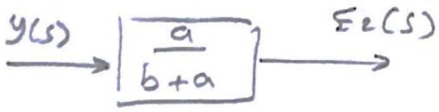


\+



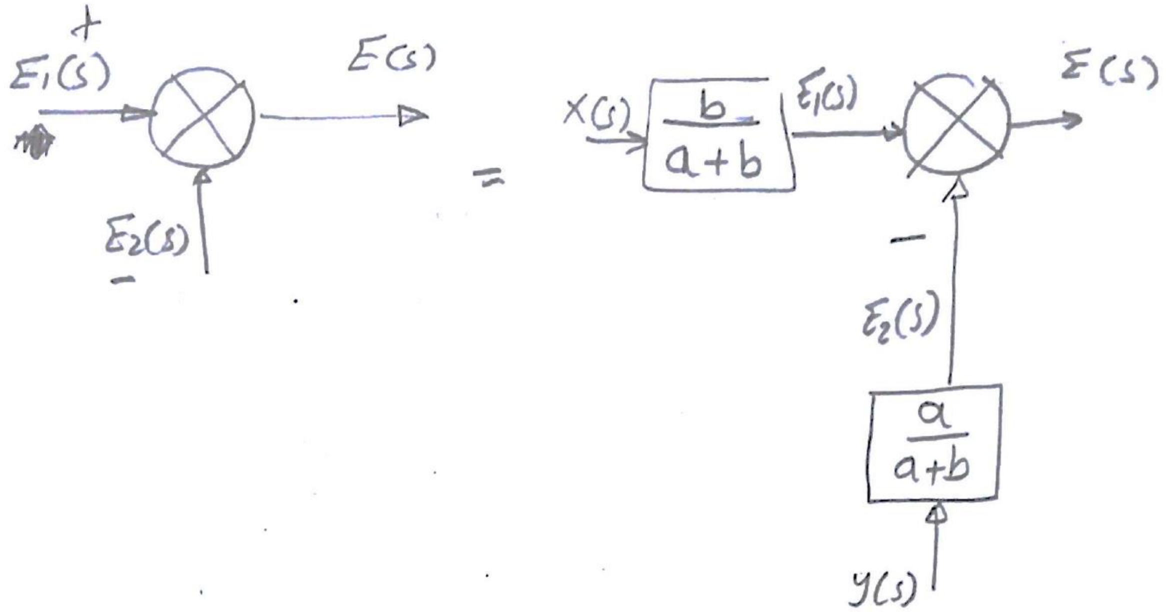
$$\frac{E_2(s)}{Y(s)} = \frac{a}{a+b}$$

$$E_2(s) = \frac{a}{b+a} Y(s)$$



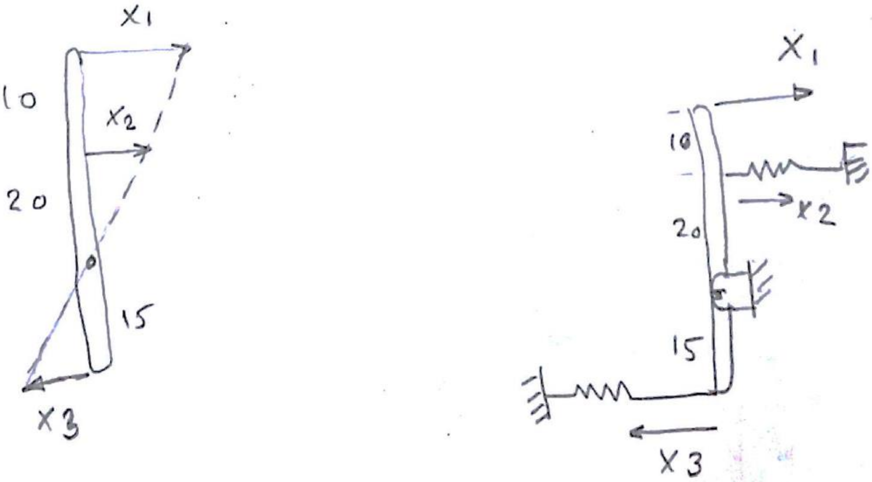
$$\therefore\ E(s) = E_1(s) - E_2(s)$$

$$= \frac{b}{a+b} X(s) - \frac{a}{a+b} Y(s)$$



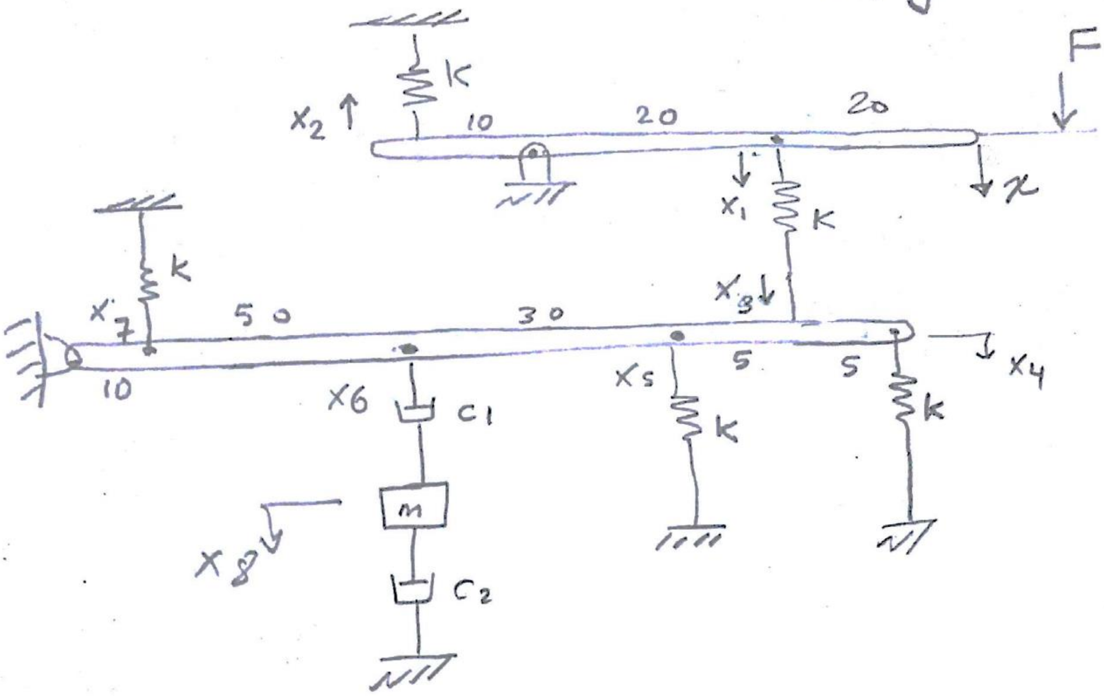
Example 4-4

Find $\frac{X_2}{X_1}$ and $\frac{X_3}{X_1}$ of the Lever shown

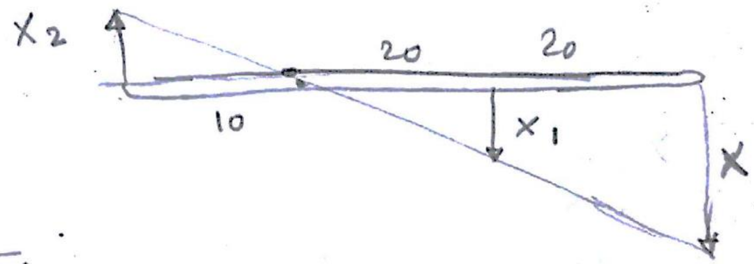


$$\frac{X_2}{X_1} = \frac{20}{30} \quad , \quad \frac{X_3}{X_1} = \frac{15}{30}$$

Example 4.5 Find T.F For the system shown in the figure.



solution

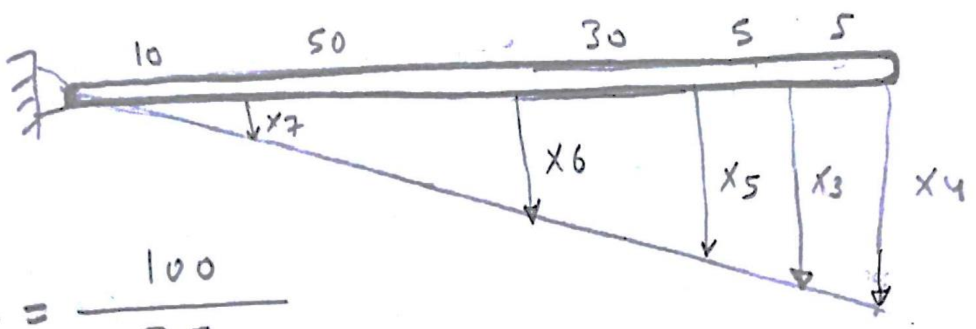


$$\frac{X_1}{X} = \frac{20}{40}$$

$$\boxed{X_1 = \frac{1}{2} x}$$

$$\frac{X_2}{X} = \frac{10}{40}$$

$$\boxed{X_2 = \frac{1}{4} x}$$



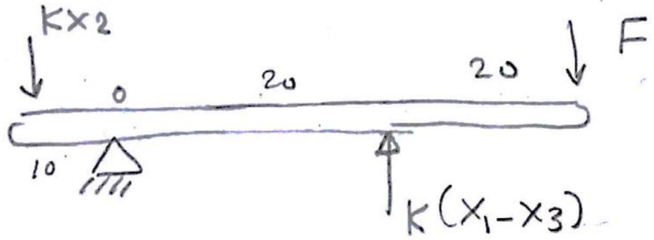
$$\frac{x_4}{x_3} = \frac{100}{95}$$

$$\boxed{x_4 = \frac{100}{95} x_3} \quad \text{------ (3)}$$

$$\frac{x_5}{x_3} = \frac{90}{95} \Rightarrow \boxed{x_5 = \frac{90}{95} x_3} \quad \text{--- (4)}$$

$$\frac{x_6}{x_3} = \frac{60}{95} \Rightarrow \boxed{x_6 = \frac{60}{95} x_3} \quad \text{--- (5)}$$

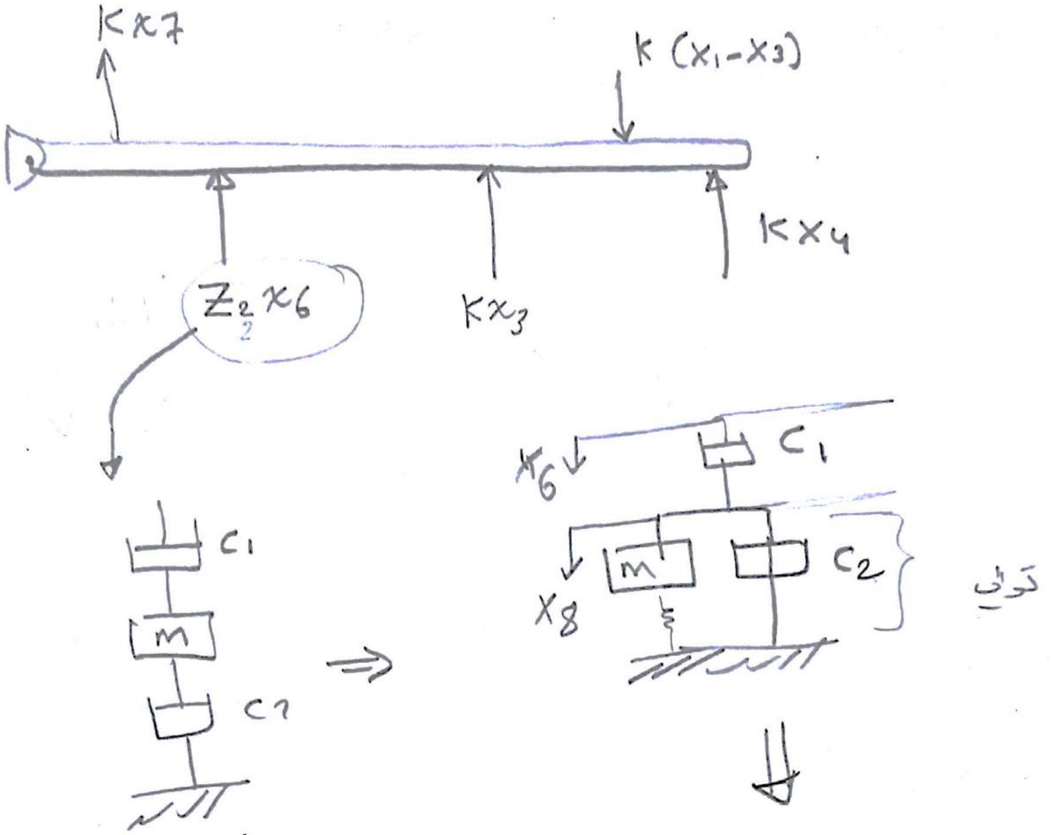
$$\frac{x_7}{x_3} = \frac{10}{95} \Rightarrow \boxed{x_7 = \frac{10}{95} x_3} \quad \text{--- (6)}$$



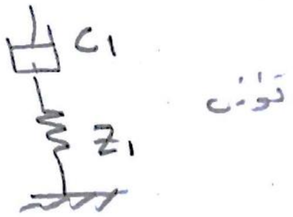
$$\Sigma M_o = 0$$

$$F * 40 - K(x_1 - x_3) * 20 - K x_2 * 10 = 0$$

$$\boxed{40F - (Kx_1 - Kx_3)20 - 10Kx_2 = 0} \quad \text{--- (7)}$$

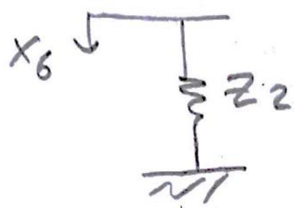


$$Z_1 = c_2 s + m s^2$$



$$\frac{1}{Z_2} = \frac{1}{Z_1} + \frac{1}{c_1 s}$$

$$\frac{1}{Z_2} = \frac{c_1 s + Z_1}{Z_1 c_1 s} \Rightarrow \boxed{Z_2 = \frac{Z_1 c_1 s}{Z_1 + c_1 s}}$$



$$\Sigma M_o = 0$$

$$K(x_4 - x_3) * 95 - K x_4 * 100 - K x_5 * 90 - Z_2 x_6 * 60$$
$$- K x_7 * 10 = 0 \quad \text{--- (8)}$$

$x_2 = \frac{1}{4} X$ $\quad x_1 = \frac{1}{2} X$ نعوض من معادلة (7)

$$40F - K\left(\frac{1}{2}x - x_3\right)20 - K\left(\frac{1}{4}X\right) * 10 = 0$$

$$40F - 10Kx + 20Kx_3 - \frac{10}{4}Kx = 0 \qquad \div 10$$

$$4F - Kx + 2Kx_3 - \frac{1}{4}Kx = 0$$

$$2Kx_3 = \frac{1}{4}Kx + Kx - 4F = 0 \qquad \div 2K$$

$$x_3 = \frac{1}{8}X + \frac{X}{2} - \frac{2F}{K}$$

$$\boxed{X_3 = \frac{5}{8}x - \frac{2F}{K}} \quad \text{--- (9)}$$

من معادلة (8)

$$K\left(\frac{1}{2}x - x_3\right) * 95 - K\left(\frac{100}{95}x_3\right) * 100 - K\left(\frac{90}{95}x_3\right)90$$
$$- \left(Z\frac{60}{95}x_3\right) * 60 - \left(K\frac{10}{95}x_3\right) * 10 = 0$$

$$\frac{95}{2}Kx - 95Kx_3 - \frac{100\times100}{95}Kx_3 - \frac{90\times90}{95}Kx_3$$

$$-\frac{60\times60}{95}Zx_3 - \frac{10\times10}{95}Kx_3 = 0$$

$$\frac{95}{2}Kx = 95Kx_3 + \frac{100\times100}{95}Kx_3 + \frac{90\times90}{95}Kx_3$$

$$+\frac{60\times60}{95}Zx_3 + \frac{10\times10}{95}Kx_3$$

$$\frac{95}{2}Kx = \left[95K + \frac{100\times100}{95}K + \frac{90\times90}{95}K + \frac{60\times60}{95}Z\right.$$

$$\left. + \frac{10\times10}{95}K\right]x_3$$

$$\frac{95}{2}Kx = \left[220.63K + 25.263Z\right]x_3$$

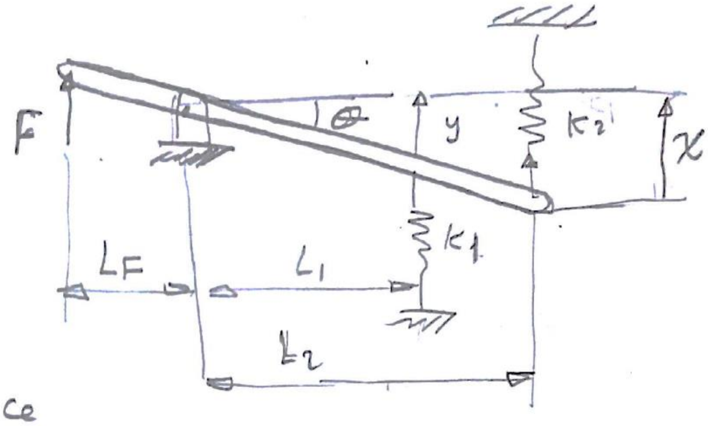
$$\frac{95}{2}Kx = \left[220.63K + 25.263Z\right]\left[\frac{5}{8}x - \frac{2F}{K}\right]$$

$$\frac{x}{F} = \boxed{\quad}$$

y

Example 4-6

For the Lever shown the variation in the applied force F. and the variation in spring position x. The horzental line represents the reference position of the lever.



a) - determine the equation relating F and x

b) - Determine the relationship between t and θ where $t = FL_f$ and $x = L_2\theta$

(a)

$$\frac{y}{x} = \frac{L_1}{L_2} \Rightarrow y = \frac{L_1}{L_2} x$$

$$\Sigma M_o = 0$$

$$FL_f = k_1 y L_1 + k_2 x L_2$$

$$FL_f = k_1 \frac{L_1^2}{L_2} x + k_2 x L_2 \Rightarrow F = \left(\frac{k_1 \frac{L_1^2}{L_2} + k_2 L_2}{L_F}\right) X$$

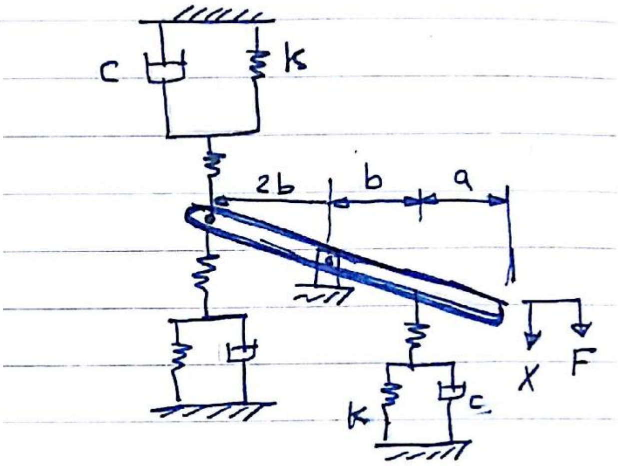
(b) $X = L_2\theta$ and $t = FL_f$

$$t = k_1 \frac{L_1^2}{L_2}(L_2\theta) + k_2 (L_2\theta) L_2$$

$$t = (k_1 L_1^2 + k_2 L_2^2)\theta$$

(Ex - 4-7)
Find T.F for the system shown in the figure.

H.W



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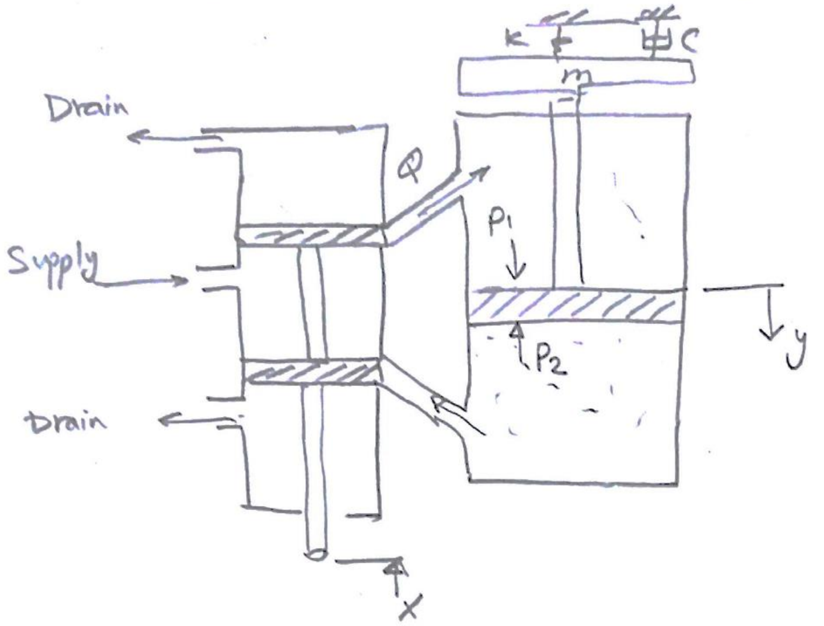
Hydraulic systems



① when load = 0

$P_1 = P_2$

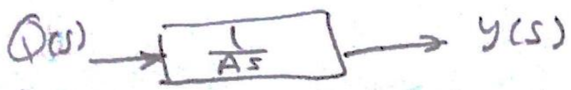
1- valve

$Q_{(s)} = C\, X_{(s)}$ Q = flow rate m^3/sec

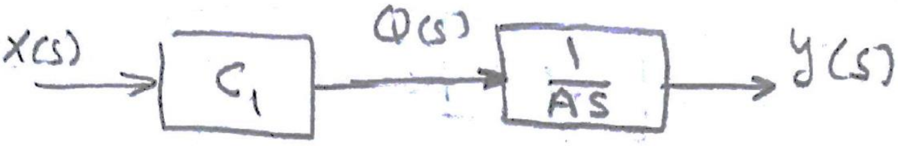
2- cylinder

$Q = A y'$

$Q_{(s)} = A S\, y_{(s)}$



overall Block diagram



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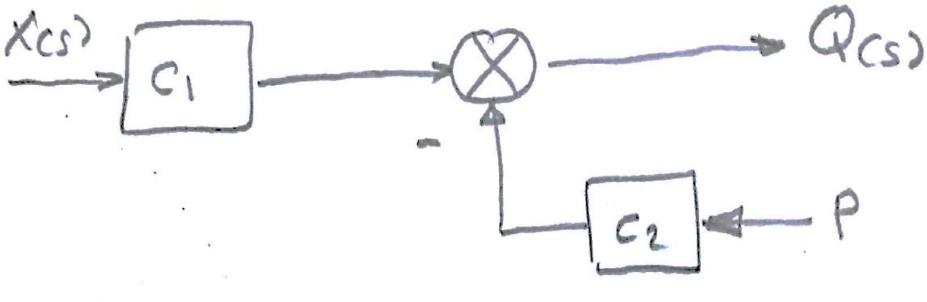
(2) when load ≠ 0

$P_1 \neq P_2$

(a) valve

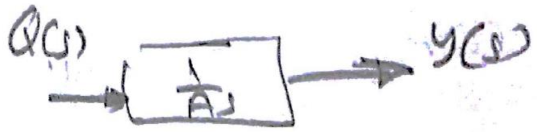
$Q = C_1 x - C_2 P$ where $P = \frac{F}{A}$

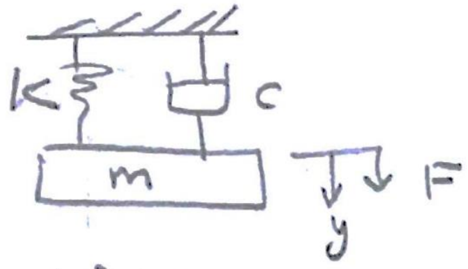
لما زادت P تقل Q



b (cylinder)

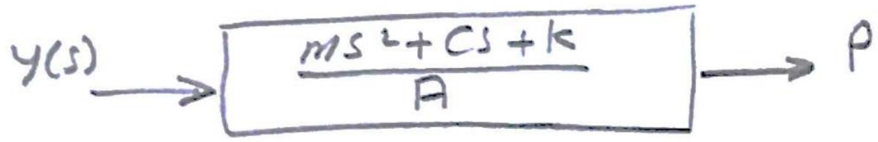
$y_{(s)} = \frac{1}{AS} Q(s)$



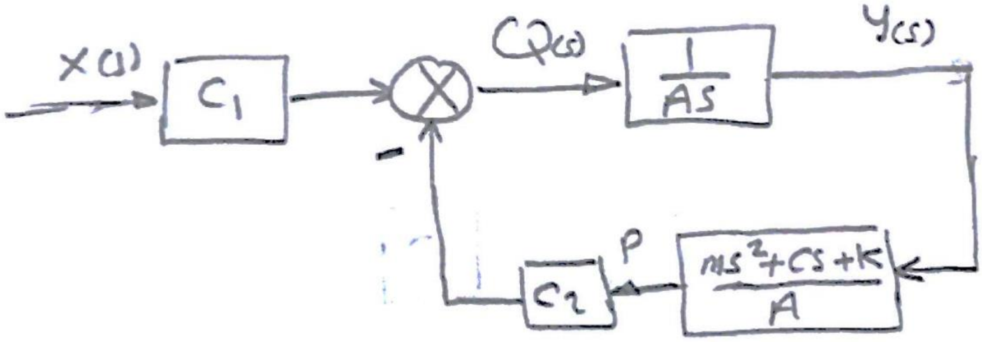


$$F = PA = (ms^2 + Cs + k)\, y(s)$$

$$P = \frac{ms^2 + Cs + k}{A}\, y(s)$$



<u>overall Block diagram</u>



Lecture Five

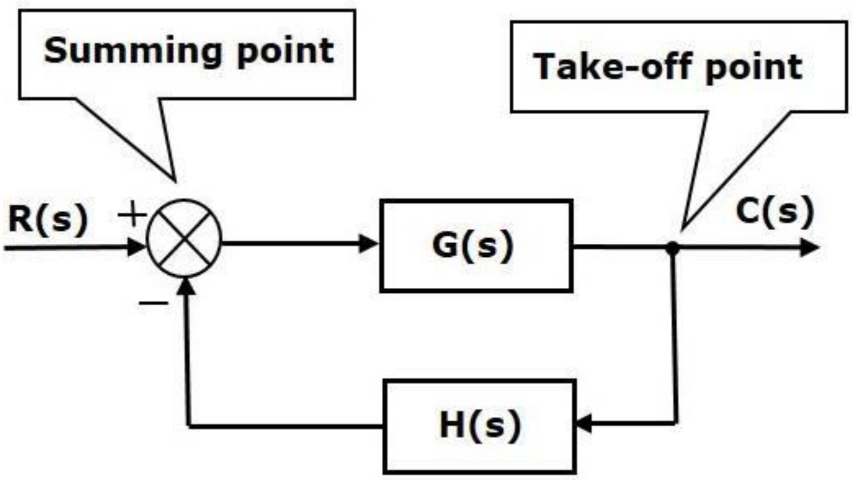
Block Diagram

Control system contains number of component the function of each component can be represented by diagram call block diagram. Block diagram is a technique used to give a perspective view of the functioning of a system, showing an overall picture of the interconnections among various components and subsystems by the direction of signal flow, which is not available from a purely abstract mathematical representation.

In control system the transfer function concept is very important; as it describes the input-output relationships of components and subsystems. The transfer function is a mathematical model; it does not give any information about the physical nature of the actual system. However, by knowing the transfer function, the response of the system when subjected to various inputs can be thoroughly investigated.

5-1 Basic Elements of Block Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point. Let us consider the block diagram of a closed loop control system as shown in the following figure to identify these elements.



The above block diagram consists of two blocks having transfer functions G(s) and H(s). It is also having one summing point and one take-off point. Arrows indicate the direction of the flow of signals. Let us now discuss these elements one by one.

6-1-1 Block

The transfer function of a component is represented by a block. Block has single input and single output. The following figure shows a block having input X(s), output Y(s) and the transfer function G(s).

Transfer function, $$G(s) = \frac{Y(s)}{X(s)}$$

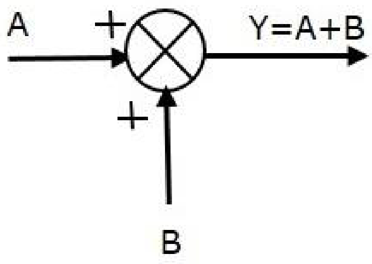
$$\Rightarrow Y(s) = G(s)X(s)$$

6-1-2 Summing Point (comparator)

The summing point is represented with a circle having cross (X) inside it. It has two or more inputs and single output. It produces the algebraic sum of the inputs. It also performs the summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of the inputs. Let us see these three operations one by one.

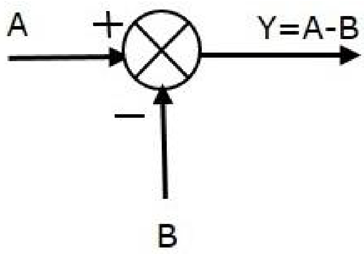
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B have a positive sign. So, the summing point produces the output, Y as **sum of A and B**.

i.e., $Y = A + B$.



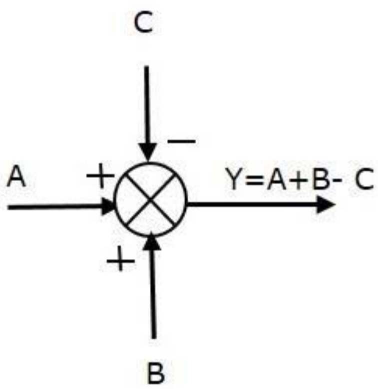
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y** as the **difference of A and B**.

$Y = A + (-B) = A - B.$



The following figure shows the summing point with three inputs (A, B, C) and one output (Y). Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as

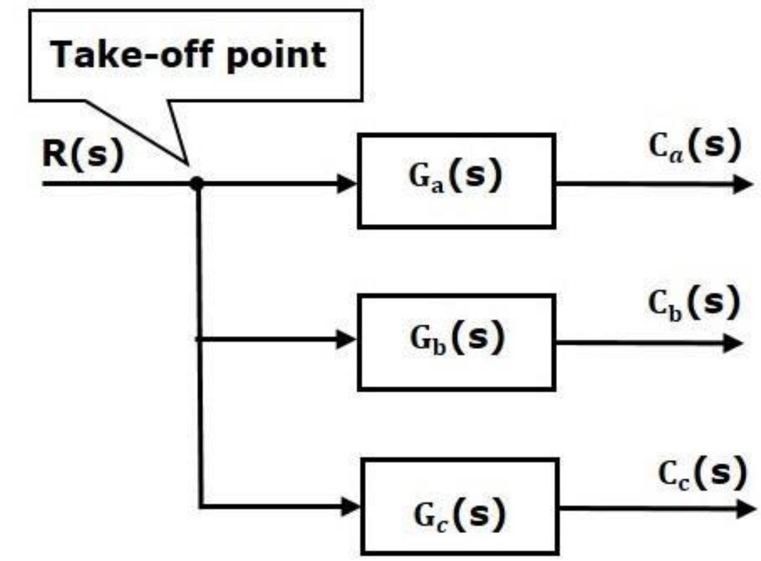
$Y=A+B+(-C)=A+B-C.$



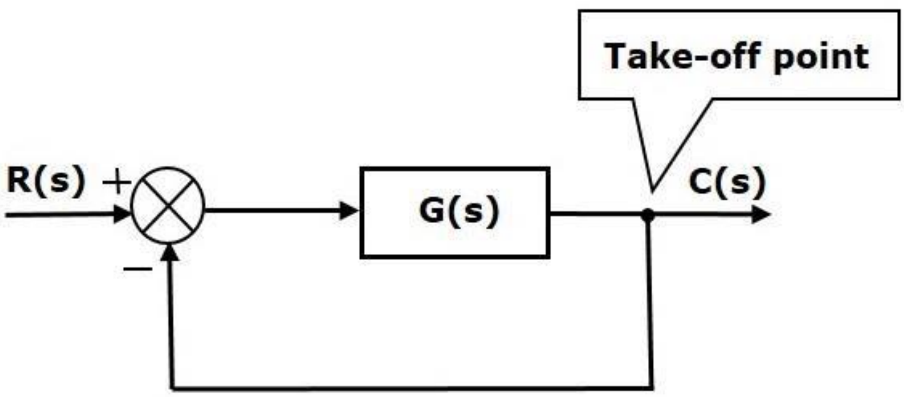
6-1-3 Take-off Point

The take-off point is a point from which the same input signal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks, summing points.

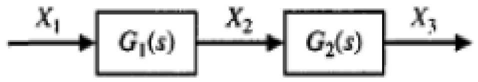
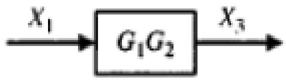
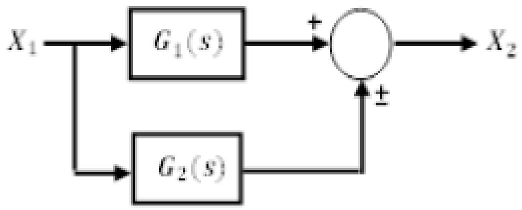
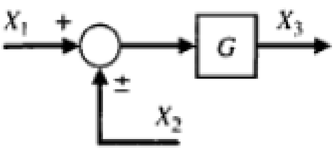
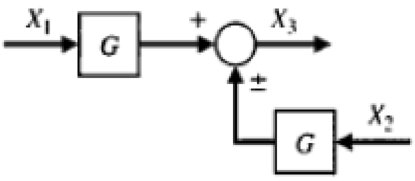
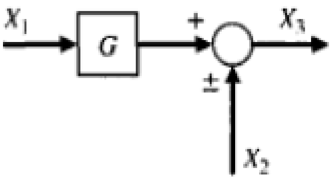
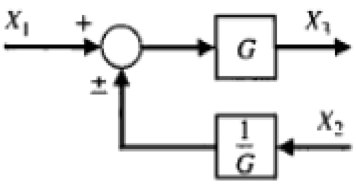
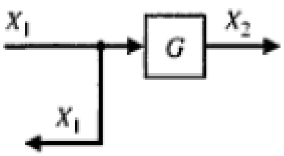
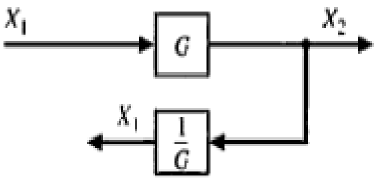
In the following figure, the take-off point is used to connect the same input, R(s) to two more blocks.

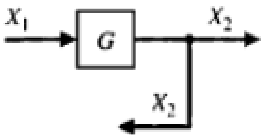
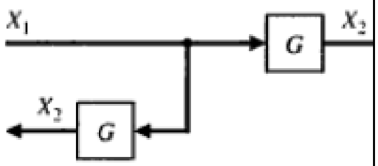
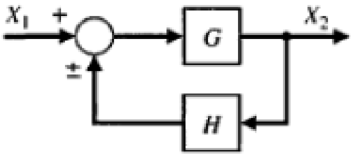
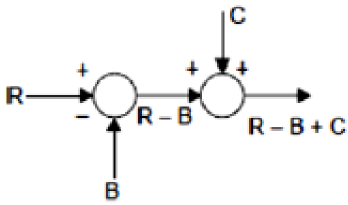
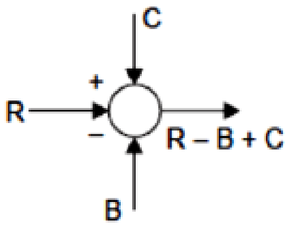
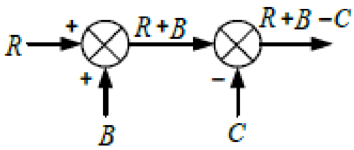
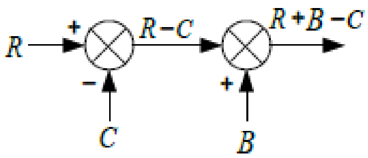


In the following figure, the take-off point is used to connect the output $C(s)$, as one of the inputs to the summing point.



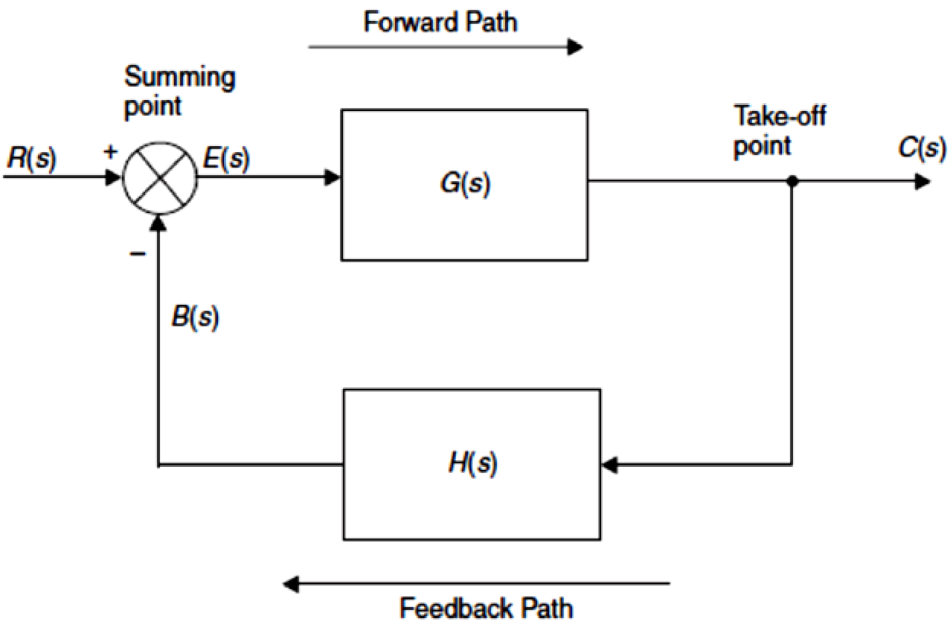
6-2 Block Diagram Reduction Rules

Transformation	Original Diagram	Equivalent Diagram
1- Combining blocks in series		
2- Combining blocks in parallel		
3- Moving a comparator after a block		
4- Moving a comparator before a block		
5- Moving a pick-off point after a block		

6- Moving a take-off point before a block		
7- Eliminating a feedback loop		
8- combining of comparators		
9- Changing between comparators		

Notice:

The comparator cannot jump over a take-off point and the opposite is true.



However, to find the *closed-loop transfer function* for the above figure the output *C(s)* and input *R(s)* are related as follows:

$$C(s) = G(s)E(s) \dots\dots\dots\dots\dots\dots\dots \quad 1$$

$$B(s) = H(s)C(s) \dots\dots\dots\dots\dots\dots\dots\dots \quad 2$$

$$E(s) = R(s) - B(s) \dots\dots\dots\dots\dots\dots\dots \quad 3$$

By substituting equation 2 and 3 in equation 1:

$$C(s) = G(s)\{R(s) - H(s)C(s)\}$$

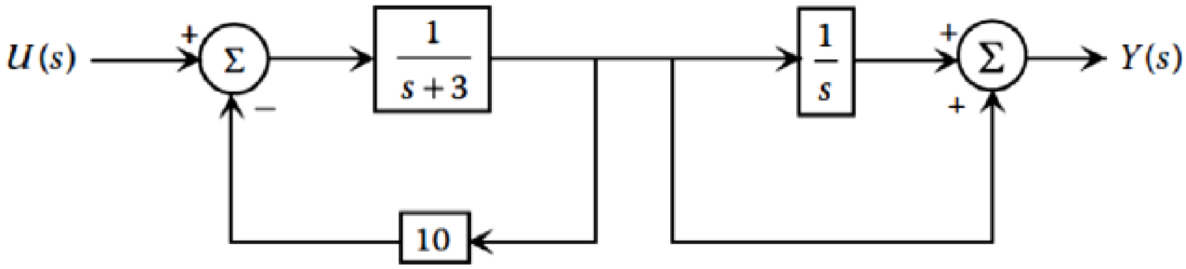
$$C(s) = G(s)R(s) - G(s)H(s)C(s)$$

$$C(s)\{1 + G(s)H(s)\} = G(s)R(s)$$

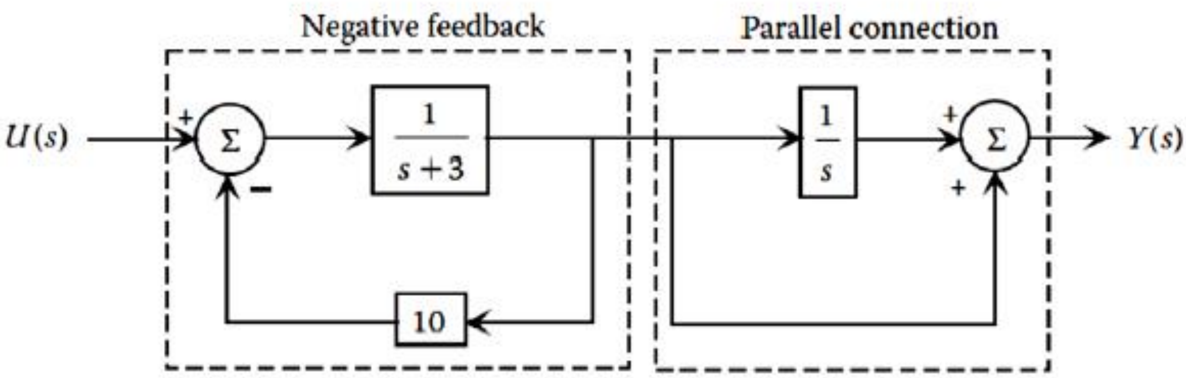
Thus

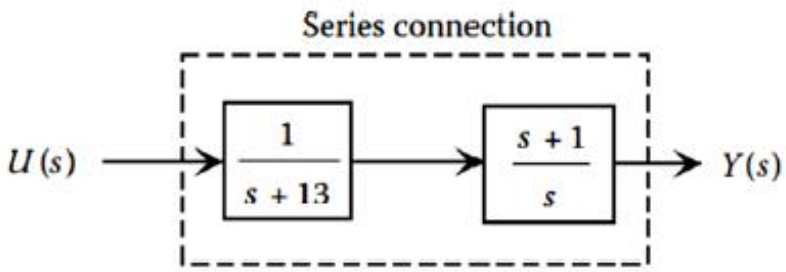
$$Closed-loop\ Transfer\ function\ = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

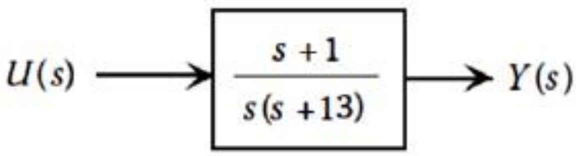
Example 5-1: Using reduction techniques, simplify the block diagram shown in the figure below to a single block with input $U(s)$ and output $Y(s)$, and determine the overall transfer function $Y(s)/U(s)$.



Solution:



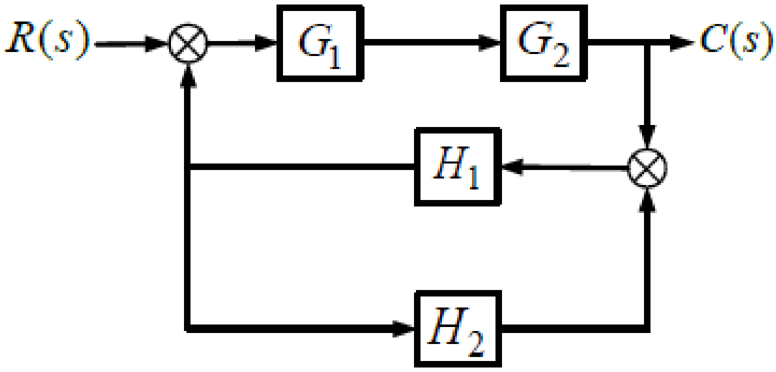




Thus, the overall transfer function can be easily found as

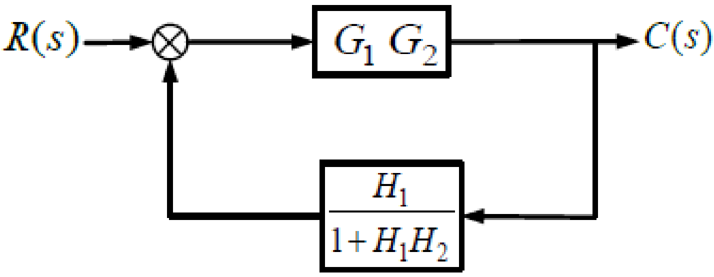
$$\frac{Y(s)}{U(s)} = \frac{s+1}{s(s+13)}$$

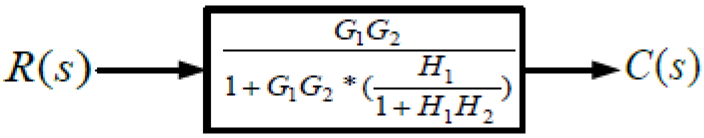
Example 5-2: Simplify the control system shown below and obtain the transfer function *C(s)/R(s)*.



Solution:

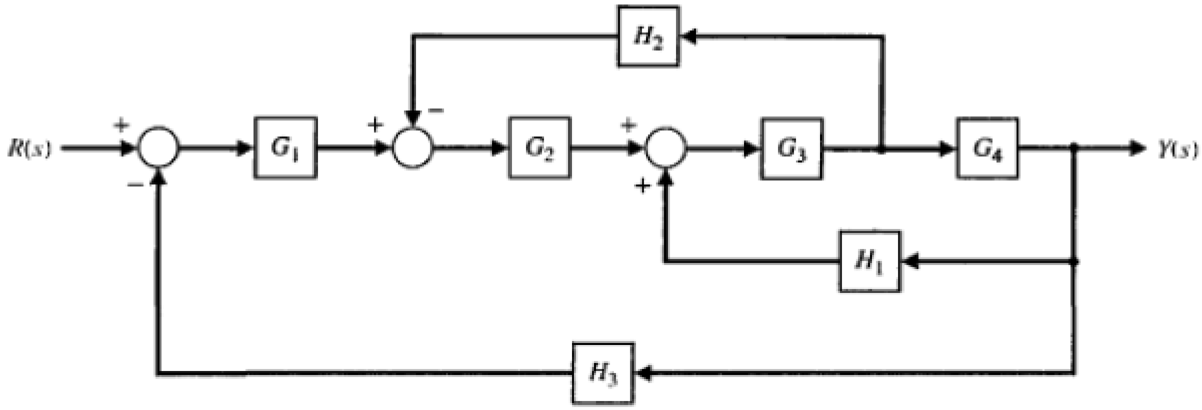
From the figure it can be noticed that *G1* and *G2* are in series. Also, *H1* and *H2* are consisting a closed-loop system, therefore, they can be reduced using eliminating rule as shown in the following.



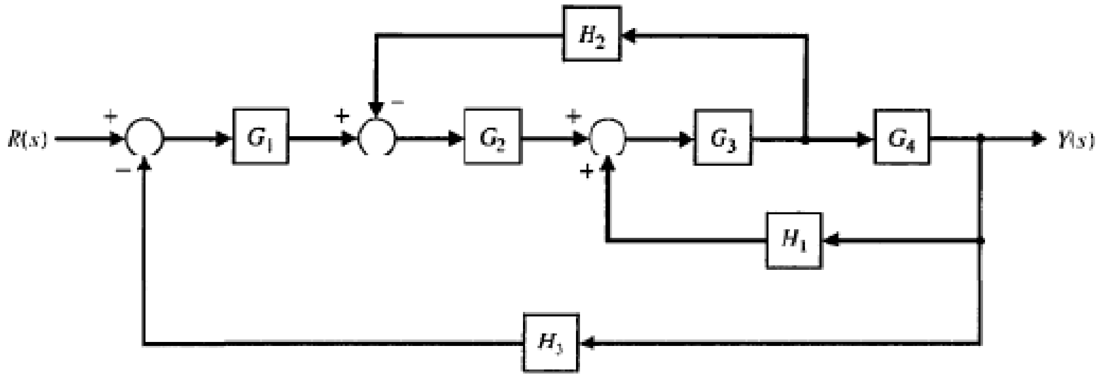


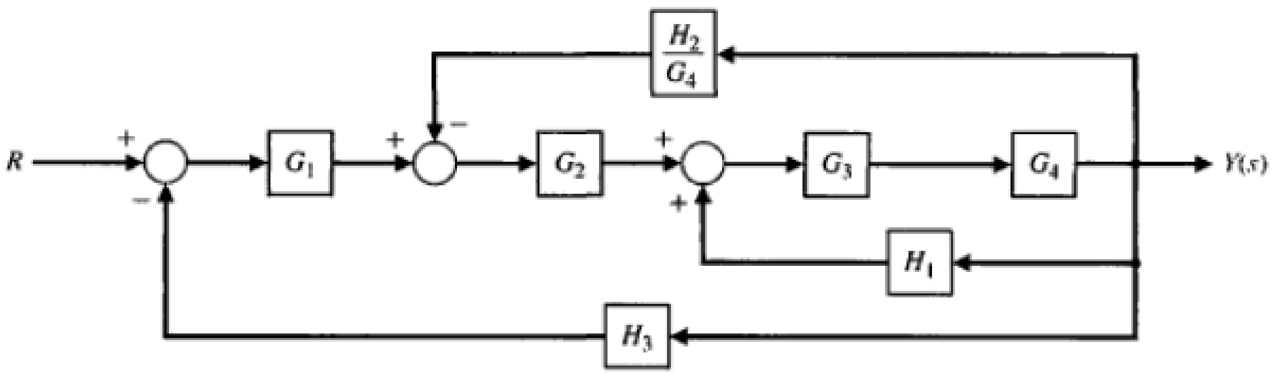
$$\therefore \qquad \frac{C(s)}{R(s)} = \frac{G_1G_2}{1+\frac{G_1G_2H_1}{1+H_1H_2}} = \frac{G_1G_2(1+H_1H_2)}{1+H_1H_2+G_1G_2H_1}$$

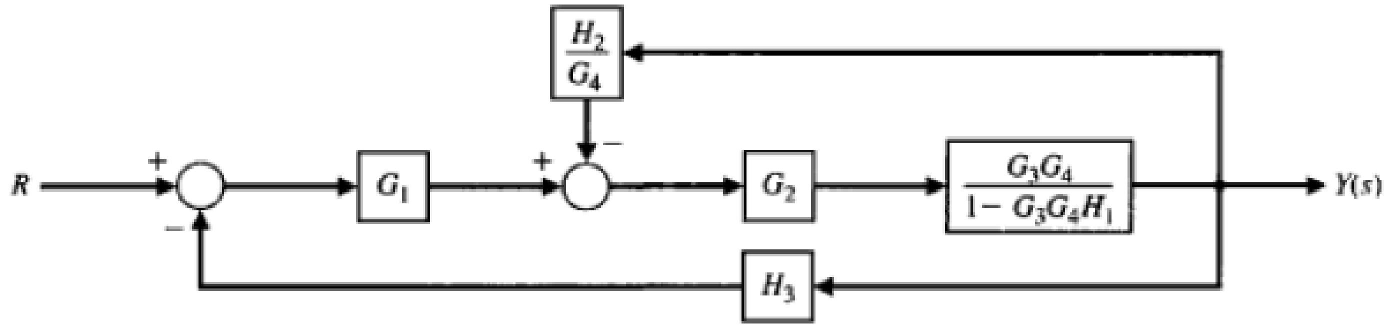
Example 5-3 : Simplify the block diagram shown in the figure below

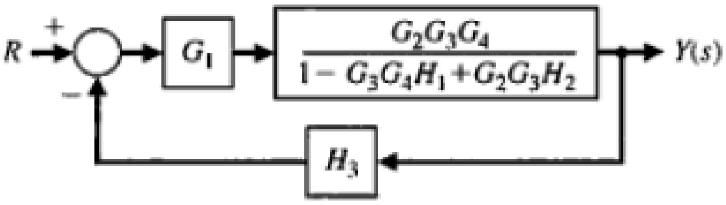


solution

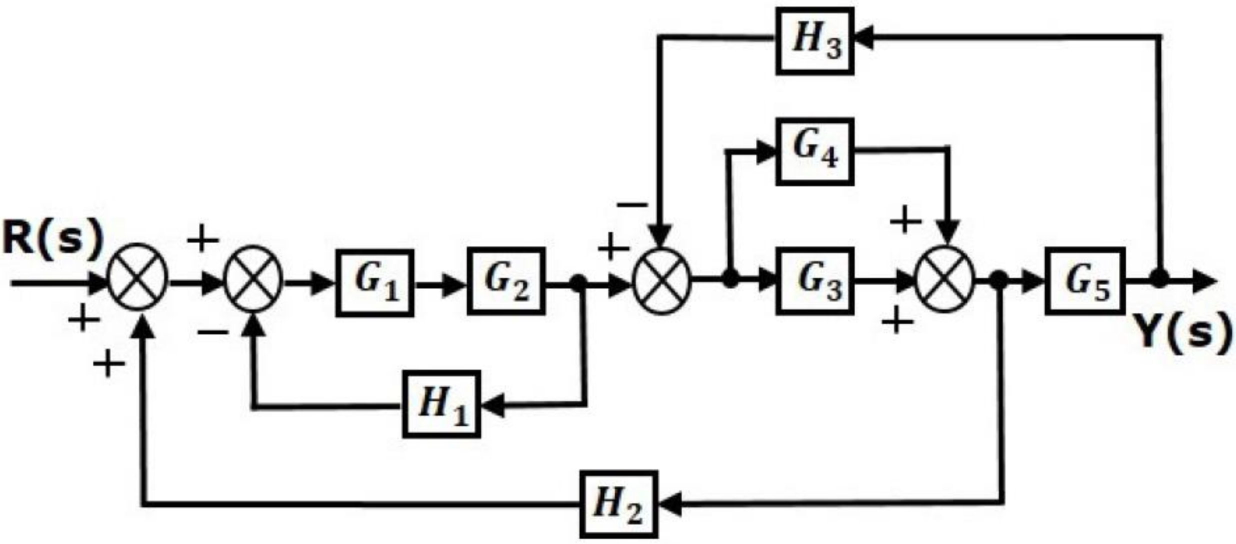




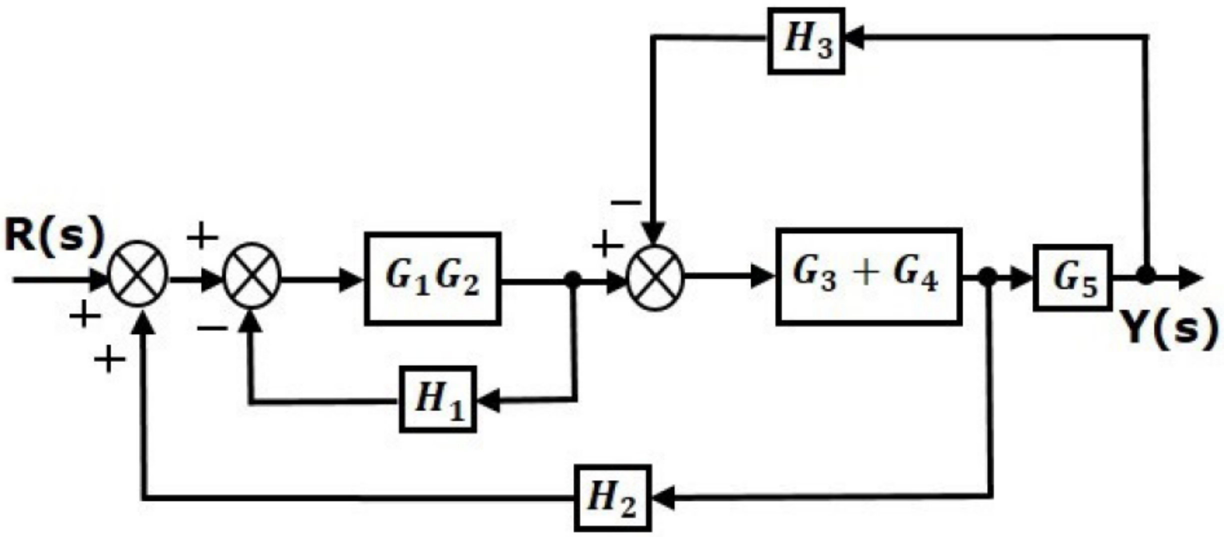




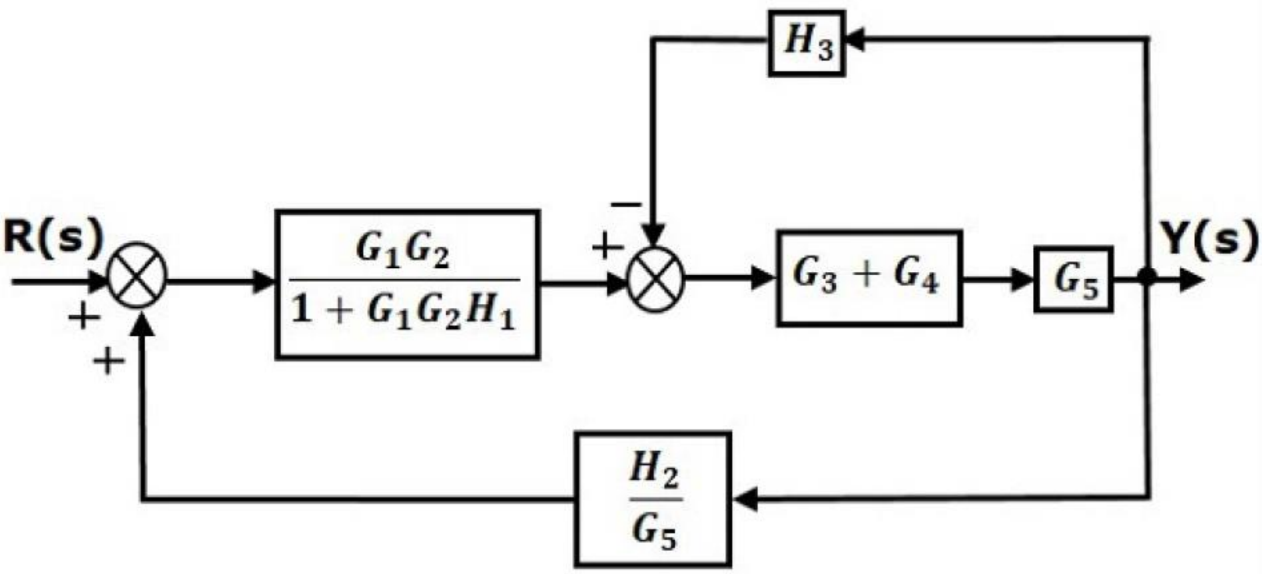
Example 5-4 Consider the block diagram shown in the following figure. Let us simplify (reduce) this block diagram using the block diagram reduction rules.



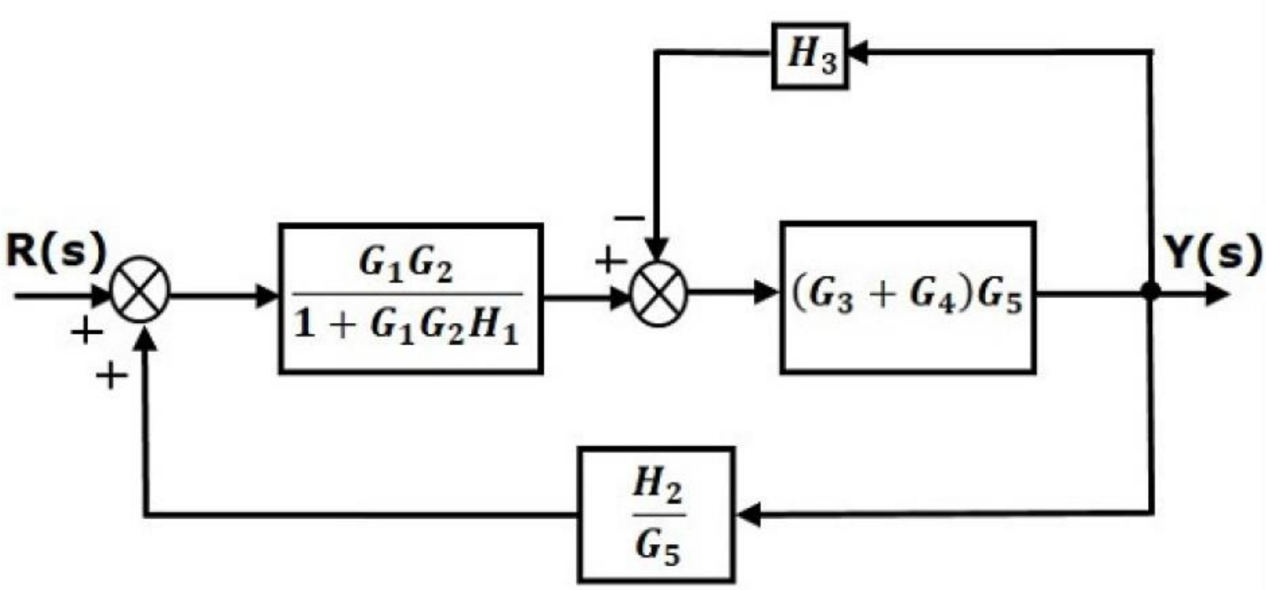
Step 1 − Use Rule 1 for blocks G_1 and G_2. Use Rule 2 for blocks $G_3$and G_4 The modified block diagram is shown in the following figure.



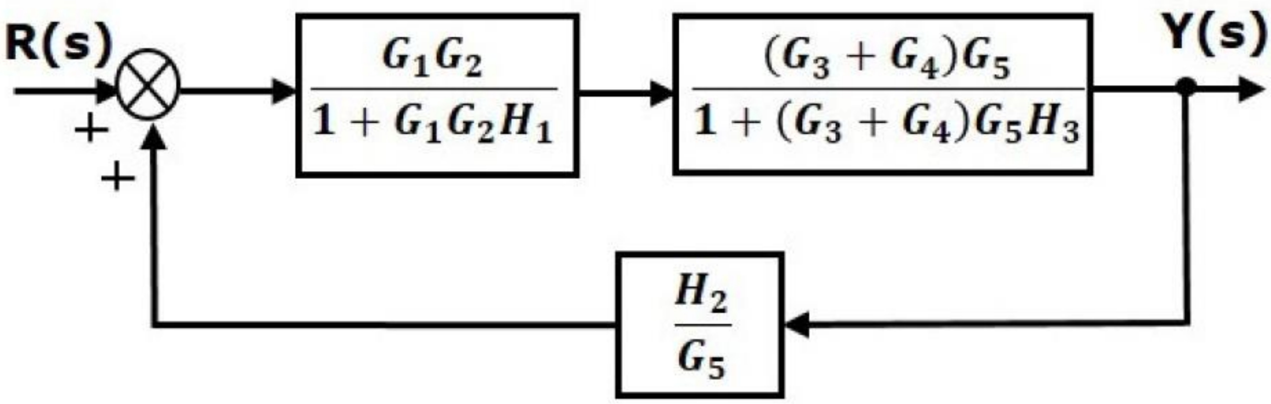
Step 2 − Use Rule 3 for blocks G_1G_2 and H_1. Use Rule 4 for shifting take-off point after the block G_5. The modified block diagram is shown in the following figure.



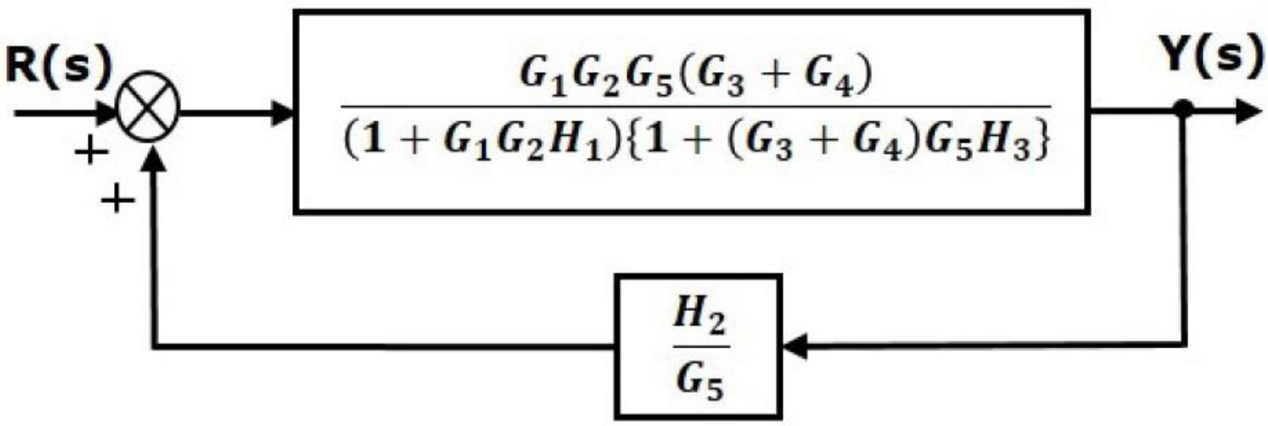
Step 3 − Use Rule 1 for blocks ($G_3 + G_4$) and G_5. The modified block diagram is shown in the following figure.



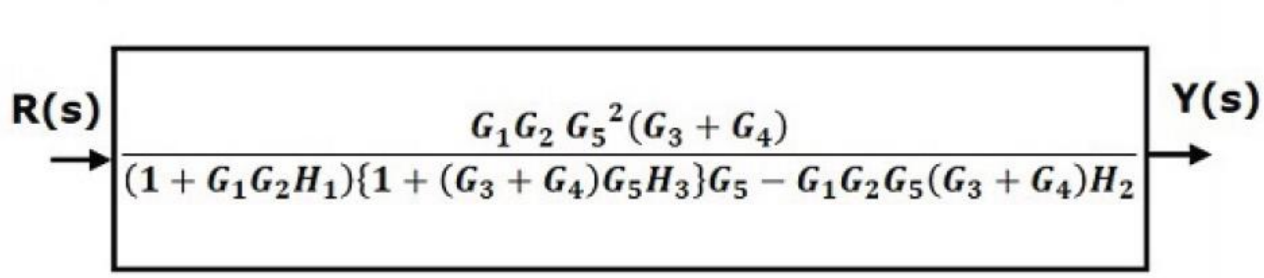
Step 4 −Use Rule 3 for blocks ($G_3 + G_4$) G_5 and H_3. The modified block diagram is shown in the following figure.



Step 5 − Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.



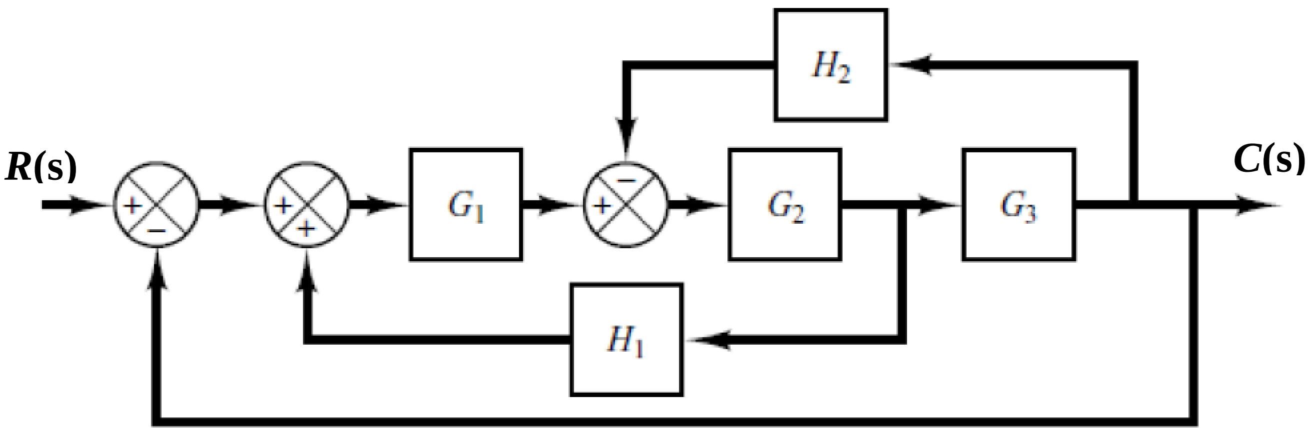
Step 6 − Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.



Therefore, the transfer function of the system is

$$\frac{Y_{(S)}}{X_{(S)}}=\frac{G_1\,G_2\,G_5^2\,(G_3+\,G_4)}{(1+G_1G_2H_1)\{\,1+(G_3+\,G_4)G_5H_3\}G_5-G_1\,G_2\,G_5\,(G_3+\,G_4)H_2}$$

Homework 5-1 Simplify the block diagram shown in the figure below and obtain the transfer function *Y(s)/R(s)*.



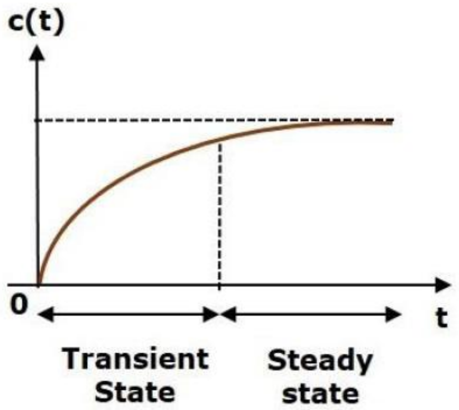
Lecture seven

Time Response Analysis

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- Transient response
- Steady state response

The response of control system in time domain is shown in the following figure.



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Where,

- $c_{tr}(t)$ is the transient response
- $c_{ss}(t)$ is the steady state response

1- Transient response:
Any system containing energy storing element like inductor, capacitor, mass and inertia etc. these energy storing element are the part of the control system and cannot be avoided. If the energy state of the systems is disturbed, then it takes a certain time to change from one state to another state. This disturbance sometimes occurs at input, sometime occurs at output and some time at both ends. The time required to change from one state to another state is known as transient time.

The part of the time response that remains even after the transient response has zero value for large values of 't' is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example

Let us find the transient and steady state terms of the time response of the control system $c(t) = 10 + 5e^{-t}$

Here, the second term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

Stander test signal

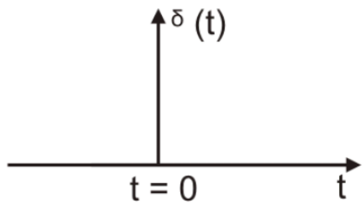
1- Impulse Input:

It is sudden change input. An impulse is infinite at t=0 and everywhere else.

$\delta(t) = 1 \qquad t = 0$

$\quad = 0 \qquad t \neq 0$

- In laplace domain we have,
  $L[r(t)] = 1$



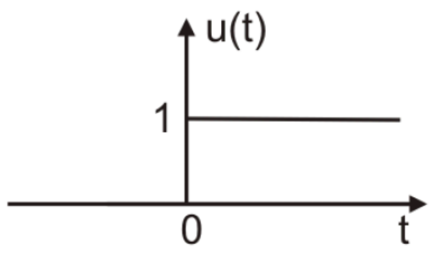
2- Step signal

- It represents a constant command such as position. Like elevator is a step input.

$u(t) = A \qquad t \geq 0$

$\quad = 0 \qquad \text{otherwise}$

$L[r(t)] = A/s$



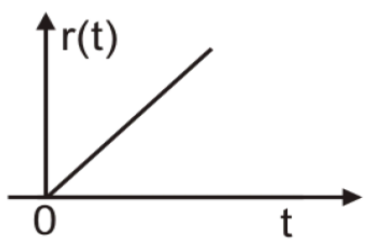
3- Ramp signal

- this represents a linearly increasing input command.
- $r(t) = At \qquad t \geq 0$,Aslope

$\quad = 0 \qquad \text{otherwise}$

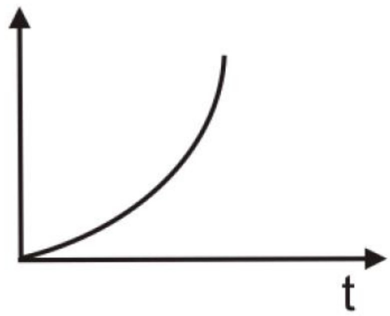
$L[r(t)] = A/s^2$

A= 1 then unit ramp

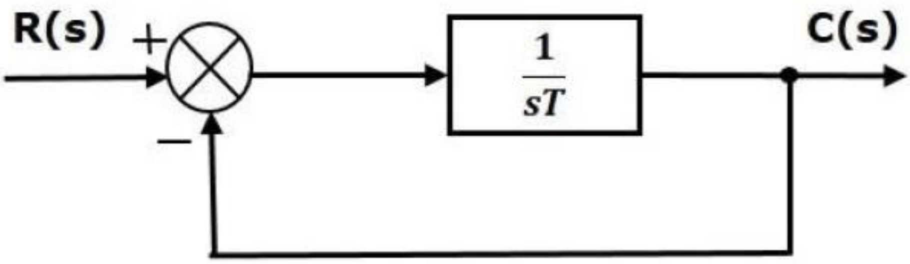


4- Parabolic signal

- Rate of change of velocity is acceleration. Acceleration is a parabolic function.
- r(t) $= At^2/2$ $\quad t \geq 0$
  $= 0$ $\quad$ otherwise
  $L[r(t)] = A/s^3$



Time response of the first order systems



$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1 + \frac{1}{sT}} = \frac{1}{sT+1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

Where,

C(s) is the Laplace transform of the output signal c(t),

R(s) is the Laplace transform of the input signal r(t), and

T is the time constant.

Step Response of First Order System

Consider the **unit step signal** as an input to first order system.

So, $r(t) = u(t)$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\,(sT+1)}$$

$$C(s) = \frac{1}{s} \cdot \frac{\frac{1}{T}}{\left(\frac{1}{T} + s\right)} = \frac{\frac{1}{T}}{s\left(s + \frac{1}{T}\right)}$$

$$C(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{T}}$$

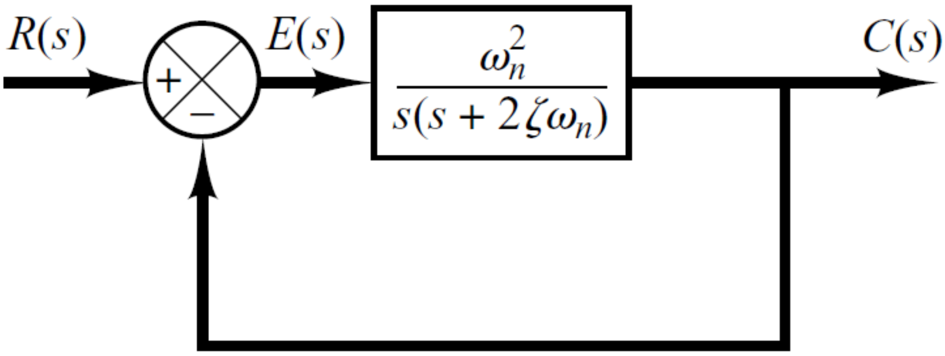
$$A = \frac{\frac{1}{T}}{s\left(s + \frac{1}{T}\right)} \cdot s \Bigg|_{s=0} = 1$$

$$B = \frac{\frac{1}{T}}{s\left(s + \frac{1}{T}\right)} \left(s + \frac{1}{T}\right) \Bigg|_{s=-\frac{1}{T}} = -1$$

$$C(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{T}}$$

$$\therefore \quad C(t) = 1 - e^{-\frac{t}{T}}$$

Time response of the second order systems



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

This form is called the ***standard form*** of the second-order system. Where ζ is the damping ratio and ωn is the natural frequency. However, the dynamic behaviour of the second-order system can then be described in terms of two parameters ζ and ωn.

Damping ratio	Type of damping	Type of denominator
$0 < \zeta < 1$	Under damping	Complex
$\zeta = 1$	Critical damping	Repeated
$\zeta > 1$	Over damping	Real
$\zeta = 0$	No damping	Real = Complex

We shall now solve for the response of the system shown in the above figure to a **unit-step input** considering the above mentioned cases.

1- For unit step input:

$$\because \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

A- No damping ($\zeta = 0$):

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \quad \rightarrow \quad C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

$$s^2 + \omega_n^2 = 0 \quad \rightarrow \quad s_{1,2} = \pm j\omega_n$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2}{s - j\omega_n} + \frac{K_3}{s + j\omega_n}$$

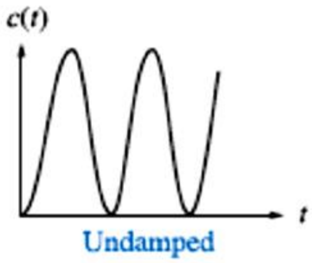
$$K_1 = \frac{\omega_n^2}{s(s - j\omega_n)(s + j\omega_n)} * s|_{s=0} = 1$$

$$K_2 = \frac{\omega_n^2}{s(s - j\omega_n)(s + j\omega_n)} * (s - j\omega_n)|_{s=j\omega_n} = -\frac{1}{2}$$

$$K_3 = \frac{\omega_n^2}{s(s - j\omega_n)(s + j\omega_n)} * (s + j\omega_n)|_{s=-j\omega_n} = -\frac{1}{2}$$

$$C(s) = \frac{1}{s} - \frac{1}{2}\frac{1}{(s - j\omega_n)} - \frac{1}{2}\frac{1}{(s + j\omega_n)}$$

$$\therefore \mathrm{C(t)} = 1 - \frac{1}{2}(e^{\omega_n t} + e^{-\omega_n t}) = 1 - \cos\omega_n t$$



B- Under damping case ($0 < \zeta < 1$) (Complex Poles):

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$C(s) = \frac{K_1}{s} + \frac{K(a+jb)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \rightarrow \textbf{Compar with} \quad s^2 + 2as + (a^2 + b^2) = 0$$

$$2a = 2\zeta\omega_n \rightarrow a = \zeta\omega_n \quad ; \quad a^2 + b^2 = \omega_n^2 \rightarrow b^2 = \omega_n^2 - \zeta^2\omega_n^2$$

$$b = \omega_n\sqrt{(1-\zeta^2)} = \omega_d$$

Comparing with

$$\boxed{s = -a + jb}$$

$$\therefore s = -\zeta\omega_n + j\omega_n\sqrt{(1-\zeta^2)}$$

$$K(a+jb) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} * (s^2 + 2\zeta\omega_n s + \omega_n^2)\Big|_{s=-\zeta\omega_n + j\omega_n\sqrt{(1-\zeta^2)}}$$

$$K(a+jb) = \frac{\omega_n^2}{-\zeta\omega_n + j\omega_n\sqrt{(1-\zeta^2)}}$$

$$|K(a+jb)| = \frac{\sqrt{(\omega_n^2)^2}}{\sqrt{(-\zeta\omega_n)^2 + \left(j\omega_n\sqrt{(1-\zeta^2)}\right)^2}} = \omega_n$$

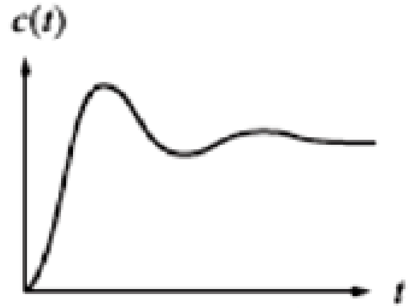
$$\alpha = tan^{-1}\frac{0}{\omega_n^2} - tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{-\zeta} = tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta}$$

$$K_1 = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} * s\Big|_{s=0} = 1$$

Comparing with

$$f(t) = -\frac{1}{b}|K(a+jb)|\, e^{-at}\sin(bt+\alpha)$$

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t + tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta})$$



Underdamped

C- *Critical damping case ($\zeta = 1$) (Repeated poles):*

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \quad \rightarrow \quad C(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)}$$

$$C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2} \quad \rightarrow \quad C(s) = \frac{K_1}{s} + \frac{K_2}{(s + \omega_n)^2} + \frac{K_3}{(s + \omega_n)}$$

$$K_1 = \frac{\omega_n^2}{s(s + \omega_n)^2} * s|_{s=0} = 1 \,; \quad K_2 = \frac{1}{(1-1)!}\frac{\omega_n^2}{s(s + \omega_n)^2} * (s + \omega_n)^2|_{s=-\omega_n} = -\omega_n$$

$$K_3 = \frac{1}{(2-1)!}\frac{d}{ds}\frac{\omega_n^2}{s(s + \omega_n)^2} * (s + \omega_n)^2|_{s=-\omega_n} = -1$$

$$C(s) = \frac{1}{s} - \frac{\omega_n}{(s + \omega_n)^2} - \frac{1}{(s + \omega_n)} \;; \quad C(t) = 1 - \omega_n t e^{-\omega_n t} - e^{-\omega_n t}$$

$$\therefore C(t) = 1 - e^{-\omega_n t}(\omega_n t + 1)$$

Poles and response of a second-order system when $\zeta = 1$

D- Over damping case ($\zeta > 1$) (Real poles):

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \rightarrow \quad s = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4{\omega_n}^2}}{2}$$

$$s_{1,2} = -\zeta\omega_n \pm \frac{\sqrt{(2\zeta\omega_n)^2 - 4{\omega_n}^2}}{2} = -\zeta\omega_n \pm \frac{2\omega_n\sqrt{(\zeta^2 - 1)}}{2}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)}$$

Now $C(s)$ can be written as bellow

$$C(s) = \frac{\omega_n^2}{s(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})}$$

$$\therefore C(s) = \frac{K_1}{s} + \frac{K_2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})} + \frac{K_3}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})}$$

Assume $\beta = \sqrt{\zeta^2 - 1}$

$$K_1 = \frac{\omega_n^2}{s(s + \zeta\omega_n + \omega_n\beta)(s + \zeta\omega_n - \omega_n\beta)} s|_{s=0}$$

$$K_1 = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \zeta\omega_n^2\beta + \zeta\omega_n^2\beta - \omega_n^2\beta^2} = \frac{\omega_n^2}{\omega_n^2\,(\zeta^2 - \beta^2)} = \frac{1}{\zeta^2 - (\zeta^2 - 1)} = 1$$

$$K_2 = \frac{\omega_n^2}{s(s + \zeta\omega_n + \omega_n\beta)(s + \zeta\omega_n - \omega_n\beta)} (s + \zeta\omega_n + \omega_n\beta)|_{s=-\zeta\omega_n - \omega_n\beta}$$

$$K_2 = \frac{\omega_n^2}{(-\zeta\omega_n - \omega_n\beta)(-\zeta\omega_n - \omega_n\beta + \zeta\omega_n - \omega_n\beta)} = \frac{\omega_n^2}{(-\zeta\omega_n - \omega_n\beta)(-2\omega_n\beta)}$$

$$K_2 = \frac{\omega_n^2}{2\zeta\omega_n^2\beta + 2\omega_n^2\beta^2} = \frac{1}{2\zeta\beta + 2\beta^2} \qquad \therefore K_2 = \frac{1}{2\beta(\zeta + \beta)}$$

$$K_3 = \frac{\omega_n^2}{s(s+\zeta\omega_n+\omega_n\beta)(s+\zeta\omega_n-\omega_n\beta)}(s+\zeta\omega_n-\omega_n\beta)|_{s=-\zeta\omega_n+\omega_n\beta}$$

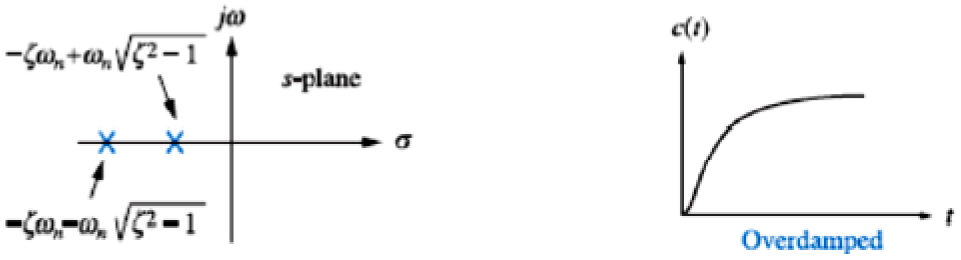
$$K_3 = \frac{\omega_n^2}{(-\zeta\omega_n+\omega_n\beta)(-\zeta\omega_n+\omega_n\beta+\zeta\omega_n+\omega_n\beta)} = \frac{\omega_n^2}{(-\zeta\omega_n+\omega_n\beta)(2\omega_n\beta)}$$

$$K_3 = \frac{\omega_n^2}{-2\zeta\omega_n^2\beta+2\omega_n^2\beta^2} = \frac{1}{-2\zeta\beta+2\beta^2} \qquad \therefore K_3 = -\frac{1}{2\beta(\zeta-\beta)}$$

$$\therefore C(s) = \frac{1}{s} + \frac{\frac{1}{2\beta(\zeta+\beta)}}{(s+\zeta\omega_n+\omega_n\beta)} - \frac{\frac{1}{2\beta(\zeta-\beta)}}{(s+\zeta\omega_n-\omega_n\beta)}$$

$$C(s) = \frac{1}{s} + \frac{\frac{1}{2\beta(\zeta+\beta)}}{(s+\omega_n(\zeta+\beta))} - \frac{\frac{1}{2\beta(\zeta-\beta)}}{(s+\omega_n(\zeta-\beta))}$$

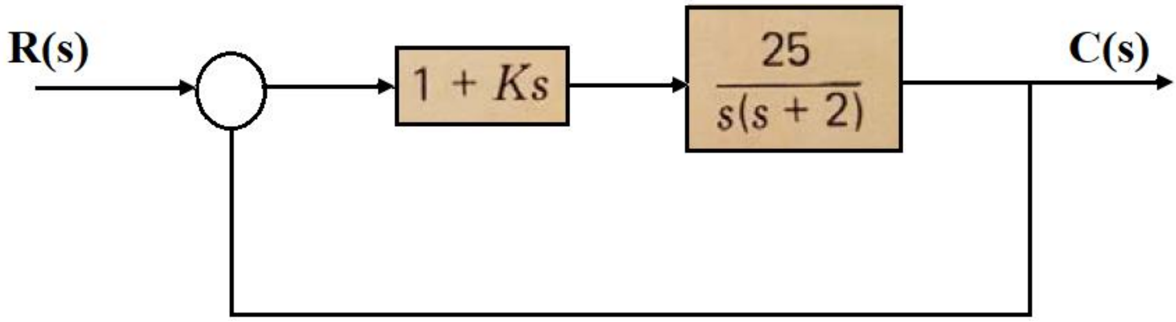
$$C(t) = 1 + \frac{1}{2\beta(\zeta+\beta)}e^{-\omega_n(\zeta+\beta)t} - \frac{1}{2\beta(\zeta-\beta)}e^{-\omega_n(\zeta-\beta)t}$$



Poles and response of a second-order system when $\zeta > 1$

Example: 1

To improve the transient behaviour of a system a controller with proportional and derivative action is added, as shown in the figure below. Determine the value of K such that the resulting system will have ζ=0.5. Also, what is the response of the resulting system to a unit step input.



Solution:

$$\frac{C(s)}{R(s)} = \frac{25(1 + Ks)}{s^2 + (25K + 2)s + 25}$$

$$s^2 + (25K + 2)s + 25 = 0 \quad \rightarrow \textit{compare with} \quad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = 25 \quad \rightarrow \quad \omega_n = 5\frac{rad}{s}; \qquad 2\zeta\omega_n = 25K + 2 \quad \rightarrow \quad 2 * 0.5 * 5 = 25K + 2$$

$$\therefore K = 0.12$$

$$\therefore \frac{C(s)}{R(s)} = \frac{25(1 + 0.12s)}{s^2 + (25 * 0.12 + 2)s + 25} = \frac{3s + 25}{s^2 + 5s + 25}$$

$$C(s) = \frac{3s + 25}{s(s^2 + 5s + 25)} \quad \rightarrow \qquad C(s) = \frac{C_1}{s} + \frac{K(a + jb)}{s^2 + 5s + 25}$$

$$C_1 = \frac{3s+25}{s(s^2+5s+25)} * s|_{s=0} = 1$$

$$s^2+5s+25 = 0 \quad \rightarrow \textbf{compare with} \qquad s^2+2as+(a^2+b^2) = 0$$

$$2a = 5 \quad \rightarrow \quad a = 2.5; \qquad a^2+b^2 = 25 \ \rightarrow \quad b = 4.33$$

$$s = -2.5 + 4.33j$$

$$K(a+jb) = \frac{3s+25}{s(s^2+5s+25)} * (s^2+5s+25)|_{s=-2.5+4.33j}$$

$$K(a+jb) = \frac{3*(-2.5+4.33j)+25}{-2.5+4.33j} = \frac{17.5+12.99j}{-2.5+4.33j}$$

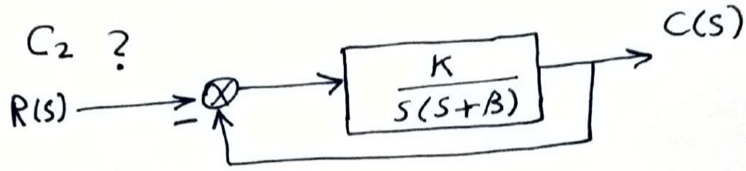
$$|K(a+jb)| = \frac{\sqrt{17.5^2+12.99^2}}{\sqrt{-2.5^2+4.33^2}} = 4.36$$

$$\alpha = tan^{-1}\frac{12.99}{17.5} - tan^{-1}\frac{4.33}{-2.5} = 96.58^o$$

$$C(t) = 1 + \frac{1}{4.33} * 4.36 * e^{-2.5t}\sin(4.33t + 96.58)$$

Example 2

For the system shown in Figure determine (K) and (β) such that the response to a unit impulse has the form $c(t) = C_1 e^{-t} + C_2 e^{-4t}$, evaluate C_1 and C_2 ?



Sol.:

$$\therefore \frac{C(s)}{R(s)} = \frac{K}{s^2+\beta s+K} \quad ; R(s) = 1$$

$$\therefore C(s) = \frac{K}{s^2+\beta s+K}$$

$$\because c(t) = C_1 e^{-t} + C_2 e^{-4t}$$

$$\therefore C(s) = \frac{C_1}{s+1} + \frac{C_2}{s+4} = \frac{K}{(s+1)(s+4)}$$

$s^2+\beta s+K = 0$ comp. with $(s+1)(s+4) = 0$

$$\therefore (s^2+5s+4) = 0 \Rightarrow K = 4 \; ; \; \beta = 5$$

$$\therefore C(s) = \frac{4}{s^2+5s+4} = \frac{C_1}{s+1} + \frac{C_2}{s+4}$$

$$\therefore C_1 = \frac{4}{(s+1)(s+4)} \cdot (s+1) \Big|_{s=-1} = \frac{4}{3}$$

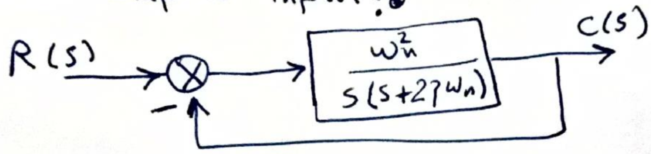
$$C_2 = \frac{4}{(s+1)(s+4)} \cdot (s+4) \Big|_{s=-4} = -\frac{4}{3}$$

$$\therefore C(s) = \frac{\frac{4}{3}}{s+1} - \frac{\frac{4}{3}}{s+4}$$

$$\therefore c(t) = \frac{4}{3} e^{-t} - \frac{4}{3} e^{-4t}$$

Example 3

Find the response equation For the second order System shown in Figure ; the system under damper has a unit Impuls input?



Sol.

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2} \quad ; \quad R(S) = 1$$

$$\therefore C(S) = \frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2}$$

$S^2+2\zeta\omega_n S+\omega_n^2$ Comp. with $S^2+2aS+(a^2+b^2)$

$$\therefore \quad 2a = 2\zeta\omega_n \Rightarrow a = \zeta\omega_n$$

$$\omega_n^2 = a^2+b^2 \Rightarrow b = \omega_n\sqrt{1-\zeta^2}$$

$$S = -a+bj = -\zeta\omega_n + j\,\omega_n\sqrt{1-\zeta^2}$$

$$\therefore C(S) = \frac{K(a+jb)}{S^2+2\zeta\omega_n S+\omega_n^2}$$

$$\therefore K(a+jb) = \frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2}\cdot(S^2+2\zeta\omega_n S+\omega_n^2)\Big|_{S=-\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2}} = \omega_n^2$$

$$\therefore |K(a+jb)| = \omega_n^2$$

$$\alpha = 0$$

$$\therefore C(t) = -\frac{1}{b}\,|K(a+jb)|\,\sin(bt+\alpha)\cdot e^{-at}$$

$$\therefore C(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\,\sin(\omega_d t)$$

Lecture eight

8.1 Transient Response Specification

Figure 8.1 shows the plot of C(t) versus t for unit step input and different transient response specifications have also been pointed.

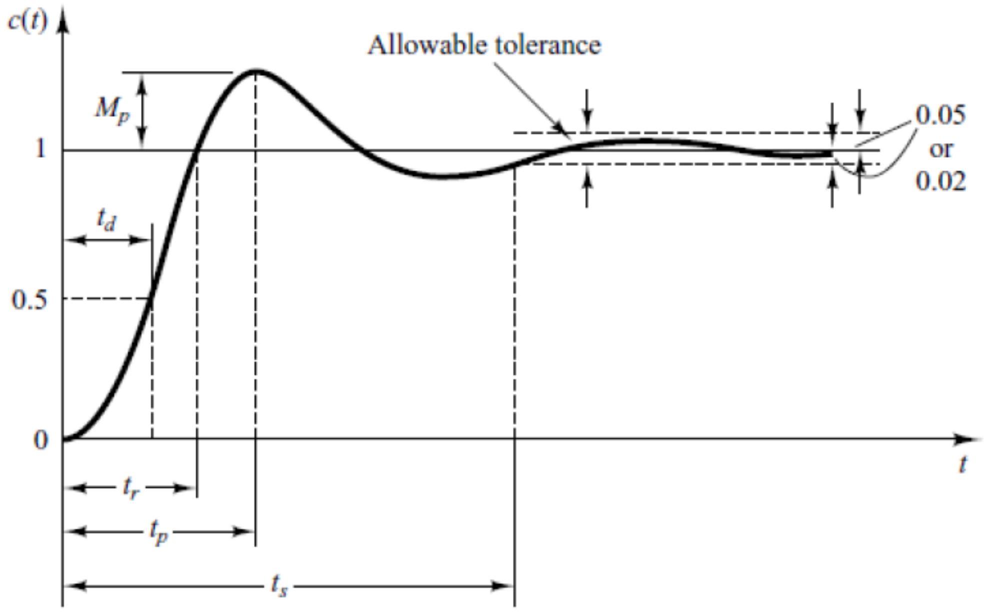


Figure 8.1

8.1.1 Delay time t_d

It is the required time for the response to reach 50% of the final value in the first attempt.

$$C(t) = \frac{1}{2} \quad \text{at} \quad t = T_d$$

$$t_d = \frac{1 + 0.5\zeta}{\omega_N}$$

8.1.2 Rise time t_r

The time required for the waveform to go from 10% of the final value to 90% of the final value for overdamped systems and from 0 to 100% of the final value for underdamped systems. It can be obtained by equating the time response function of under damping system to 1.

$$C(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t_r + tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta}) = 1$$

$$\sin(\omega_d t_r + tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta}) = 0$$

$$\because \alpha = tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta} \qquad \therefore \sin(\omega_d t_r + \alpha) = 0$$

$$\omega_d t_r + \alpha = n\pi \qquad where\ n = 1, 2, 3, \ldots..$$

$$\text{Let } n = 1 \qquad \therefore \omega_d t_r + \alpha = \pi$$

$$t_r = \frac{\pi - \alpha}{\omega_d}$$

8.1.3 Peak time t_p

The time required to reach the first, or maximum, peak. It can be found by differentiating the time response function of under damping system and equating the derivative to zero, since the peak value occur when the derivative is zero.

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\sin(\omega_d t + tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta})$$

$$\dot{C}(t) = -(\frac{1}{\sqrt{(1-\zeta^2)}}(-\zeta\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t + \alpha) + \omega_d e^{-\zeta\omega_n t}\cos(\omega_d t + \alpha)))$$

$$\frac{1}{\sqrt{(1-\zeta^2)}}(\ \zeta\omega_n e^{-\zeta\omega_n t_p}\sin(\omega_d t_p + \alpha) - \omega_d e^{-\zeta\omega_n t_p}\cos(\omega_d t_p + \alpha)) = 0$$

$$\zeta\omega_n e^{-\zeta\omega_n t_p}\sin(\omega_d t_p + \alpha) - \omega_d e^{-\zeta\omega_n t_p}\cos(\omega_d t_p + \alpha) = 0$$

$$\zeta\omega_n e^{-\zeta\omega_n t_p}\sin(\omega_d t_p + \alpha) - \omega_n\sqrt{(1-\zeta^2)}e^{-\zeta\omega_n t_p}\cos(\omega_d t_p + \alpha) = 0$$

$$\zeta\sin(\omega_d t_p + \alpha) - \sqrt{(1-\zeta^2)}\cos(\omega_d t_p + \alpha) = 0$$

$$\because \alpha = tan^{-1}\frac{\sqrt{(1-\zeta^2)}}{\zeta} \qquad \therefore \sin\alpha = \sqrt{(1-\zeta^2)} \quad \& \quad \cos\alpha = \zeta$$

$$\alpha\sin(\omega_d t_p + \alpha) - \sin\alpha\cos(\omega_d t_p + \alpha) = 0$$

$$t_p = \frac{\pi}{\omega_d}$$

8.1.4 Maximum (percent) overshoot Mp

It is the largest error between reference input and output during the transient period. The maximum overshoot occurs at the peak time tp.

$$M_p = C(t_p) - R(t) = C(t_p) - 1$$

$$C(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{(1-\zeta^2)}} \sin(\omega_d t_p + \alpha)$$

$$C(t_p) - 1 = -\frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{(1-\zeta^2)}}}}{\sqrt{(1-\zeta^2)}} \sin(\omega_d \frac{\pi}{\omega_d} + \alpha)$$

$$M_p = -\frac{e^{\frac{-\zeta\pi}{\sqrt{(1-\zeta^2)}}}}{\sin\alpha} \sin(\pi + \alpha)$$

$$M_p = \frac{e^{\frac{-\zeta\pi}{\sqrt{(1-\zeta^2)}}}}{\sin\alpha} * \sin\alpha$$

$$\boxed{M_p = e^{\frac{-\zeta\pi}{\sqrt{(1-\zeta^2)}}}}$$

8.1.5 Settling time ts

The time required for response to decrease and stay within specified percentage of its final value (within the tolerance band).

For 2% criterion $t_s = \dfrac{4}{\zeta\omega_n}$

For 5% criterion $t_s = \dfrac{3}{\zeta\omega_n}$

For no damping (ζ=0) system

8.1.6 Rise time *tr*

It has established previously that for unit step and no damping

$$C(t) = 1 - \cos\omega_n t \qquad \text{Let} \qquad C(t) = 1 \quad and\; t = t_r$$

$$1 = 1 - \cos\omega_n t_r \quad \rightarrow \quad \cos\omega_n t_r = 0 \quad \rightarrow \quad \omega_n t_r = \frac{\pi}{2}$$

$$t_r = \frac{\pi}{2\omega_n}$$

8.1.7 Peak time t_p

$$C(t) = 1 - \cos\omega_n t \qquad \text{Let} \qquad t = t_p \qquad and \qquad \frac{dC(t_p)}{dt} = 0$$

$$0 = \omega_n \sin\omega_n t_p \quad \rightarrow \quad \omega_n t_p = \pi$$

$$t_p = \frac{\pi}{\omega_n}$$

8.1.8 Maximum overshoot M_p

$$C(t) = 1 - \cos\omega_n t \qquad Let \qquad t = t_p = t_p = \frac{\pi}{\omega_n}$$

$$C(t_p) = 1 - \cos\omega_n t_p \quad \rightarrow \quad C(t_p) - 1 = -\cos(\omega_n * \frac{\pi}{\omega_n})$$

$$M_p = C(t_p) - 1 = -\cos\pi \qquad ;\ \cos\pi = -1$$

$$M_p = 1$$

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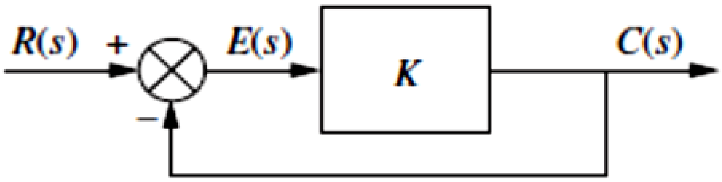
8.2 Steady state error and response

Consider the figure the following figure



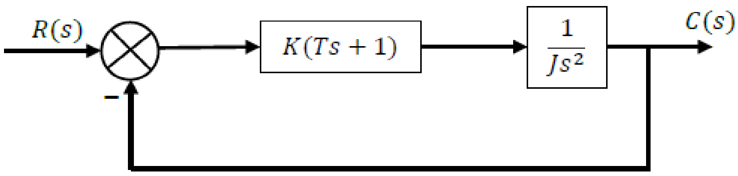
$$\textit{System error} \quad \rightarrow \quad E(s) = R(s) - C(s)$$

$$e(t) = \mathcal{L}^{-1}E(s) \quad \rightarrow \quad e(t) = R(t) - C(t)$$

$$\textit{Steady sate error} \quad \rightarrow \quad E_{s.s} = \lim_{s\to 0} sE(s) \qquad e_{s.s} = \lim_{t\to\infty} e(t)$$

$$\textit{Steady sate response} \quad \rightarrow \quad C_{s.s} = \lim_{s\to 0} sC(s) \qquad c_{s.s} = \lim_{t\to\infty} C(t)$$

Example 8-1 in the figure below the time *T=3 sec* and the ratio of torque to inertia $\frac{K}{J} = \frac{2}{9}\,\frac{N}{kg.m}$ Find the natural frequency and damping ratio.



Solution:

$$\frac{C(s)}{R(s)} = \frac{K(Ts+1)}{Js^2 + K(Ts+1)} \qquad \div J$$

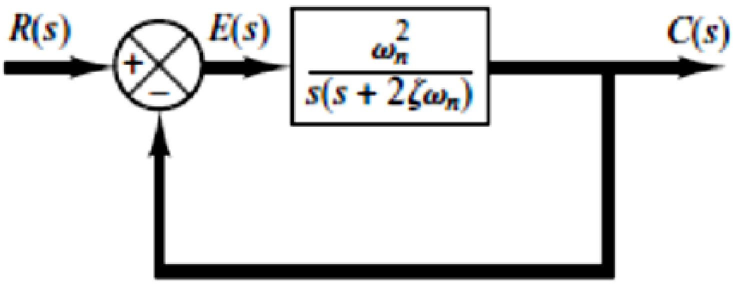
$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}(Ts+1)}{s^2 + \frac{K}{J}(Ts+1)} = \frac{(\frac{2}{3}s + \frac{2}{9})}{s^2 + \frac{2}{3}s + \frac{2}{9}}$$

$$s^2 + \frac{2}{3}s + \frac{2}{9} = 0 \quad \rightarrow \textit{Compar with} \qquad s^2 + 2\,\zeta\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = \frac{2}{9} \quad \rightarrow \quad \omega_n = 0.471\ rad/s$$

$$2\,\zeta\omega_n = \frac{2}{3} \quad \rightarrow \quad 2\zeta * 0.41 = \frac{2}{3} \qquad \therefore \zeta = 0.707$$

Example 8-2: Consider the system shown in the figure below, where ζ=0.6 and ωn=5 rad/s. Find the rise time $\boldsymbol{tr}$, peak time $\boldsymbol{tp}$, maximum overshoot $\boldsymbol{Mp}$, and settling time $\boldsymbol{ts}$ when the system is subjected to a unit-step input.



Solution:

$$Rise\ time:\ t_r = \frac{\pi - \alpha}{\omega_d};\quad \alpha = tan^{-1}\frac{\sqrt{1-0.6^2}}{0.6} \quad \rightarrow \quad \alpha = 0.927;$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5 * \sqrt{1-0.6^2} = 4\,rad/s$$

$$\therefore t_r = \frac{\pi - 0.927}{4} = 0.55\,sec$$

$$Peak\ time:\ t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.875\,sec$$

$$Maximum\ overshoot:\ M_p = e^{\frac{-\zeta\pi}{\sqrt{(1-\zeta^2)}}} = e^{\frac{-0.6*\pi}{\sqrt{(1-0.6^2)}}} = 0.095$$

The maximum percent overshoot is thus 9.5%.

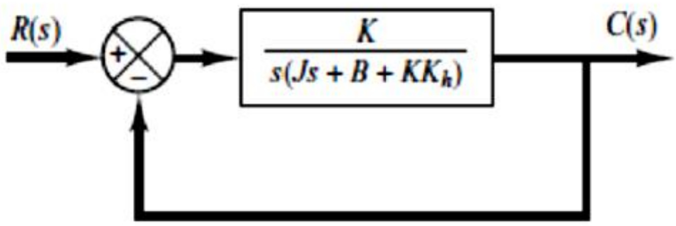
Settling time t_s: For the 2% criterion, the settling time is

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.6*5} = 1.33\,sec$$

For the 5% criterion,

$$t_s = \frac{3}{\zeta\omega_n} = \frac{3}{0.6*5} = 1\,sec$$

Example 8-3: For the system shown in the figure below, determine the values of gain K and the constant K_h so that the maximum overshoot in the unit-step response is $M_p = 0.2$ and the peak time is 1 sec. With these values of K and K_h, obtain the rise time and settling time. Assume that J=1 kg-m^2 and B=1 $N.m.s/r$.



Solution:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + KK_h + K} = \frac{K/J}{s^2 + \frac{(B + KK_h)s}{J} + \frac{K}{J}}$$

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{(1-\zeta^2)}}} \quad \rightarrow \quad 0.2 = e^{\frac{-\zeta*\pi}{\sqrt{(1-\zeta^2)}}} \rightarrow \ln 0.2 = \frac{-\zeta * \pi}{\sqrt{1-\zeta^2}}$$

$$\zeta = 0.456$$

$$Peak\ time:\ t_p = \frac{\pi}{\omega_d} \quad \rightarrow \quad 1 = \frac{\pi}{\omega_d} \quad \rightarrow \quad \omega_d = 3.14\, rad/sec$$

$$\therefore \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53 \frac{rad}{s}$$

$$s^2 + \frac{(B + KK_h)s}{J} + \frac{K}{J} = 0 \rightarrow \textbf{Compare with} \quad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\because \omega_n = \sqrt{\frac{K}{J}} \qquad \therefore K = \omega_n^2 J = 12.5\ N.m$$

$$2\zeta\sqrt{\frac{K}{J}} = \frac{(B + KK_h)}{J} \quad \rightarrow \quad \zeta = \frac{B + KK_h}{2\sqrt{KJ}}$$

$$0.456 = \frac{1 + 12.5K_h}{2\sqrt{12.5 * 1}} \quad \rightarrow \quad \therefore K_h = 0.178$$

$$Rise\ time:\ t_r = \frac{\pi - \alpha}{\omega_d}; \quad \alpha = tan^{-1}\frac{\sqrt{1-0.456^2}}{0.456} = 1.09$$

$$t_r = \frac{\pi - 1.09}{3.14} = 0.65$$

Settling time t_s: For the 2% criterion,

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.456 * 3.53} = 2.48\ sec$$

For the 5% criterion,

$$t_s = \frac{3}{\zeta\omega_n} = \frac{3}{0.456 * 3.53} = 1.86\ sec$$

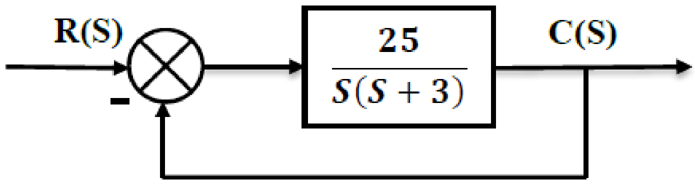
Example 10-4: For the system shown below where the input is unit-step:

Determine the following:

1 (The damping ratio ζ, the natural frequency ω_n, the damped frequency ω_d

2 (The time response $C(t)$ and error $e(t)$.

3) Steady state response ($Cs.s$) and steady state error ($Es.s$).



Solution:

$$T.F = \frac{C(s)}{R(s)} = \frac{25}{s^2 + 3s + 25}$$

1)

$s^2 + 3s + 25 = 0 \quad \rightarrow \quad \textbf{Compar with} \quad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\therefore\ \omega_n^2 = 25 \quad \rightarrow \omega_n = 5\ \frac{rad}{s};$

$\because 2\zeta\omega_n = 3 \quad \rightarrow \quad 2 * 5 * \zeta = 3 \quad \rightarrow \quad \therefore \zeta = 0.3$

$\because \omega_d = \omega_n\sqrt{1-\zeta^2} \quad \rightarrow \quad \therefore \omega_d = 5\sqrt{1-0.3^2} = 4.77\ \frac{rad}{s}$

2)

When the input is unit step $$C(s) = \frac{25}{s(s^2+3s+25)}$$

$\because \zeta = 0.3$ $\quad \therefore$ *the system is under damping and thus the poles are complex*

$$C(s) = \frac{K_1}{s} + \frac{K(a+jb)}{s^2+3s+25}$$

$$K_1 = \frac{25}{s(s^2+3s+25)} s|_{s=0} = 1$$

$$s^2+3s+25 = 0 \quad \rightarrow \textbf{Compar with} \quad s^2+2as+(a^2+b^2) = 0$$

$$2a = 3 \rightarrow a = 1.5; \qquad 1.5^2 + b^2 = 25 \rightarrow b = 4.77$$

$$s = -a + jb \rightarrow s = -1.5 + 4.77j$$

$$K(a+jb) = \frac{25}{s(s^2+3s+25)} * (s^2+3s+25)|_{s=-1.5+4.77j}$$

$$K(a+jb) = \frac{25}{-1.5+4.77j}; \qquad |K(a+jb)| = \frac{\sqrt{(25)^2}}{\sqrt{(-1.5)^2+(4.77)^2}} = 5$$

$$\alpha = tan^{-1}\frac{0}{25} - tan^{-1}\frac{4.77}{-1.5} = 72.54$$

Thus

$$C(s) = \frac{1}{s} - \frac{\frac{25}{-1.5+4.77j}}{s^2+3s+25}$$

and comparing with

$$f(t) = \frac{1}{b}\,|K(a+jb)|\, e^{-at}\sin(bt+\alpha)$$

Will get

$$C(t) = 1 - \frac{1}{4.77} * 5 * e^{-1.5t}\sin(4.77t + 72.54)$$

$$e(t) = R(t) - C(t) = 1 - (1 - \frac{5}{4.77} * e^{-1.5t}\sin(4.77t + 72.54))$$

$$e(t) = 1.04\, e^{-1.5t}\sin(4.77t + 72.54)$$

3)

$$E(s) = R(s) - C(s) = \frac{1}{s} - \frac{25}{s(s^2 + 3s + 25)}$$

$\boldsymbol{Steady\ sate\ error} \quad \rightarrow \qquad E_{s.s} = \lim_{s \to 0} sE(s) \qquad e_{s.s} = \lim_{t \to \infty} e(t)$

$$E_{s.s} = \lim_{s \to 0} s * (\frac{1}{s} - \frac{25}{s(s^2 + 3s + 25)}) = 1 - \frac{25}{25)} = 0$$

$\boldsymbol{Steady\ sate\ response} \quad \rightarrow \qquad C_{s.s} = \lim_{s \to 0} sC(s) \qquad C_{s.s} = \lim_{t \to \infty} C(t)$

$$C_{s.s} = \lim_{t \to \infty} \left(1 + \frac{1}{4.77} * 5 * e^{1.5t} \sin(4.77t + 72.54)\right) \quad \rightarrow \quad C_{s.s} = 1$$

Homework :

For a system having G(s)= $\frac{25}{s(s+10)}$ **and unity feedback find**

1- ω_n

2- ζ

3- t_p

4- M_p

Lecture Nine

System Stability

In the previous lectures it has been established that the systems have to pass through a short, transient period before getting settled. However, to find out whether the system under analysis will reach to its planned steady state or not the stability analysis has to be conducted, to explore whether the system is stable or unstable. A control system is stable if and only if all closed-loop poles lie in the left-half of the s-plane.

9.1 The Routh criterion stability

In this criterion the coefficients are arranged in an array which is known as Routh's array. For the characteristic equation

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \dots\dots = 0$$

$$\begin{array}{c|cc} s^n & a_0 & a_2 \\ s^{n-1} & a_1 & a_3 \\ s^{n-2} & b_1 & b_2 \\ s^{n-3} & c_1 & c_2 \end{array}$$

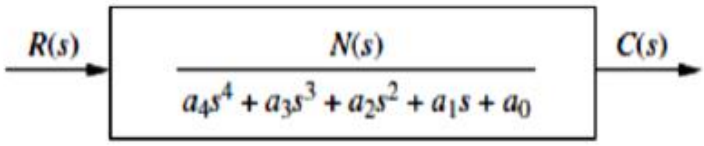


Table 11-1: Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Notice:

If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.

Example 9-1: Examine the stability of the following equation using Routh-Hurwitz method.

$$s^3 + 6s^2 + 11s + 6 = 0$$

Solution:

s^3	1	11	0
s^2	6	6	0
s^1	10	0	
s^0	6		

$$b_1 = \frac{-\begin{vmatrix} 1 & 11 \\ 6 & 6 \end{vmatrix}}{6} = \frac{(6-66)}{6} = 10 \quad ; \qquad b_2 = \frac{-\begin{vmatrix} 1 & 0 \\ 6 & 0 \end{vmatrix}}{6} = 0$$

Since there is no sign change in the first column, thus, the system is stable.

Example 9-2: Examine the stability of the following equation using Routh-Hurwitz method.

$$s^4 + 2s^3 + 6s^2 + 7s + 5 = 0$$

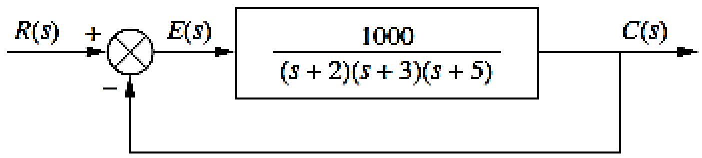
Solution:

s^4	1	6	5	0
s^3	2	7	0	
s^2	2.5	5		
s^1	3			
s^0	5			

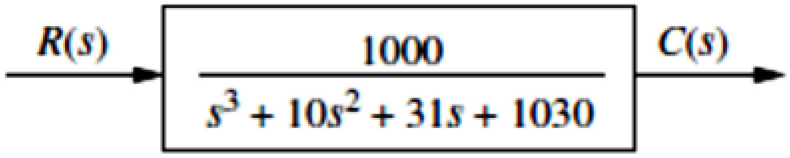
$$b_1 = \frac{-\begin{vmatrix} 1 & 6 \\ 2 & 7 \end{vmatrix}}{2} = 2.5 \quad ; \quad b_2 = \frac{-\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix}}{2} = 5 \quad ; \quad c_1 = \frac{-\begin{vmatrix} 2 & 7 \\ 2.5 & 5 \end{vmatrix}}{2.5} = 3$$

Since there is no sign change in the first column, thus, the system is stable.

Example 9-3: Make the Routh table for the system shown in the figure below.



Solution:



s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

In the above table there are two sign changes in the first column. The first sign change occurs from 1 in the s^2 row to -72 in the s^1 row. The second occurs from -72 in the s^1 row to 103 in the s^0 row. Thus, the above system is unstable since two poles exist in the right half-plane.