University of Technology

Mechanical Engineering Dep.

Aircraft engineering \ Fourth year stage

AIRCRAFT STABILITY AND CONTROL

استقرارية و سيطرة الطائرة

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Lecture 1

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1.1. Brief outline of historical developments

1.1.1 Early developments

The first attempts to study the stability of vehicles in flight were made by Sir George Cayley (1774-1857) who also carried out experiments on models of gliders with horizontal tail and rudder.

.By the 1880's, I.C. engines were available which were lighter than the earlier engines. However, inadequate understanding of stability and control delayed the first successful flight of a powered vehicle.

Otto Lilienthal (1848-1896) during 1890-1895 and Wilbur Wright (1867- 1912) and Orville Wright (1871-1948) during 1900-1903 carried out a number of experiments on hang gliders and gliders, which gave a better understanding of the stability and control. This led to the first successful flight on Dec.17, 1903. The name of this airplane was Wright flyer (Fig.1.1). It had a canard surface ahead of the wings for control of the pitching motion, vertical rudder for directional control while control in roll was obtained by warping the wings

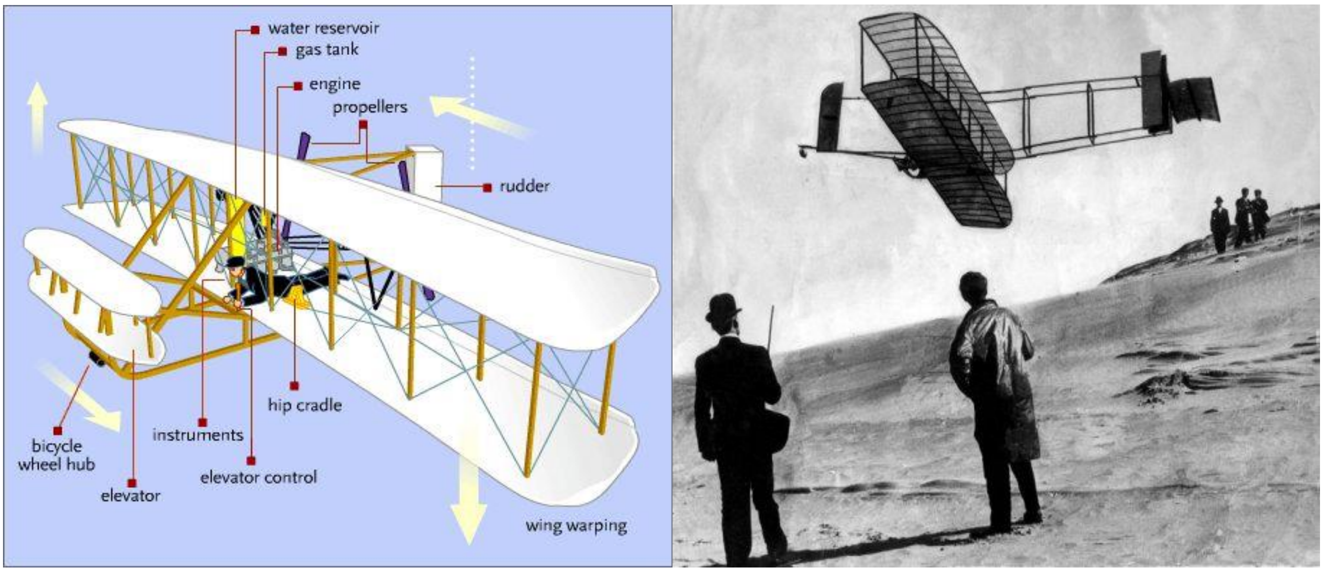
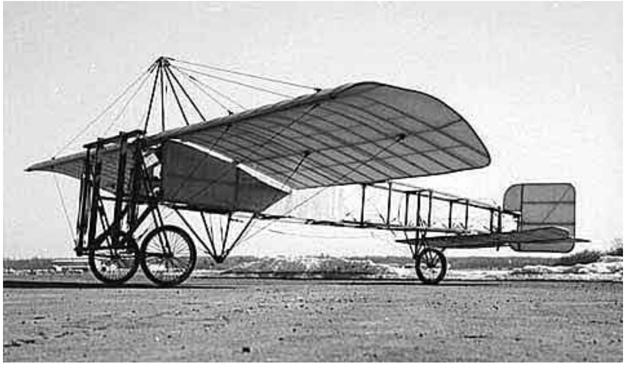


Figure 1.1 Two views of Wright Flyer

The first airplane with ailerons (Fig.1.2) was built in 1907 by Louis Blériot (1872-1936). It was also a monoplane. The first airplane with horizontal tail at the rear (Fig.1.3) was constructed in 1909 by A. Verdon-Roe (1877-1970).



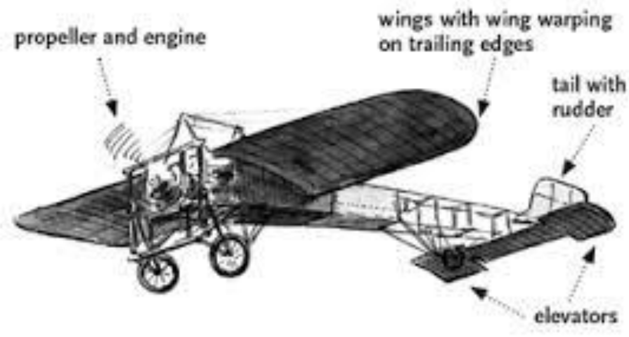


Figure 1.2 Two views of Louis Blériot's airplane

As regards the theoretical analysis, F.W. Lanchester (1868-1946) gave ideas about stability in his book entitled "Aerodonetics" published by Archibald Constable in 1908. He also mentioned about motion following longitudinal disturbance and called it phugoid.

Figure 1.3 Airplane of A. Verdon-Roe

In 1911, G H Bryan published a book entitled 'Stability in aviation', published by Macmillan in which he presented the mathematical analysis of the flight following a disturbance. It may be added that in the equilibrium state the resultant forces and moments acting on the airplane are zero. Any event altering this state is a disturbance. It could be for example, (a) movement of airplane controls by the pilot or (b) inputs beyond pilot's control like gust of air. The equations derived by Bryan still form the basis of stability analysis.

1.1.2 Subsequent developments

In the 1930's, the flying qualities of the airplane were studied. These (flying qualities) are based on the opinion of the pilots regarding the amenability of the airplane to perform chosen tasks with precision and without undue effort on the part of the pilot. These were correlated to features of the motion like frequency of oscillation, damping etc. and finally to the geometric features of the airplane like area of horizontal tail, area of vertical tail and dihedral.

In the 1940's automatic control of airplanes became possible. An airplane with automatic control has sensors to detect the linear and angular accelerations and changes in flight path. Once the changes have been detected, the control surfaces are deflected automatically depending on the quantity sensed and the corrections needed. An airplane with automatic control is equivalent to an airplane with a different level of stability. By changing the ratio of input to the output of the automatic control system, it was possible in 1950's to have airplanes with variable stability.

Figure 1.4 Supersonic airplanes(MIG-29)

Supersonic flight became possible in 1950's after gaining an understanding of the changes in drag coefficient; lift coefficient and pitching moment coefficient when flight Mach number (M) changes from subsonic to supersonic. These changes also affect the stability of the airplane. It was also understood that the adverse effects of these changes can be alleviated by use of wing sweep (Fig.1.4).

In 1980's airplanes with fly-by-wire technology were available. In this technique the movement of the control stick or pedals by the pilot is transmitted to a digital computer. The input to the computer is processed along with the characteristics of the airplane and the actuators of the controls are operated so as to give optimum performance.

Recent developments include relaxed static stability and control configured vehicle (CCV). Relaxed static stability is used in fighter airplanes to improve their performance. The light combat aircraft (LCA) designed and developed in India has this feature. In a control configured vehicle, the control surfaces and flaps are automatically deployed when the airplane changes from one flight to another. With CCV the structural weight, size of the wing and size of control surfaces can be reduced to an optimum level while achieving greater maneuverability of the airplane.

1.2. Basic concepts about airplane stability and control

The equilibrium equations for flight give the lift and the thrust required during the flight. Subsequent analysis of these equations gives important items of performance. It may be pointed out that these analyses tacitly assume that the airplane will continue to fly in the equilibrium state. However, in actual practice it is noticed that among the various equilibrium states that we can imagine, some are not observable. To illustrate this, consider the following example.

One can imagine a chalk piece to rest in equilibrium on its narrow rounded end on a smooth horizontal table. However, no one has seen this equilibrium. The reason for this is that while imagining the equilibrium it is tacitly assumed that the chalk piece is rotationally symmetric about the center point of the rounded end and that there are no disturbances (e.g. small current of air). On the other hand, the chalk piece can be made to stand on the table on its flat, broad end. It will remain standing even in the presence of a small current of air like a gentle blowing. Of course, blowing hard at the chalk piece will topple it. This brings us to the following important observations.

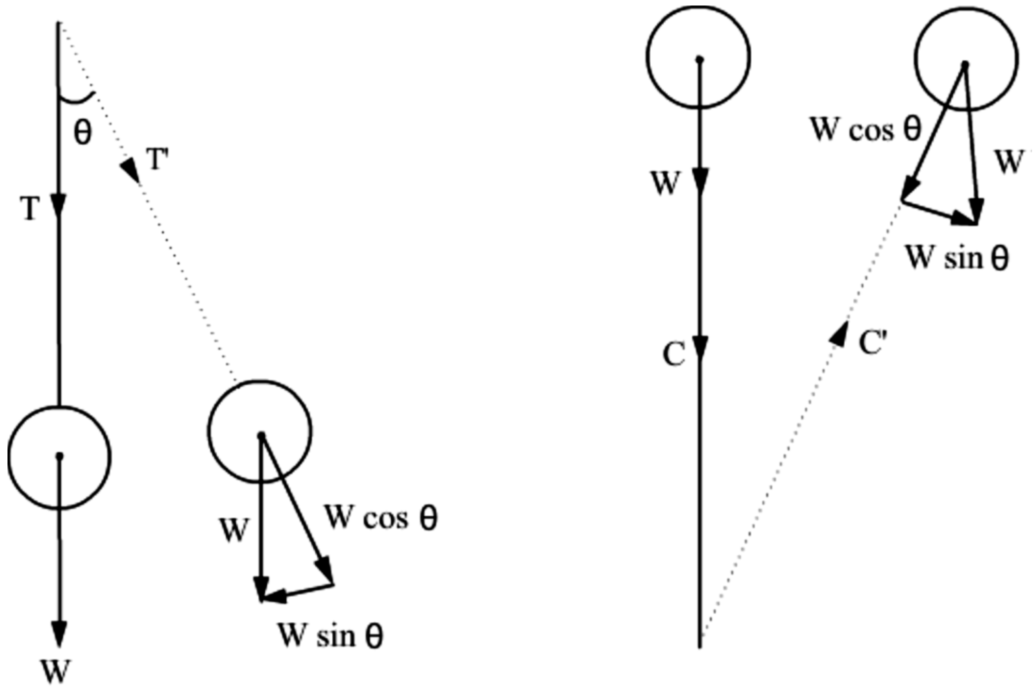
(a) There are equilibrium states from which, when a system is disturbed slightly over a short period, it will return to the equilibrium state. In other case it will not. The former are termed stable states of equilibrium and the later as unstable states of equilibrium.

(b) When the disturbance is large, the system may not come back to the equilibrium state. Further, analysis of the case of large disturbance is more complicated.

1.2.1. Stable, unstable and neutrally stable states of equilibrium

To explain the concepts of stable and unstable equilibrium, let us consider the example of a pendulum.

Figure 1.5a shows the pendulum in a state referred to as ‘A’. In this state, the weight (W) of the bob is supported by the tension (T) in the rod. Let the pendulum be disturbed, so that it makes an angle θ to the original position. In this disturbed position, the weight of the bob has components Wcosθ and Wsinθ. The component $W \cos \theta$ is balanced by the tension (T’) in the rod whereas the unbalanced component W sin θ causes the pendulum to move towards the undisturbed position. While returning to the equilibrium position, the bob may overshoot that position. However, when there is friction at the hinge and/or damping due to the medium in which the pendulum moves, it (pendulum) will eventually come back to its original equilibrium position. Thus, the equilibrium ‘A’ is a case of stable equilibrium.



(a) Bob at the bottom – state ‘A’ **(b) Bob at the top – state ‘B’**

Figure 1.5 Equilibrium states and stability of a pendulum

In equilibrium state 'B' as shown in Fig.1.5 (b), the weight of the bob is balanced by compression (C) in the rod. Let the pendulum be disturbed, so that it makes an angle θ to the original position. In this disturbed position, the weight of the bob has components $W\,cos\theta$ and $W\,sin\theta$. The component $W\,sin\theta$ in this case tends to move the pendulum away from its equilibrium position. Hence, equilibrium 'B' is unstable.

Apart from the stable and unstable equilibrium, there is a third state called neutrally stable equilibrium. It is defined as follows.

If a system, when disturbed slightly from its equilibrium state, stays in the disturbed position (neither returns to the equilibrium position nor continues to move away from it), then, it is said to be in neutrally stable equilibrium. In the above example of the pendulum, if the static friction at the hinge is very large, then, on being disturbed from the equilibrium position, it will remain in the disturbed position.

1.2.2. types of motions following a disturbance – subsidence, divergence, neutral stability, damped oscillation, divergent oscillation and undamped oscillation

After a system has been disturbed from its equilibrium position, its subsequent motion will be like any one on the six types shown in Fig.1.6. For the sake of the subsequent discussion, it is assumed that initially the disturbance is positive.

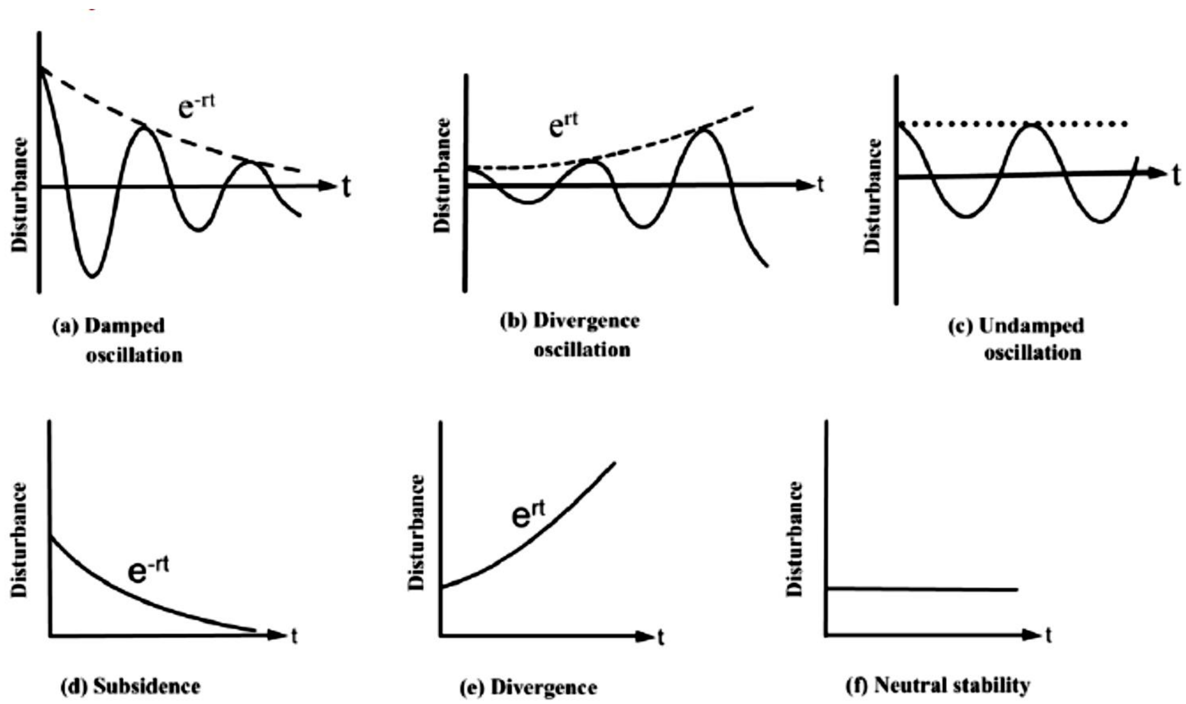


Figure 1.6 Types of motion following a disturbance

i) Figure 1.6a shows a damped oscillation. In this case the system while returning to the equilibrium position goes beyond the undisturbed state towards the negative side. However, the amplitude on the negative side is smaller than the original disturbance and it (amplitude) decreases continually with every oscillation. Finally, the system returns to the equilibrium position. The time taken to return to the equilibrium position depends on the damping in the system. An example of this is the motion of pendulum (Fig.1.5 a) when there is friction at the hinge or the pendulum moves in a fluid (air or water). The friction at the hinge or that between the bob and the fluid results in damping.

ii) Figure 1.6b shows the divergent oscillation. In this case, also the system shows an oscillatory response but the amplitude of the oscillation increases with each oscillation and the system never returns to the equilibrium position. It may even lead to the disintegration of the system. Divergent oscillations are seldom encountered. The practical systems are designed such that they do not get into divergent oscillations.

iii) Figure 1.6c shows the undamped oscillation. In this case, also the system shows an oscillatory response but the amplitude of the oscillation remains unchanged and the system never returns to the equilibrium position. An example of this situation is the ideal case of the pendulum motion (Fig.1.5a), when the hinge is frictionless, and the pendulum oscillates in the vacuum.

iv) When a system returns to its equilibrium position without performing an oscillation, the motion is said to be a subsidence (Fig.1.6d). An example of this could be the motion of a door with a hydraulic damper. In the equilibrium position the door is closed. When someone enters, the equilibrium of the door is disturbed. When left to itself the door returns to the equilibrium position without performing an oscillatory motion.

v) Conversely, when the system continuously moves away from the equilibrium position, the motion is called divergence (Fig.1.6e).

vi) If the system stays in the disturbed position (Fig.1.6f), then the system is said to have neutral stability.

I.3. **Static stability and dynamic stability**

In the cases illustrated by Fig.1.6 a, b, c and d, it is observed that, as soon as the system is disturbed, it tends to return to the undisturbed position. Such systems are called statically stable.

When the tendency of the system, after the disturbance, is to stay in the disturbed position, then it is said to have neutral static stability.

Even when the system has a tendency to go towards the undisturbed position (cases 1.6a, b, c and d), it may not return to the equilibrium position as in the cases shown in Fig.1.6 b & c namely divergent oscillation and undamped oscillation. Only when the system finally returns to the equilibrium position, the system is said to be dynamically stable. Otherwise, it is dynamically unstable. With this criterion, the damped oscillation and subsidence are the only dynamically stable cases.

Remarks:

i) The definitions of the terms static stability and dynamic stability are as follows:

Static Stability: A system is said to be statically stable when a small disturbance causes forces and moments that tend to move the system towards its undisturbed position. If the forces and moments tend to move the system away from the equilibrium position, then the system is said to be statically unstable. In the case of a system having neutral static stability, no forces or moments are created as a result of the disturbance.

Dynamic Stability: A system is said to be dynamically stable if it eventually returns to the original equilibrium position after being disturbed by a small disturbance.

ii) It is obvious from the above discussion that for a system to be dynamically stable, it must be statically stable. Table 1.1 categories the cases in Fig.1.6 as regards the static stability and dynamic stability.

Case	Figure	Static stability	Dynamic stability
Damped oscillation	1.6a	Yes	Yes
Divergent oscillation	1.6b	Yes	No
Undamped oscillation	1.6c	Yes	No
Subsidence	1.6d	Yes	Yes
Divergence	1.6e	No	No
Neutral stability	1.6f	No	No

Table 1 Static and dynamic stability

iii) iii) The distinction between static stability and dynamic stability is of special significance in aeronautical applications as the analysis of static stability is much simpler than that of the dynamic stability. This can be explained as follows. The disturbance to an airplane in flight due to a gust may change its angle of attack (α) or sideslip (β) or bank (φ) or the thrust output. Now, these changes may produce changes in aerodynamic forces and moments. If these forces and moments tend to bring the airplane to the original state, then the airplane is statically stable. Thus, to assess the static stability, one needs only to examine the aerodynamic / propulsive forces and moments brought about at the time the disturbance is applied. On the other hand, to examine the dynamic stability of the airplane, one has to consider the subsequent motion which involves accelerations and hence, the inertia forces. Further, the dynamic stability analysis requires solution of the equations of motion taking into account the changes, with time, in aerodynamic forces and moments due to changes in α,β, φ , and the linear and angular velocities etc. of the airplane. These quantities denoting changes in aerodynamic forces and moments due to aforesaid changes are called aerodynamic / stability derivatives. Hence, in aeronautical engineering practice first the static stability is ensured by providing adequate areas of horizontal tail and vertical tail and the dihedral angle. Subsequently, the dynamic stability analysis is carried out to ensure that there is adequate damping.

I.4. Recapitulation of some terms – body axes system, earth fixed axes systems, attitude, angle of attack and angle of sideslip

At this stage a brief discussion on body axes system, attitude, angle of attack and angle of sideslip would be helpful

I. Body axes system

To formulate and solve a problem in dynamics we need a system of axes. To define such a system, we note that an airplane is nearly symmetric in geometry and mass distribution about a plane which is called the plane of symmetry (Fig.1.7). This plane is used for defining the body axes system. Figure 1.8 shows a system of axes (X_b Y_b Z_b) fixed on the airplane which moves with the airplane and hence called body axes system. The origin 'O' of the body axes system is the center of gravity (c.g.) of the body which, by assumption of symmetry, lies in the plane of symmetry. The axis X_b is taken as positive in the forward direction. The axis Z_b is perpendicular to X_b in the plane of symmetry, positive downwards. The axis Y_b is perpendicular to the plane of symmetry such that X_b Y_b Z_b is a right handed system.

Figure 1.8 also shows the forces and moments acting on the airplane and the components of linear and angular velocities. The quantity V is the velocity vector. The quantities X, Y, Z are the components of the resultant aerodynamic force, along X_b, Y_b *and* Z_b axes respectively. L' , M, N are the rolling moment, pitching moment and yawing moment respectively about X_b, Y_b *and* Z_b; the rolling moment is denoted by L' to distinguish it from lift (L) . Figure 1.8 also shows the positive directions of L' , M *and* N. The convention is that an aerodynamic moment is taken positive in clock-wise sense when looking along the axis about which the moment is taken. u,v,w are the components , along X_b, Y_b *and* Z_b of the velocity vector (V). The angular velocity components are indicated by p,q,r.

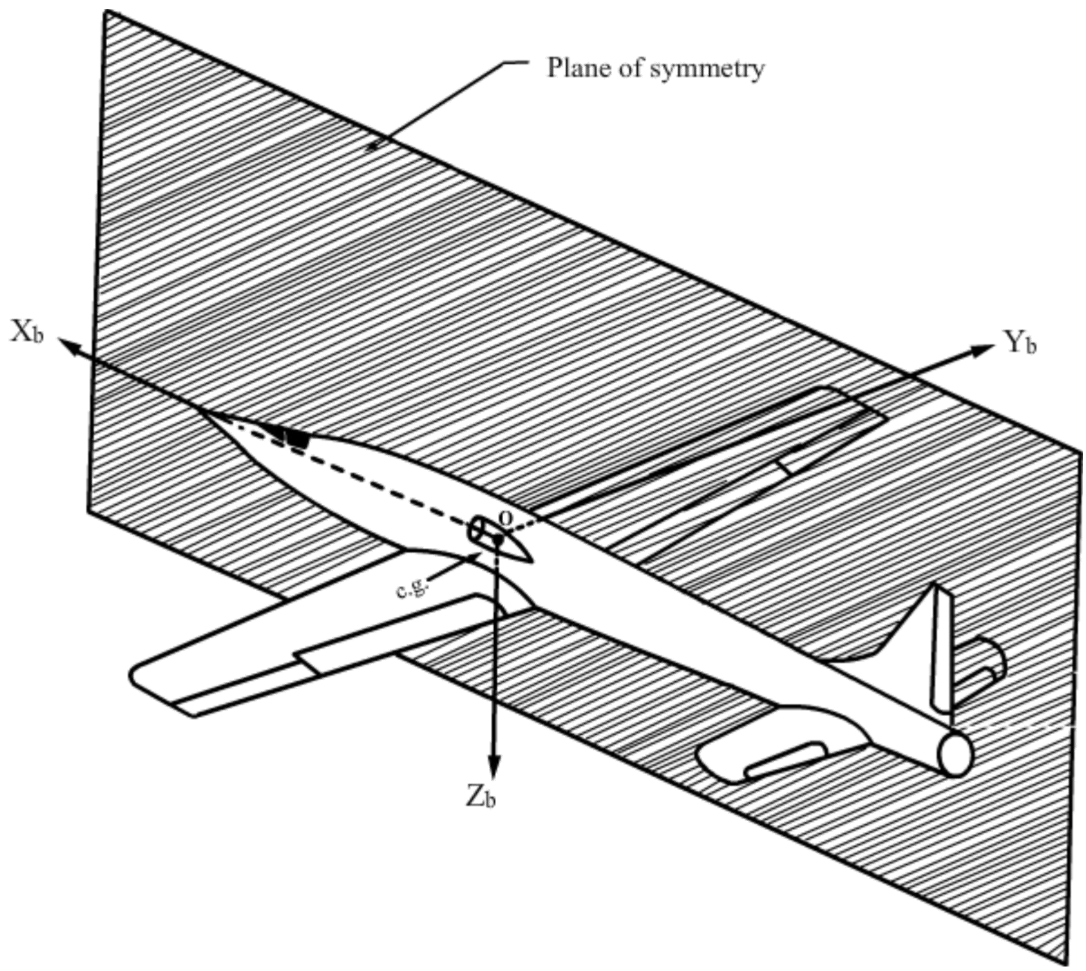


Fig.1.7 Plane of symmetry and body axis system

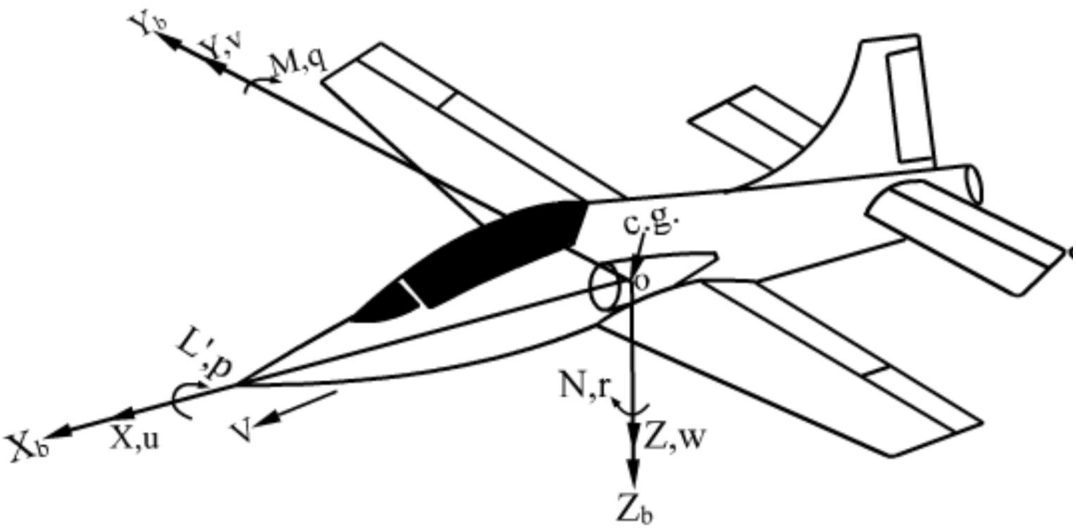


Fig.1.8 Body axes system, forces, moments and linear and angular velocities

II. Earth fixed axes system

In flight dynamics a frame of reference attached to the earth is taken as a Newtonian frame (Fig.1.9).

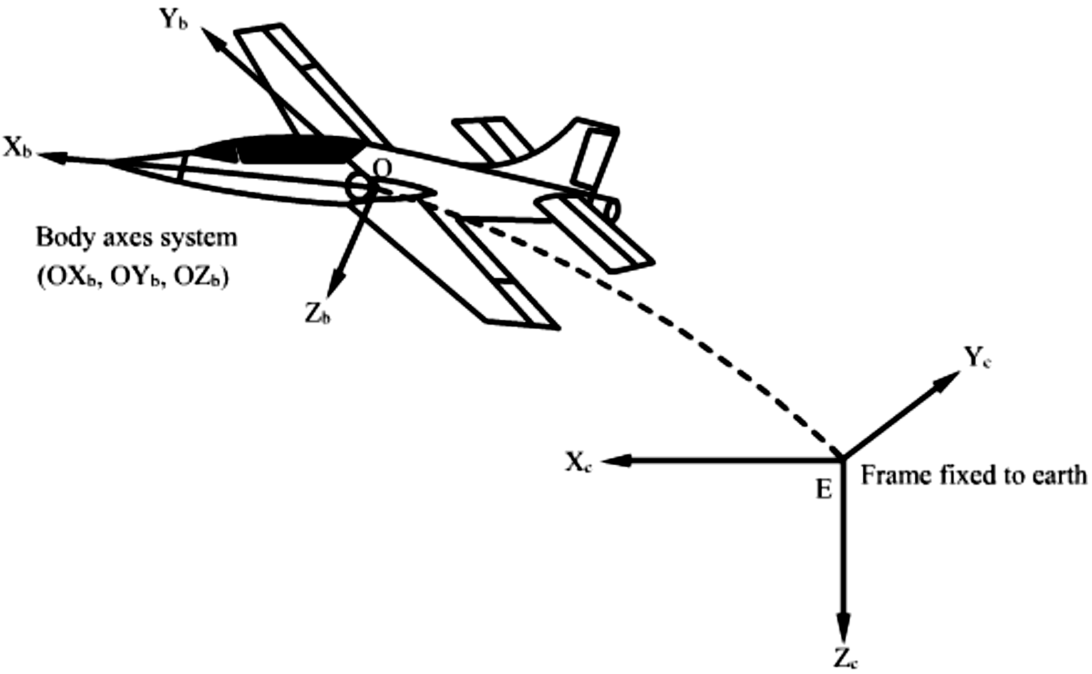


Fig.1.9 Earth fixed and body fixed co-ordinate systems

III. Attitude

In this course the airplane is treated as a rigid body has a six degrees of freedom and hence, six coordinates are needed to describe the position of the airplane with respect to an earth fixed system. In flight dynamics, the six coordinates employed to prescribe the position are (a) the three coordinates describing the instantaneous position of the c.g. of the airplane with respect to the earth fixed system and (b) the attitude of the airplane described by the angular orientations of X_b Y_b Z_b system with respect to the X_e Y_e Z_e system. This is done with the help of Euler angles. In next sections it is shown that to arrive at the X_b Y_b Z_b system, we need to rotate the X_e Y_e Z_e system through only three angles which are called Euler angles.

At this stage, simpler cases are considered. When an airplane climbs along a straight line its attitude is given by the angle 'γ' between the axis X_b and the horizontal (Fig.1.10). When an airplane executes a turn, the projection of the X_b axis, in the horizontal plane makes an angle 'Ψ' with reference to fixed horizontal axis (Fig.1.11). When an airplane is banked, the axis Y_b makes an angle 'φ' with respect to the horizontal and the axis Z_b makes an angle 'φ' with vertical (Fig.1.12).

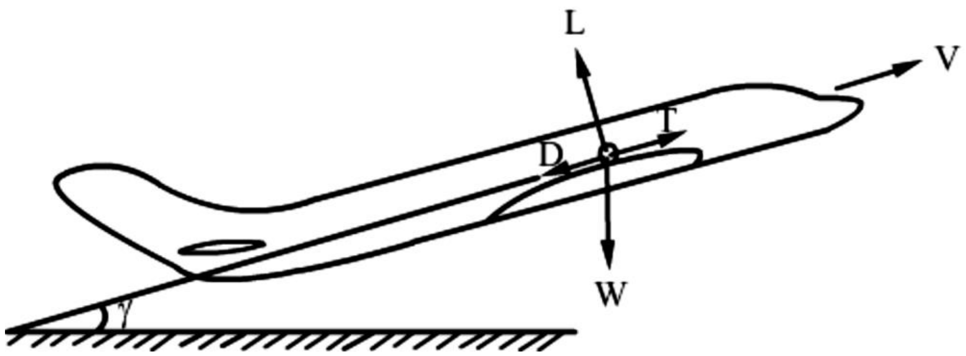


Fig.1.10 Airplane in a climb

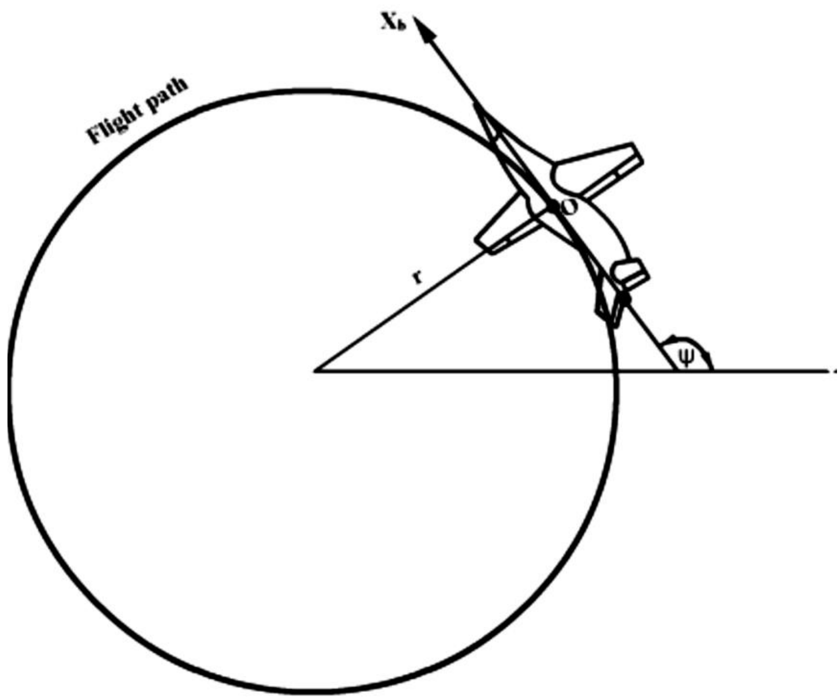


Fig.1.11 Airplane in a turn – view from top

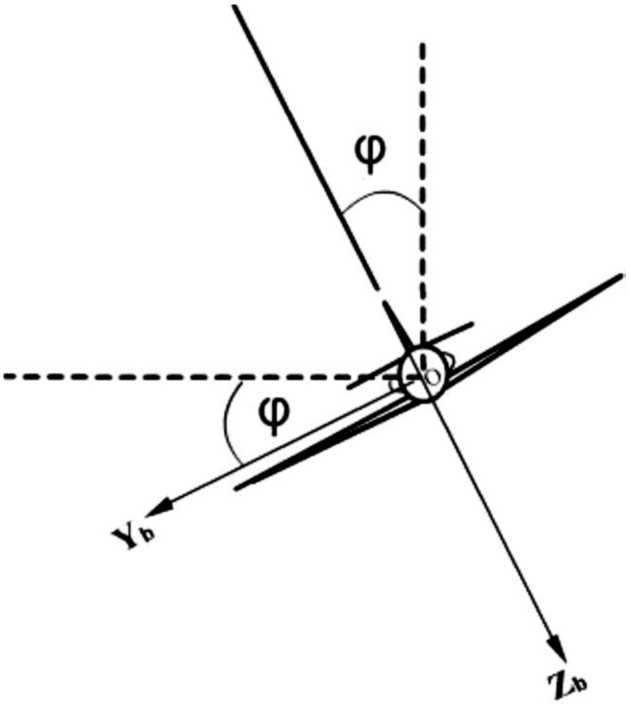


Fig.1.12 Angle of bank (φ)

IV. Flight path:

The flight path, also called the trajectory, means the path or the line along which the c.g. of the airplane moves. The tangent to this curve at a point gives the direction of flight velocity at that point on the flight path. The relative wind is in a direction opposite to that of the flight velocity.

V. Angle of attack and angle side slip:

The concept of the angle of attack of an airfoil is well known. While discussing the forces acting on an airfoil, we take the chord of the airfoil as the reference line and the angle between the chord line and the relative wind is the angle of attack (α). The aerodynamic forces namely lift (L) and drag (D), produced by the airfoil, depend on the angle of attack (α) and are respectively perpendicular and parallel to relative wind direction (Fig.1.13).

In the case of an airplane, the flight path, as mentioned earlier, is the line along which c.g. of the airplane moves. The tangent to the flight path is the direction of flight velocity (V). The relative wind is in a direction opposite to the flight velocity. If the flight path is confined to the plane of symmetry, then the angle of attack would be the angle between the relative wind direction and the fuselage reference line (FRL) or X_b axis (see Fig.1.14). However, in a general case the velocity vector (V) will have components both along and perpendicular to the plane of symmetry. The component perpendicular to the plane of symmetry is denoted by 'v'. The projection of the velocity vector in the plane of symmetry would have components u and w along X_b and Z_b axes (Fig.1.15). With this background, the angle of sideslip and angle of attack are defined below.

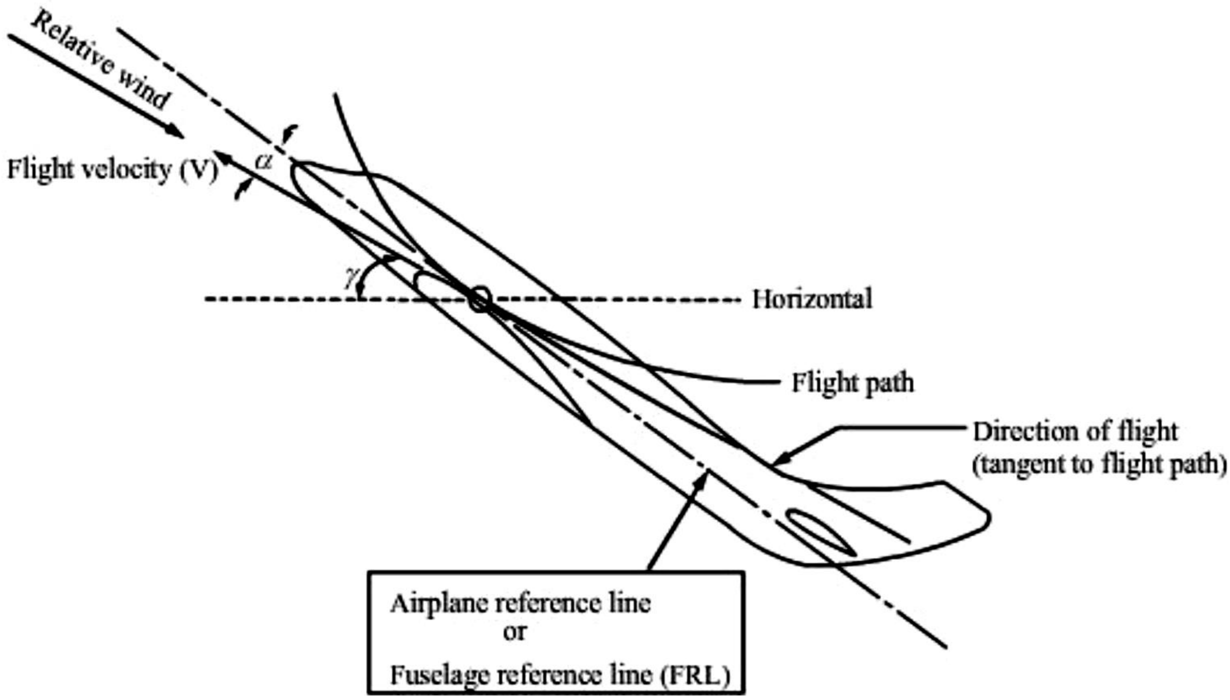
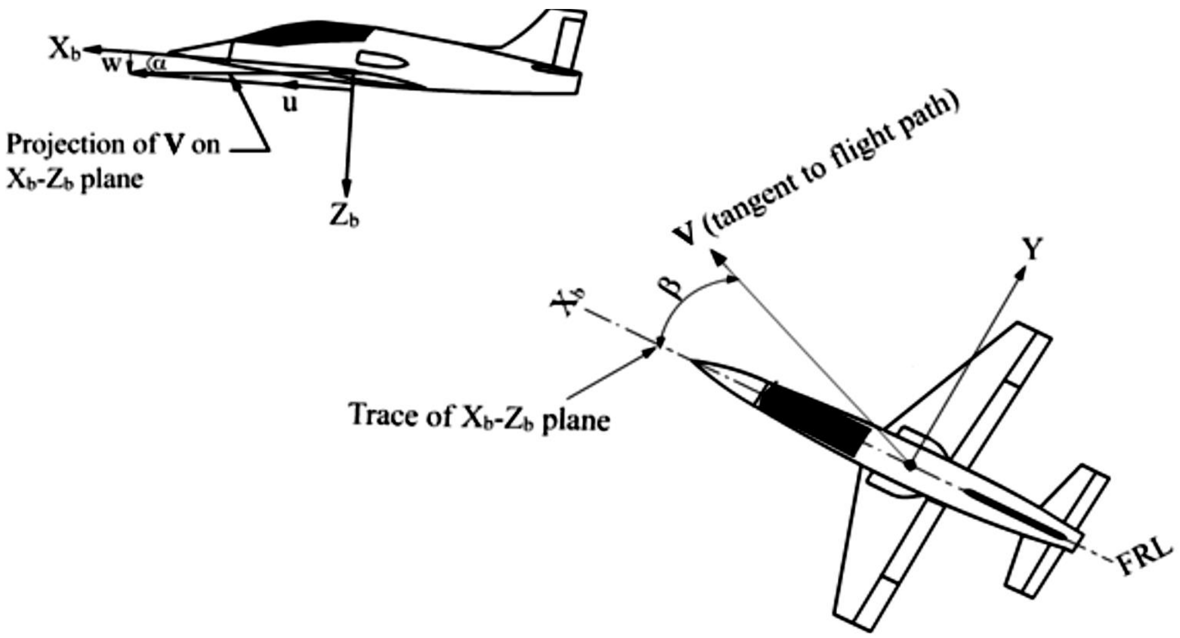


Fig.1.14 Flight path in the plane of symmetry



Velocity components in a general case and definition of angle of attack and sideslip

The angle of sideslip (β) is the angle between the velocity vector (**V**) and the plane of symmetry i.e.

$$\beta = \sin^{-1}\left(\frac{v}{|V|}\right) \text{ ; where } |\mathbf{V}| \text{ is the magnitude of } \mathbf{V}.$$

The angle of attack (α) is the angle between the projection of velocity vector (V) in the X_b - Z_b plane or

$$\boldsymbol{\alpha = \tan^{-1}\frac{w}{u} = \sin^{-1}\frac{w}{\sqrt{|V|^2 - v^2}} = \sin^{-1}\frac{w}{\sqrt{u^2 - w^2}}}$$

Remark:

It is easy to show that, if V denotes magnitude of the velocity (V) , then

$$\boldsymbol{u = V\cos\alpha\cos\beta \,;\; v = V\sin\beta;\;\; w = V\sin\alpha\cos\beta}\,.$$

Chapter 1 : Introdution

Lecture 2

Topics

1.5 Definitions

1.6 Longitudinal Stability Derivatives

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1.5. Definitions

One of the difficulties encountered by students of flight mechanics relates to sign conventions, physical units, and nomenclature involving multiple subscripting. The conventions used in the following material will follow closely to the guidelines established jointly by the American National Standards Institute (ANSI) and AIAA. These conventions are used in a large part by industry and researchers in the United States.

1.5.1. Dynamic pressure: Velocity may be expressed as either true, V_t, or equivalent, V_e; the relationship to each other is from the dynamic pressure expression

$$Q = \frac{1}{2}\rho V_t^2 = \frac{1}{2}\rho_o V_e^2$$

Where

$\rho = atmospheric\ density\ at\ altitude\ above\ sea\ level$

$\rho_o = ideal\ sea - level\ atmospheric\ density$

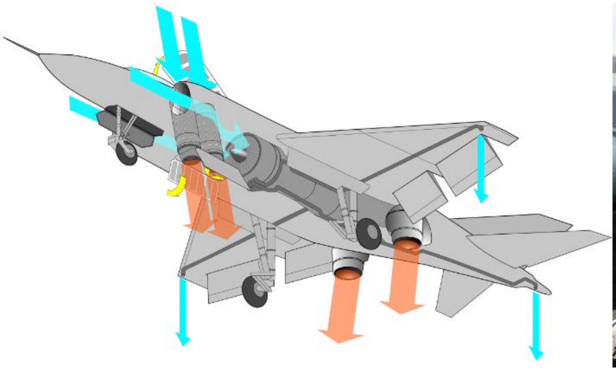
$V_t = true\ airspeed\ (TAS)$

$V_e = equivalent\ airspeed\ (EAS)$

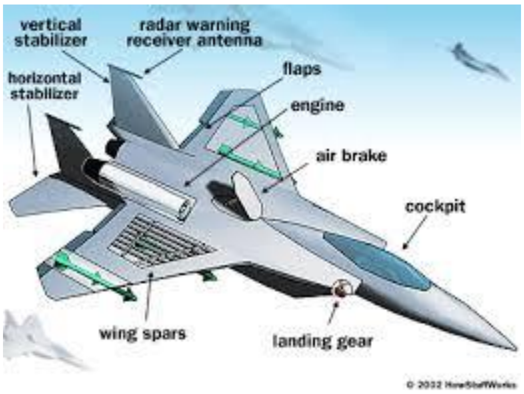
$$V_t = \sqrt{\frac{\rho_o}{\rho}}\, V_e$$

1.5.2. Control Systems

In addition to conventional control systems such as ailerons, elevators, and rudders, external control of the forces and moments acting on the vehicle can also occur from a variety of other physical devices including 1) thrust vectoring by a jet engine's exhaust, 2) speedbrakes/spoilers mounted on the fuselage or wing, 3) horizontal tail mounted forward of the wing (canard surface), 4) vee tail to replace horizontal and vertical tail(s) aft of the main wing lifting surface, 5) vertical fin on the underside of the fuselage for direct side-force control, and 6) variable wing camber.



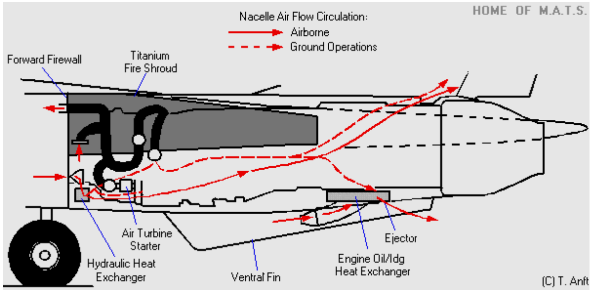
thrust vectoring by a jet engine's exhaust



speedbrakes/spoilers mounted on the fuselage or wing

horizontal tail mounted forward of the wing (canard surface)

vee tail



vertical fin on the underside of the fuselage for direct side-force control

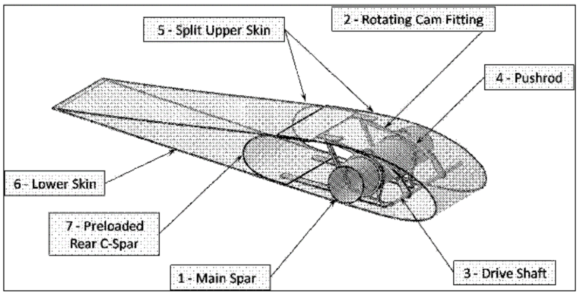


FIG. 1

variable wing camber

For the most part, the material described in this lectures will assume a conventional control system. The conventional controls shown in Fig. 1.16 include the aileron (mounted on the wing trailing edge), the elevator (mounted on the horizontal tail), and the rudder (mounted on the vertical tail). These control surfaces produce their aerodynamic force and moment inputs due to a change in camber of a lifting surface.

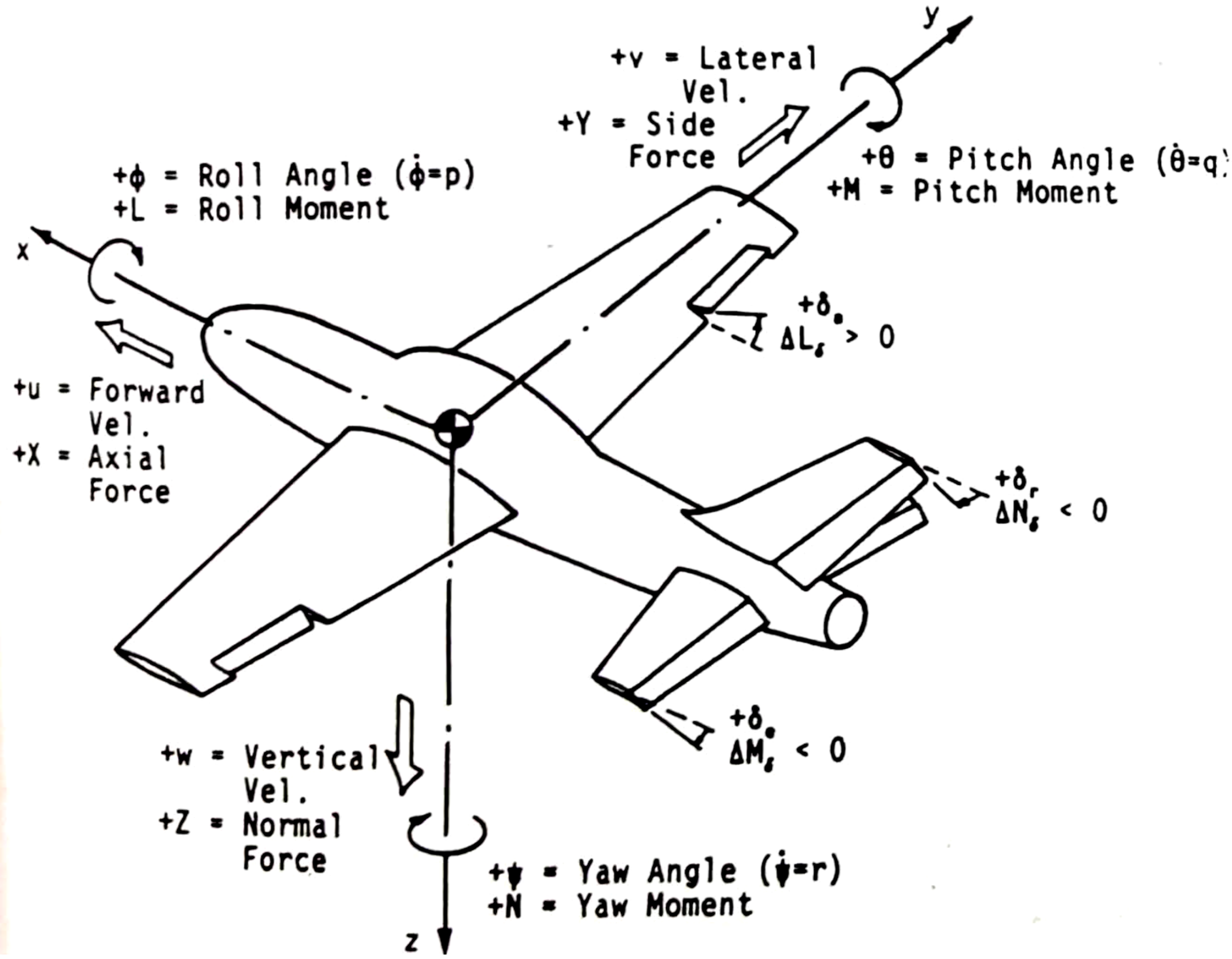


Figure (1.16) Coordinate sign convention.

Longitudinal control deflection δ_e is shown in Fig. 1.16 as positive with the trailing edge down. If the longitudinal control were behind the aircraft e.g., one would expect the resulting pitch moment M about the c.g. to be negative in sign. The rate of change of pitching moment with elevator deflection is described by the sensitivity term of $\Delta M_\delta = \partial M / \partial \delta_e$ The two force sensitivity terms, ΔX_δ and ΔZ_δ, are important in dynamics but are overshadowed by the importance of ΔM_δ.

Lateral control deflection δ_a is shown as positive with right-hand aileron trailing edge up and left-hand aileron trailing edge down. It will be assumed that the deflection pattern is antisymmetric, which results in wing span loads producing a rolling moment, i.e., positive aileron control tends to lower the right-hand wing in the direction of increasing roll (bank) angle ϕ. Although lateral control produces yawing moments and side forces on the airframe, the dominant term is the rate of change of rolling moment with aileron deflection, described by the sensitivity term $\Delta L_{\delta} = \partial L / \partial \delta_a$.

Directional control deflection δ_r, is shown in Fig. 1.16 as positive with the rudder's trailing edge rotating to the left and will normally produce a positive side force with a corresponding negative yawing moment about the c.g. The prime influence of rudder control is the production of a yawing moment as described by the sensitivity term $\Delta V_{\delta} = \partial N / \partial \delta_r$. This stability derivative, in addition to the associated rolling moment and side-force sensitivities, will vary in value (and sign) depending on airplane angle of attack.

1.5.3. Dimensionless Aerodynamic Coefficients

The dimensional stability derivatives to be used in the equations of motion have as a basis the dimensionless coefficients defined as follows:

$$C_X = \frac{X}{QS};$$

$$C_Y = \frac{Y}{QS};$$

$$C_Z = \frac{Z}{QS};$$

$$C_l = \frac{roll\ moment, L}{QSb};$$

$$C_m = \frac{pitch\ moment, M}{QSb};$$

$$C_n = \frac{yaw\ moment, N}{QSb};$$

Where

$Q = dynamic\ pressure = 0.5p(u^2 + v^2 + w^2) = 0.5\,\gamma p M^2$

$\rho = air\ density\ at\ specified\ altitude$

$p = static\ pressure\ at\ specified\ altitude$

$M = Mach\ number\ of\ aircraft$

$y = ratio\ of\ specific\ heats, normally\ 1.4\ for\ air$

$S = wing\ area$

$b = wing\ span$

$c = wing\ chord$

Two other dimensionless force coefficients, which are commonly used in aerodynamic and performance analyses, are the airplane **lift and drag coefficients**. The use of L as a subscript for lift force should not be confused with the notation of L to indicate a rolling moment,

$$C_L = \frac{lift\ force}{QS};\ positive\ up\ and\ normal\ to\ the\ velocity\ vector$$

$$C_D = \frac{drag\ force}{QS};\ positive\ aft\ and\ parallel\ to\ the\ velocity\ vector$$

1.5.4. Center of Gravity (CG) is the point where the **weight** of the aircraft is balanced.

1.5.5. Neutral Point (NP) is the point where the **aerodynamic forces** generated by the wing and tail are balanced $\left(\frac{x}{c}\right)_{NP}$.

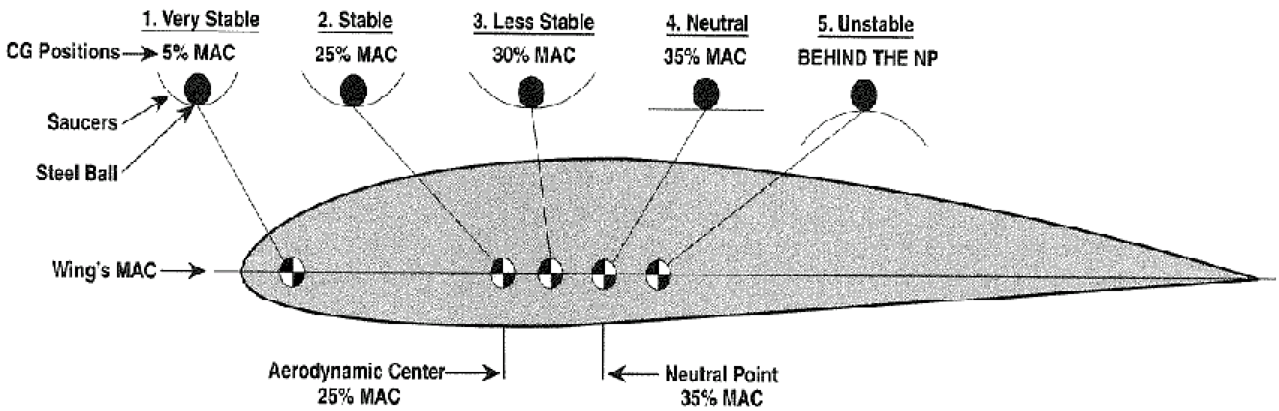
1.5.6. The Maneuver Point, M_P is defined as the CG position at which, under steady, pull-up maneuver. (in which the velocity and angle of attack (α) are held constant)

To investigate static stability and control for accelerated flight, use is made of a pull-up. Of interest is the elevator angle required to make an n-g turn or pull-up. There is a cg position where the elevator angle per g goes to zero, making the airplane too easy to maneuver. This cg position is called the maneuver point. There is another maneuver point associated with the stick force required to make an n-g pull-up.

Static Margin when Placing CG 5% - 15% of MAC $\left(\frac{x}{c}\right)_{cg}$ in front of NP creates longitudinal (pitch) stability.

A lower margin produces less stability and greater elevator authority, while a higher margin creates more stability and less elevator authority. Too much static margin results in elevator stall at take-off and landing.

MAC: Mean aerodynamic chord.



$(\boldsymbol{\Delta x/c})_{ac}$ = location of the wing-body aerodynamic center (a.c.) relative to the c.g. and normalized with respect to the wing reference chord length

The term contributing to the M_α derivative may be estimated directly from the airframe's dimensionless static stability derivative. Because the complete aircraft lift from α acts at an x location along the fuselage axis of symmetry described as the neutral point, the static stability derivative can be expressed relative to the complete airframe lift-curve slope as

$$\frac{\partial C_m}{\partial \alpha} = \left(\frac{\partial C_L}{\partial \alpha}\right)_{a/c}\left[\left(\frac{x}{c}\right)_{np} - \left(\frac{x}{c}\right)_{cg}\right] = \left(\frac{\partial C_L}{\partial \alpha}\right)_{a/c}\left(\frac{\Delta x}{c}\right)_{np}$$

where it is assumed in above Eq. that l) both c.g. and neutral point locations are relative to the leading edge of the reference chord and 2) the body-axis coordinate system.

longitudinally stable aircraft corresponds to $\left(\frac{\Delta x}{c}\right)_{np}$ having a negative sign, which implies that the c.g. is forward of the neutral point. In this case, a positive angle-of-attack perturbation results in a restorative (stable) pitching moment about the c.g. from the lift change that acts at the neutral point. A related term, the static margin, is described as

$$Static\ margin = -\left(\frac{\Delta x}{c}\right)_{np}$$

Chapter 2: Static Stability and Control

Lecture 3

Topics

2.1. Longitudinal Control for Velocity Change

Static stability associated with an aircraft's longitudinal axis normally is addressed by the sign of the C_{m_α} stability derivative. A minus sign implies that a perturbation increase in angle of attack from the flight trim setting will result in a negative pitching moment being developed, where $\Delta C_m = C_m \Delta\alpha$ will be in a direction to restore the aircraft α back to α_{trim}. The degree of longitudinal static stability for an aircraft will depend on the location of the aircraft c.g. relative to the neutral point.

Design problems associated with the c.g. location are legendary; two early examples are as follows.

1. An early U.S. Air Force jet fighter/trainer had many internal fuel tanks, which required careful fuel management by the pilot throughout the flight mission in order that the aircraft remain within acceptable c.g. boundaries.
2. An early supersonic jet bomber required careful fuel management during transition between subsonic and supersonic flight.

Aircraft longitudinal control brings many requirements into play. These requirements include the following.

1. The aircraft should be trimmable at a given value of C_L, c.g. location, and airframe configuration (e.g., flaps up, flaps down, flaps down with gear extended, etc.).
2. The flight velocity should increase by pilot application of nose-down (+δ) longitudinal control and decrease by a reversal of the control direction.
3. The aircraft must have a maneuvering capability to do either pull-ups or turns subject to staying within the maneuvering (V-n) enyelope.
4. Recovery from transient maneuvers, such as incurred by aircraft wake and turbulent gust encounters, must be possible using longitudinal control.

2.1.1 Longitudinal Control for Velocity Change

The trim velocity of an aircraft determines the equilibrium lift coefficient, which in turn is established by the use of longitudinal control. There are two independent variables, α and δ, which are determined by satisfying equilibrium in lift and pitching moment for a given aircraft configuration. A graphical sketch

(Fig. 2.2) shows the C_m and α variation with C_L for fixed values of longitudinal control as might be obtained from wind-tunnel tests on a scale model of the aircraft. The zero-lift pitching-moment coefficient, when control $\delta = 0$, is denoted as C_{mo} in Fig. 2.2. The zero-lift intercept for *a* C_m-C_L curve corresponding to a positive trim C_L value must be a positive value for a statically stable aircraft. The α = const line on the C_m-C_L plot is shown for various control angles. It is not at a constant value of airplane C_L due to a ΔC_L term arising from $(C_{L_\delta}\Delta\delta)$. A stable stall break also is shown in Fig. 3.1, which implies that when the airplane a reaches and exceeds α_{stall}, a negative change in pitching moment occurs to lower the aircraft's nose for the initiation of stall recovery. A stall demonstration is a dynamic maneuver used in the determination of the minimum flying speed for a given airplane weight, configuration, and e.g. position.

It is customary in data plots of C_m *vs* C_L to show (-) C_m toward the right, as in Fig. 2.2.

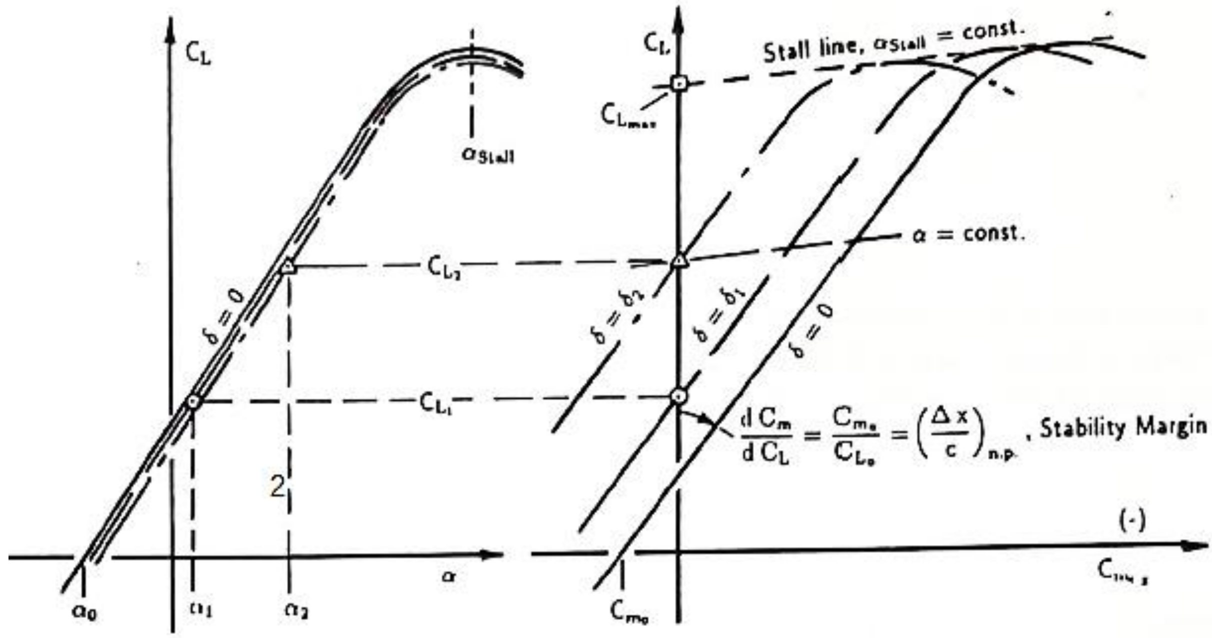


Fig. 2.2 Sketch of the C_m and α variation with C_L.

The initial trim point at C_{L1} may be found by considering the static equilibrium relations in C_m and C_L using Fig. 2.2 to clarify the expressions.

Moment equilibrium is given by

$$0 = C_{m_{cg}} = C_{m_0} + C_{m_\alpha}(\alpha_1 - \alpha_0) + C_{m_\delta}\delta_1$$

Lift equilibrium is given by

$$C_{L_1} = C_{L_\alpha}(\alpha_1 - \alpha_0) + C_{L_\delta}\delta_1$$

Where $\alpha_o = \alpha$ for zero lift when control $\delta = 0$ and C_{m_o} is the pitching-moment coefficient when $\alpha = \alpha_o$ and $\delta = 0$. Note that C_{mo} is invariant with respect to c.g. position.

The two equilibrium equations may be combined in matrix form as

$$\begin{bmatrix} C_{m_\alpha} & C_{m_\delta} \\ C_{L_\alpha} & C_{L_\delta} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \delta_1 \end{Bmatrix} = \begin{Bmatrix} C_{m_\alpha}\alpha_0 - C_{m_0} \\ C_{L_1} + C_{L_\alpha}\alpha_0 \end{Bmatrix}$$

or symbolically as

$$[A] \begin{Bmatrix} \alpha_1 \\ \delta_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ C_{L_1} \end{Bmatrix} + \begin{Bmatrix} C_{m_\alpha}\alpha_0 - C_{m_0} \\ C_{L_\alpha}\alpha_0 \end{Bmatrix} \quad \text{……(2.1)}$$

A solution for α_1 and δ_1 at the trim value of C_L, is found by premultiplying both sides of Eq. (2.l) by the inverse of the matrix [A], i.e.,

$$\begin{Bmatrix} \alpha_1 \\ \delta_1 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} 0 \\ C_{L_1} \end{Bmatrix} + [A]^{-1} \begin{Bmatrix} C_{m_\alpha}\alpha_0 - C_{m_0} \\ C_{L_\alpha}\alpha_0 \end{Bmatrix} \quad \text{….(2.2)}$$

where

$$[A]^{-1} = \frac{1}{\Delta}\begin{bmatrix} C_{L_\delta} & -C_{m_\delta} \\ -C_{L_\alpha} & C_{m_\alpha} \end{bmatrix}$$

and

$$\Delta = \det[A] = C_{m_\alpha} C_{L_\delta} - C_{m_\delta} C_{L_\alpha}$$

It can be seen from Eq. (2.2) that trim at a second value of lift coefficient, C_{L2} , will require α and δ to satisfy Eq. (2.2) by the following relation

$$\begin{Bmatrix} \alpha_2 \\ \delta_2 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} 0 \\ C_{L_2} \end{Bmatrix} + [A]^{-1} \begin{Bmatrix} C_{m_\alpha}\alpha_0 - C_{m_0} \\ C_{L_\alpha}\alpha_0 \end{Bmatrix}$$

The determination of the α and δ values at a given trim C_L, assumed that thrust induced lift force and pitching moments were negligible. Coupling of the trim relation Eq. (2.2) would be handled by introducing thrust-related stability derivatives. The thrust values used with these derivatives would primarily depend on maintaining x-axis force equilibrium and would reflect whether the aircraft was in level flight or in a climb/descent condition.

<u>Example 2.1</u>

An aircraft described by

W/S = 2394.012 N/m^2 **Stick fixed neutral point at 0.35c**

$C_{m_o} = +0.02$ $C_{m_\delta} = -0.75\, rad^{-1}$ $C_{L_\alpha} = +5.0\, rad^{-1}$

$C_{L_\delta} = +0.25\, rad^{-1}$ $\alpha_o = -0.01745\, rad$ $V_{trim_1} = 128.611 m/s^2$[1]

$V_{trim_2} = 192.6\, m/s^2$

Initial c.g. position corresponds to a 5.0% static margin,

Assume that linearity prevails and that both compressibility effects and propulsion system interactions are negligible.

Determine

1) α and δ for a given initial trim velocities and c.g. position and ;

2) the variation of δ required to change air speed at a given c.g. position.

Solution:

The dynamic pressure at trim is

$$Q_1 = \frac{1}{2}\rho_o V^2 = 0.5 * 1.225 * 128.611^2 = 10131.46\, N/m^2$$

The trim lift coefficient is

$$C_{L_1} = \frac{\left(\frac{W}{S}\right)}{Q_1} = \frac{2394.012}{10131.46} = 0.2363$$

Because the lift coefficient at trim point equal the weight

---

[1] Equivalent airspeed (**EAS**)

Then obtain $C_{m_\alpha =}$

$$C_{m_\alpha =} C_{L_\alpha}\left(\frac{\Delta x}{c}\right)_{np} = 5.0\, rad^{-1} * -0.05 = -0.25\, rad^{-1}$$

Expanding the terms in accord with Eq. (2.2) for the assumed aircraft trim condition yields

$$\begin{Bmatrix}\alpha_1\\ \delta_1\end{Bmatrix} = [A]^{-1}\begin{Bmatrix}0\\ 0.2363\end{Bmatrix} + [A]^{-1}\begin{Bmatrix}-0.0156\\ -0.0873\end{Bmatrix}$$

where

$$[A]^{-1} = \begin{Bmatrix}0.0678 & 0.2034\\ -1.3559 & -0.0678\end{Bmatrix} rad$$

Completion of the matrix operations gives

$$\begin{Bmatrix}\alpha_1\\ \delta_1\end{Bmatrix} = \begin{Bmatrix}0.0293\\ 0.0111\end{Bmatrix} rad = \begin{Bmatrix}1.68\\ 0.64\end{Bmatrix} deg$$

Reconsider the previous steps for ***V = 192.6 m/s²*** and find α_2 and δ_2

for **(x/c)$_{cg}$ = 0.30** from Eq. (2.2),

$$Q_2 = \frac{1}{2}\rho_o V^2 = 0.5 * 1.225 * 92.6^2 = 5252.04\, N/m^2$$

$$C_{L_2} = \frac{\left(\frac{W}{S}\right)}{Q_2} = \frac{2394.012}{5252.04} = 0.4558$$

$$\begin{Bmatrix}\alpha_2\\ \delta_2\end{Bmatrix} = [A]^{-1}\begin{Bmatrix}0\\ 0.4558\end{Bmatrix} + [A]^{-1}\begin{Bmatrix}-0.0156\\ -0.0873\end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_2 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 0.0739 \\ -0.0038 \end{Bmatrix} rad = \begin{Bmatrix} 4.23 \\ -0.22 \end{Bmatrix} deg$$

Application of Eq. (2.2) to two C_L values gives the change in both α and δ to change the aircraft trim C_L from C_{L1} *to* C_{L2} • Here

$$\begin{Bmatrix} \alpha_2 - \alpha_1 \\ \delta_2 - \delta_1 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} 0 \\ C_{L2} - C_{L1} \end{Bmatrix} \ldots(2.3)$$

Or

$$\begin{Bmatrix} \Delta\alpha \\ \Delta\delta \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} 0 \\ \Delta C_L \end{Bmatrix} \ldots..(2.4)$$

Dividing all column elements in Eq. (3.4) by ΔC_L and taking the limit as ΔC_L tends to zero leads to a sensitivity analysis; i.e.,

$$\begin{Bmatrix} \dfrac{d\alpha}{dC_L} \\ \dfrac{d\delta}{dC_L} \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 1/\Delta \begin{Bmatrix} -C_{m\delta} \\ C_{m\alpha} \end{Bmatrix} \ldots(2.5)$$

The controllability term of interest is the slope of the δ vs V curve shown in Fig. 2.3 at the trim velocity. Here

$$\frac{d\delta}{dV} = \frac{d\delta}{dC_L} . \frac{dC_L}{dV} = -\frac{2C_L}{V} . \frac{C_{m\alpha}}{\Delta} \ldots.(2.6)$$

where

$$C_L = -\frac{C_L}{QS} = \frac{C_L}{\frac{\Delta\,\rho V^2}{2} S} = \frac{C_L V^{-2}}{\frac{\rho}{2} S}$$

$$\frac{dC_L}{dV} = \frac{-2C_L V^{-3}}{\frac{\rho}{2} S} = \frac{-2C_L}{\frac{\rho V^2}{2} S.V} = -\frac{2C_L}{V}$$

Conclusions from above example 2.1

The analyses just shown were extended to a velocity range of V from 180 knot to 320 knot with trim at V = $\mathbf{128.611\ m/s^2}$ and for e.g. locations at 0.30c, 0.25c, and 0.20c. Figure 2.3 summarizes the effect of e.g. location on longitudinal control demands when changing airspeed from a trim position for the assumed aircraft in steady flight.

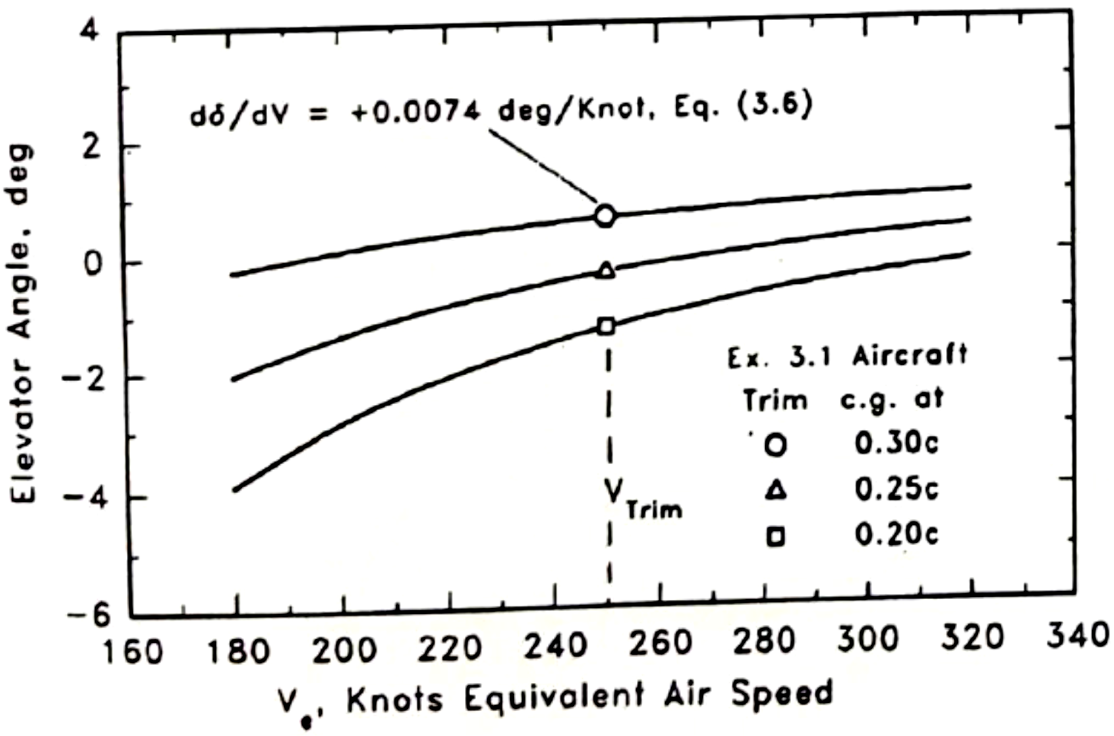


Fig. 2.3 Longitudinal control for airspeed change.

The airspeed changes from level-flight trim shown in Fig. 2.3 will result in the airframe either descending or ascending unless thrust changes are made. Example 2.1 assumed that thrust related derivatives were zero; i.e., $\frac{\partial C_L}{\partial T} = 0$ and $\frac{\partial C_m}{\partial T} = 0$. Also, the matrix [A] in Example 2.1 varied only with respect to the influence of c.g. position on the $C_{m\alpha}$ derivative in order to illustrate first principles.

When Eq. (2.6) is applied to Example 2.1 at the trim velocity of V = $\mathbf{128.611}\ \boldsymbol{m/s^2}$ with a 5% static margin, the $d\delta/dV$ derivative determination provides a slope of +0.0074 deg/kn as noted in Fig. 2.3. A low $d\delta/dV$ gradient may make holding a trim velocity difficult.

Chapter 2 : Static Stability and Control

Lecture 4

Topics

2.1.2. Maneuvering Flight

An important use of the longitudinal control is to provide an aircraft with a maneuvering capability. Both steady, vertical pull-ups and constant altitude banked turns will be considered in this section, subject to the constraint that the aircraft's velocity remains constant while in steady curvilinear flight (i.e., thrust An important use of the longitudinal control is to provide an aircraft with a maneuvering capability. Both steady, vertical pull-ups and constant altitude banked turns will be considered in this section, subject to the constraint that the aircraft's velocity remains constant while in steady curvilinear flight (i.e., thrust may be added to compensate for the increased drag due to the α increase). Stability derivatives due to thrust changes .also are assumed as zero. The assumption of steady flight implies that all transient dynamic aircraft responses, such as might be encountered by an abrupt entry into a pull-up, are neglected. Only the pitch damping term will have a bearing upon the analysis, whereas the influence of terms such as $d\alpha/dt$ and dq/dt will not be present. The constant altitude, turning flight will be presented to show the difference in the elevator per g from that obtained by steady pull-ups. Although either dimensional or dimensionless stability derivatives could be used in the analysis, the dimensionless form will be used initially followed by a conversion to the dimensional form.

2.12.1 pull-up maneuver

Figure 2.4 shows an idealized aircraft in a steady, vertical pull-up maneuver. Equilibrium of the z forces relative to the body axes may be expressed as

$$m(g + Vq) = QS\left[C_{L_\alpha}\alpha + C_{L_\delta}\delta\right] + QS\left[\left(C_{L_\alpha} + C_D\right)\Delta\alpha + C_{L_q}(qc/2V) + C_{L_\delta}\Delta\delta\right] \quad (2.7)$$

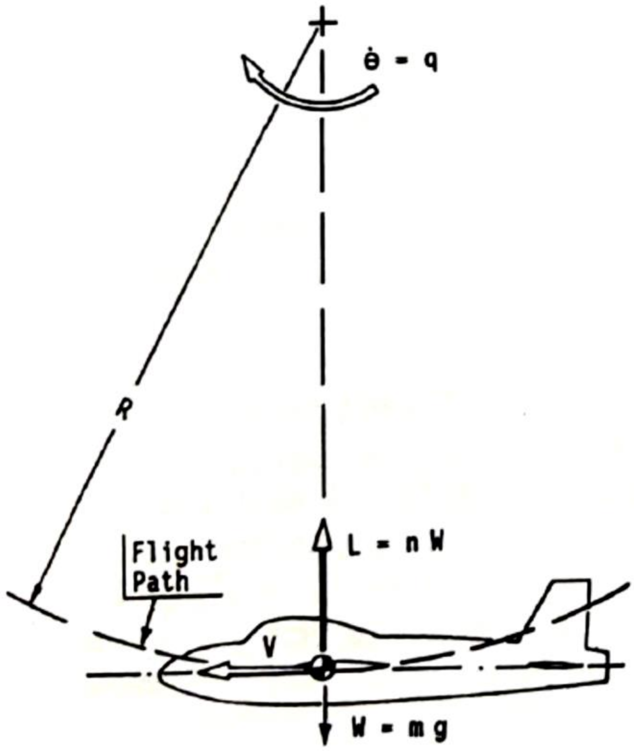


Fig. 2.3 Aircraft in a steady, vertical pull-up maneuver

It will be noted from Fig. 2 .3 that the positive change of normal acceleration Δa_n, which acts in the positive z direction is related to aircraft load factor n by the following statement:

$$\Delta a_n = -(n-1)g = -Vq \quad \ldots(2.8)$$

Because steady, unaccelerated level flight corresponds to

$$W = mg = QS\left[C_{L_\alpha}\alpha + C_{L_\delta}\delta\right]$$

$$0 = QS\left[\left(C_{L_\alpha} + C_D\right)\Delta\alpha + C_{L_q}\frac{c}{2V}q + C_{L_\delta}\Delta\delta\right] - mVq \quad ..(2.9)$$

Dividing through Eq. (2.9) by (-m) converts the equation into dimensional stability derivative form, i.e.,

$$0 = Z_\alpha \Delta\alpha + (V + Z_q)q + Z_\delta \Delta\delta \quad \ldots(2.10)$$

the dimensional stability derivatives are

$$Z_\alpha = -\frac{QS}{m}(C_{L_\alpha} + C_D)$$

$$Z_\delta = -\frac{QS}{m}C_{L_\delta}$$

$$Z_q = -\frac{QS}{m}\frac{c}{2V}C_{L_q}$$

The change in pitching moment for the steady pull-up maneuver is given by

$$0 = C_{m_\alpha}\Delta\alpha + C_{m_q}\frac{qc}{2V} + C_{m_\delta}\Delta\delta \quad \ldots(2.11)$$

Multiplying by QSc/I_y provides the dimensional form with stability derivatives

$$0 = M_\alpha \Delta\alpha + M_q q + M_\delta \Delta\delta \quad \ldots(2.12)$$

where

$$M_\alpha = \frac{QSc}{I_y}C_{m_\alpha}$$

$$M_\delta = \frac{QSc}{I_y}C_{m_\delta}$$

$$M_q = \frac{QSc}{I_y}\frac{c}{2V}C_{m_q}$$

Equations (2.10) and (2.12) can be expressed in matrix form as

$$\begin{bmatrix} Z_\alpha & (V + Z_q) \\ M_\alpha & M_q \end{bmatrix} \begin{Bmatrix} \Delta\alpha \\ q \end{Bmatrix} = -\begin{Bmatrix} Z_\delta \\ M_\delta \end{Bmatrix} \Delta\delta \quad ..(2.13)$$

Because the pitch rate term q is related to normal acceleration by Eq. (2.8), Eq. (2.13) may be re-expressed as

$$\begin{bmatrix} Z_\alpha & -(V + Z_q)/V \\ M_\alpha & -M_q/V \end{bmatrix} \begin{Bmatrix} \Delta\alpha \\ \Delta a_n \end{Bmatrix} = -\begin{Bmatrix} Z_\delta \\ M_\delta \end{Bmatrix} \Delta\delta \quad ....(2.14)$$

When one divides both sides of Eq. (2.14) by $\Delta\delta$ takes $\Delta\delta$ to a zero limit, and premultiplies the resulting equation by the inverse. of the 2 x 2 matrix, an expression for the acceleration sens1t1v1ty due to long1tud1nal control input is obtained, i.e.,

$$\begin{Bmatrix} \frac{d\alpha}{d\delta} \\ \frac{da_n}{d\delta} \end{Bmatrix} = -\begin{bmatrix} Z_\alpha & -(V + Z_q)/V \\ M_\alpha & -M_q/V \end{bmatrix}^{-1} \begin{Bmatrix} Z_\delta \\ M_\delta \end{Bmatrix}$$

which may be readily solved to yield

$$\begin{Bmatrix} \frac{d\alpha}{d\delta} \\ \frac{da_n}{d\delta} \end{Bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -M_q/V & (V + Z_q)/V \\ -M_\alpha & Z_\alpha \end{bmatrix} \begin{Bmatrix} Z_\delta \\ M_\delta \end{Bmatrix} \quad ..(2.15)$$

where the determinant Δ is

$$\Delta = [Z_\alpha M_q - M_\alpha(V + Z_q)]/V$$

From Eq. (2.15), the acceleration in a steady, vertical pull-up will be recognized as

$$\frac{da_n}{d\delta} = \frac{V\,(Z_\alpha M_\delta - Z_\delta M_\alpha)}{[Z_\alpha M_q - M_\alpha(V + Z_q)]} \quad \ldots(2.16)$$

Equation (2.16) may alternatively be expressed in terms of elevator per g using dimensionless stability coefficients. Recognize that $da_n = -gdn$, and assume that $Cv << CL_\alpha$ and $Z_q << V$; then one finds that

$$\frac{d\delta}{dn} = -\frac{C_L C_{m_\alpha} + (gc/2V^2) C_{L_\alpha} C_{m_q}}{C_{L_\alpha} C_{m_\delta} - C_{L_\delta} C_{m_\alpha}} \quad \ldots(2.17)$$

It will be noted in Eq. (3.17) that elevator per g in a pull-up maneuver at constant airspeed is a constant, and the constant will increase in value as the e.g. moves forward (i.e., due to the increase in Cm_α).

Example 2.2

Estimate the normal acceleration sensitivity for the DC-8 aircraft operating at a cruise flight condition, i.e.,

$M = 0.84$ $\quad h = 10058\,m$ $\quad \left(\frac{x}{c}\right)_{cg} = 0.15$

$Z_\alpha = -202.5\,m/s^{-1}$ $\quad M_\alpha = -9.149 s^{-1}$ $\quad Zq = 0.0$

$Z_\delta, = -10.57 m/s^{-1}$ $\quad M_\delta = -4.59\,s^{-2}$ $\quad V = 251.4\,m/s^{-1}$

$M_q = -0.924\,s^{-1}$

Solution

Substituting this information into Eq. (2.16) and solving gives

$$\frac{da_n}{d\delta} = 83.8\frac{m}{s^2}\ .rad^{-1} \ = (0.15g/deg)$$

Note that because plus a_n represents a negative load factor by the sign convention, then $dn/d_\delta = -0.150\ g\ /deg$. Physically this corresponds to a negative load factor being produced by a trailing-edge down (+) elevator control motion.

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2.1.2.2 Finding a maneuver point mp

The M_α term in the determinant Δ of Eq. (2.15) is linearly related to the static margin$\left(\frac{\Delta x}{c}\right)_{np}$. When the c.g. is located at the airframe's neutral point, M_α becomes zero and the steady-state velocity can change without elevator control input (i.e., neutral speed stability). However, location of the c.g. at the neutral point does not result in the airframe's acceleration response from longitudinal control becoming infinite. The information contained in Δ a11ows one to define a <u>maneuver point.</u>

<u>**Definition**</u>: When the e.g. is located at the maneuver point, the airframe has infinite acceleration sensitivity to longitudinal control input commands.

On the assumption that $Z_q << V$, the denominator of Eq. (2.16) may be expressed as

$$Z_\alpha M_q - M_\alpha V = -\frac{QScV}{I_y} C_{L_\alpha}\left[\frac{\rho Sc}{4m} C_{m_q} + \left(\frac{\Delta x}{c}\right)_{np}\right] \quad ..(2.18)$$

The term in the brackets of Eq. (2.18) represents the location of the maneuver point relative to the c.g., i.e.,

$$\left(\frac{\Delta x}{c}\right)_{\text{mp}} = \left[\frac{\rho S c}{4m} C_{m_q} + \left(\frac{\Delta x}{c}\right)_{\text{np}}\right]$$

and the dimensionless distance from the neutral point is given by

$$\left(\frac{\Delta x}{c}\right)_{\text{mp}} - \left(\frac{\Delta x}{c}\right)_{\text{np}} = \frac{\rho S c}{4m} C_{m_q} \quad \ldots(2.19)$$

Because the pitch damping derivative is normally negative in sign, one finds that the maneuver point for an aircraft is located aft of the neutral point.

Example 2.3

Estimate the stability margin between the neutral and maneuver points for the DC-8 aircraft operating at a cruise flight condition, where

$\rho = 0.4107569 kg/m^3$ $C_{m_q} = -14.6$ $S = 241.5479 m^2$

$W = 10432.625\ kg$ $c = 7.0104\ m$

Solution

Substitution of this data into Eq. (2.19) yields

$$\left(\frac{\Delta x}{c}\right)_{\text{mp}} - \left(\frac{\Delta x}{c}\right)_{\text{np}} = -0.0243$$

This result states that an estimate of the maneuver point location at the assumed flight condition for the DC-8 aircraft is 2.43% of the MAC aft of the neutral point.

2.1.2.3 Steady banked flight with turn rate Ω

A sketch of an aircraft in a constant altitude, steady banked flight with turn rate Ω is shown in Fig. 2.4. A typical turn rate for an aircraft under instrument flight conditions is 180 deg/min (i.e., 2-min tum). The turn rate is a vector quantity acting in a vertical direction and may be resolved into the aircraft's body-axis oriented pitch and yaw rates at a bank angle ϕ. In addition, the centrifugal force component experienced as side force by the aircraft during a coordinated turn is related to the lateral component of weight. These three relations may be stated as

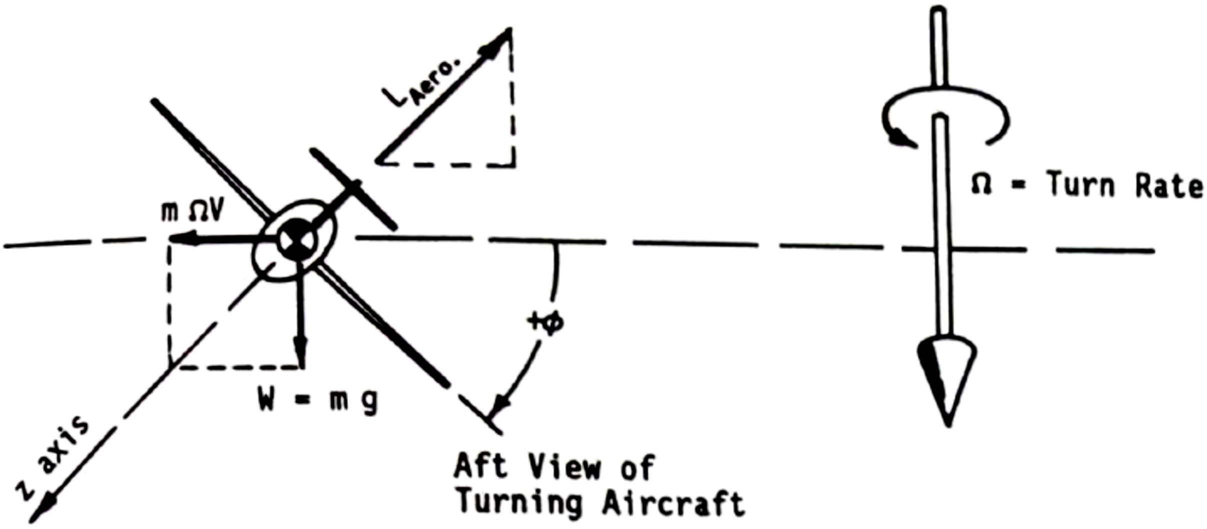

Fig. 2.4 Aircraft in constant altitude, steady turning flight.

$$r = \Omega\cos\phi \ldots\ldots(2.20)$$

$$q = \Omega\sin\phi \ldots\ldots\ldots(2.21)$$

$$mg\sin\phi \;= m\,\Omega V\cos\phi \ldots\ldots(2.22)$$

Applying Eq. (2.22) to Eq. (2.21) yields an expression for the aircraft pitch rate that is induced by the steady turn. Here

$$q = \frac{g}{V}(1-\cos^2\phi)/\cos\phi \;\ldots\ldots(2.23)$$

The aircraft's normal force (z direction) due to aerodynamic terms will be in equilibrium with the inertial components induced by the weight and centrifugal force vectors, i.e.,

$$Z + mg\cos\phi + m\Omega V\sin\phi = 0$$

which simplifies upon use·of Eq. (2.22) to

$$Z = -mg/\cos\phi$$

Load factor n is considered positive acting upwards, which corresponds to a negative Z force. Therefore

$$n = -\frac{Z}{mg} = 1/\cos\phi \ldots\ldots (2.24)$$

A steady turn at a 60-deg bank angle induces a load factor of $n = 2$, an effect that is quite apparent during flight when one extends an arm forward while executing a steeply banked turn. Application of Eq. (2.24) to Eq. (2.23) provides an alternate expression for the turn induced pitch rate in terms of load factor, i.e.,

$$q = \frac{g}{V}n\left(1 - \frac{1}{n^2}\right) = \frac{g}{V}\frac{(n-1)(n+1)}{n} \quad \ldots(2.25)$$

An approach similar to that used in developing Eq. (2.9) can be followed to describe the aerodynamic forces required to maintain equilibrium, i.e.,

$$m\Delta a_n = -m(n-1)g = QS\left[\left(C_{L_\alpha} + C_D\right)\Delta\alpha + C_{L_q}(c/2V)q + C_{L_\delta}\Delta\delta\right]$$

Dividing through by $-m$ and simplifying yields

$$Z_\alpha\Delta\alpha + Z_\delta\Delta\delta = -\left[Z_q\frac{(n+1)}{Vn} + 1\right](n-1)g \quad \ldots(2.26)$$

Substitution of Eq. (2.25) into Eq. (2 .12) corresponds to maintaining aircraft pitching-moment equilibrium during the steady turn, i.e.,

$$M_\alpha \Delta\alpha + M_\delta \Delta\delta = -M_q[(n+1)/Vn](n-1)g \quad \ldots(2.27)$$

Combining Eqs. (2 .26) and (2.27) into matrix format and solving for $\Delta\alpha$ and $\Delta\delta$ provides the following relations:

$$\begin{Bmatrix} \Delta\alpha \\ \Delta\delta \end{Bmatrix} = -\frac{(n-1)}{Vn}\frac{g}{\Delta}\begin{bmatrix} M_\delta & -Z_\delta \\ -M_\alpha & Z_\alpha \end{bmatrix}\begin{Bmatrix} Z_q(n+1)+Vn \\ M_q(n+1) \end{Bmatrix} \ldots(2.28)$$

Where

$$\Delta = Z_\alpha M_\delta - M_\alpha Z_\delta$$

Unlike a vertical pull-up maneuver, the elevator per g in a steady tum is not a linear function of load factor. The elevator required to develop load factor in the turn, from Eq. (2.28), is

$$\Delta\delta = -\frac{(n-1)}{Vn}\frac{g}{\Delta}\{Z_\alpha M_q(n+1) - M_\alpha[Z_q(n+1)+Vn]\} \quad ..(2.29)$$

The preceding relation may be re-expressed in an approximate dimensionless stability coefficient form under the assumptions that both $C_D \ll C_{L_\alpha}$ and $Z_q \ll V$.

Here

$$\Delta\delta = -\frac{(n-1)C_L}{\Delta}\left[C_{m_\alpha} + \frac{(n+1)}{2\mu n}C_{m_q}C_{L_\alpha}\right] \quad \ldots(2.30)$$

Where

C_L =lift coefficient when $\phi = 0$

μ=dimensionless mas coefficient $2m/\rho Sc$

$\Delta = C_{L_\alpha} C_{m_\delta} - C_{L_\delta} C_{m_\alpha}$

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Example 2.4

Estimate the elevator control required in steady banked turns at a 60 degree bank angle for the DC-8 aircraft operating at a cruise-flight condition and Static margin according to

$\rho = 0.4107569 kg/m^3$	$C_{m_q} = -14.6$	$S = 241.5479 m^2$
$W = 104331.8\ kg$	$c = 7.0104\ m$	$C_{L_\alpha} = 6.744\ rad^{-1}$
$C_{m_\alpha} = -2.017\ rad^{-1}$	for $\left(\frac{x}{c}\right)_{cg} = -0.15$	$C_{L_q} = 0.0$
$C_{L_\delta} = 0.352\ rad^{-1}$	$C_{m_\delta} = -1.008\ rad^{-1}$	$C_L = 0.326$

Solution

The dimensionless mass coefficient, which corresponds to the mass of the aircraft normalized with respect to the air contained in a volume defined by the product of $S\ c\ /2$, is given by

$$\mu = \frac{2 * 104331.8\ kg}{0.4107569 \frac{kg}{m^3} * 241.5479 m^2 * 7.0104\ m} = 300[.]$$

The determinant Δ. for the aircraft when $\left(\frac{x}{c}\right)_{cg} = -0.15$ is

$$\Delta = 6.744 * (-1.008) - (-2.017) * 0.352 = -6.088\ \text{rad}^{-1}$$

The load factor for the aircraft at a 60-deg bank angle from Eq. (2.24) is

$$n = 1/\cos\phi = 2$$

The elevator control required to maintain $n = 2.0$ in a steady banked turn may be estimated from Eq. (2.30) as

$$\Delta\delta = -0.212 \text{ rad } (= -6.95 \text{ deg})$$

Control angles for other bank angles may be found using the preceding two calculations as guidelines.

An estimate of control angle requirements for the aircraft with other c.g. locations requires a change in the C_{m_α} stability derivative, which can be determined once the neutral point is established.

The static margin for $\left(\frac{x}{c}\right)_{cg} = -0.15$ can be found from C_{m_α} and C_{L_α} using fallowing Equation.

$$Static\ margin = -\left(\frac{\Delta x}{c}\right)_{np} =_{-} \frac{C_{m_\alpha}}{C_{L_\alpha}} = \frac{2.017\ rad^{-1}}{6.744\ rad^{-1}} = 0.299$$

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(Home Work)

Estimate the elevator control Δδ required in Example 2.4, when the c.g. is located at 0.20c.

$ans = -0.082\ rad.$

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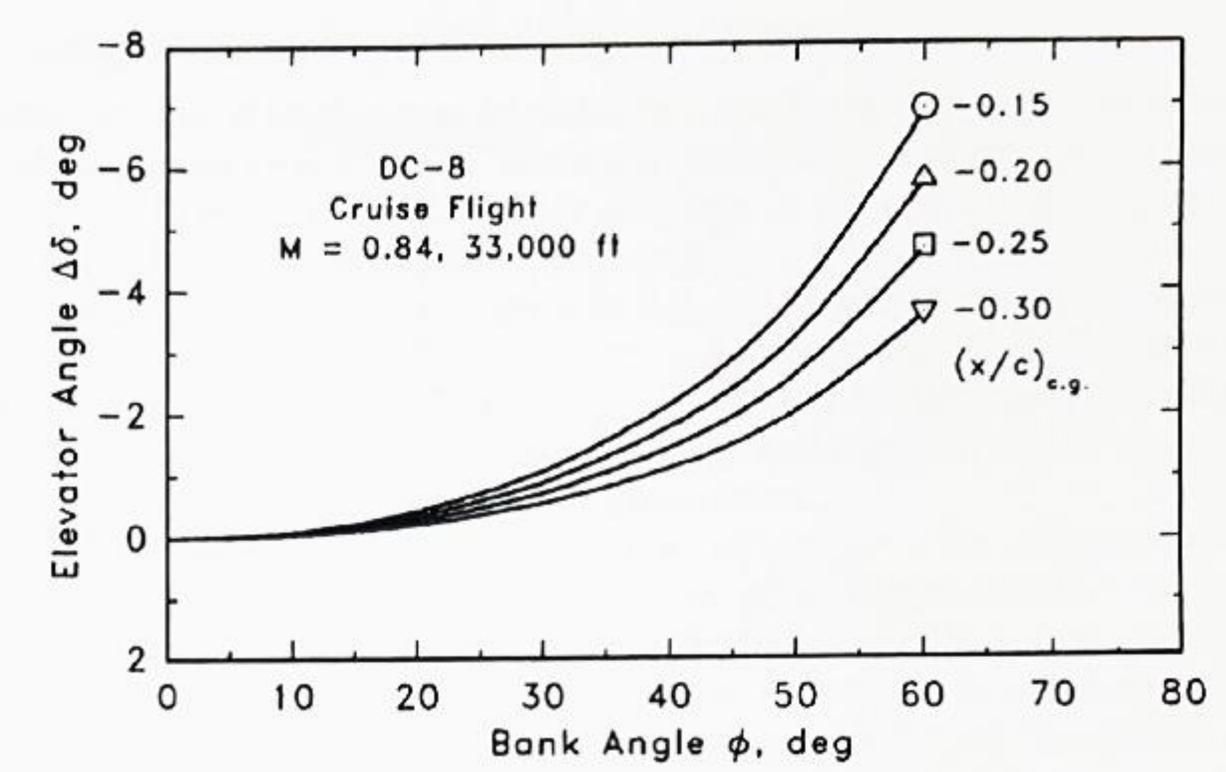


Fig. 2.5 Estimate of elevator control vs bank angle.

The estimated variation of elevator angle as a function of bank angle is shown on Fig. 2 .5 for various c.g. locations. Items to note include that

1) Longitudinal control is not linear with bank angle and ;

2) Moving the c.g. location forward places increased demands on the longitudinal control.

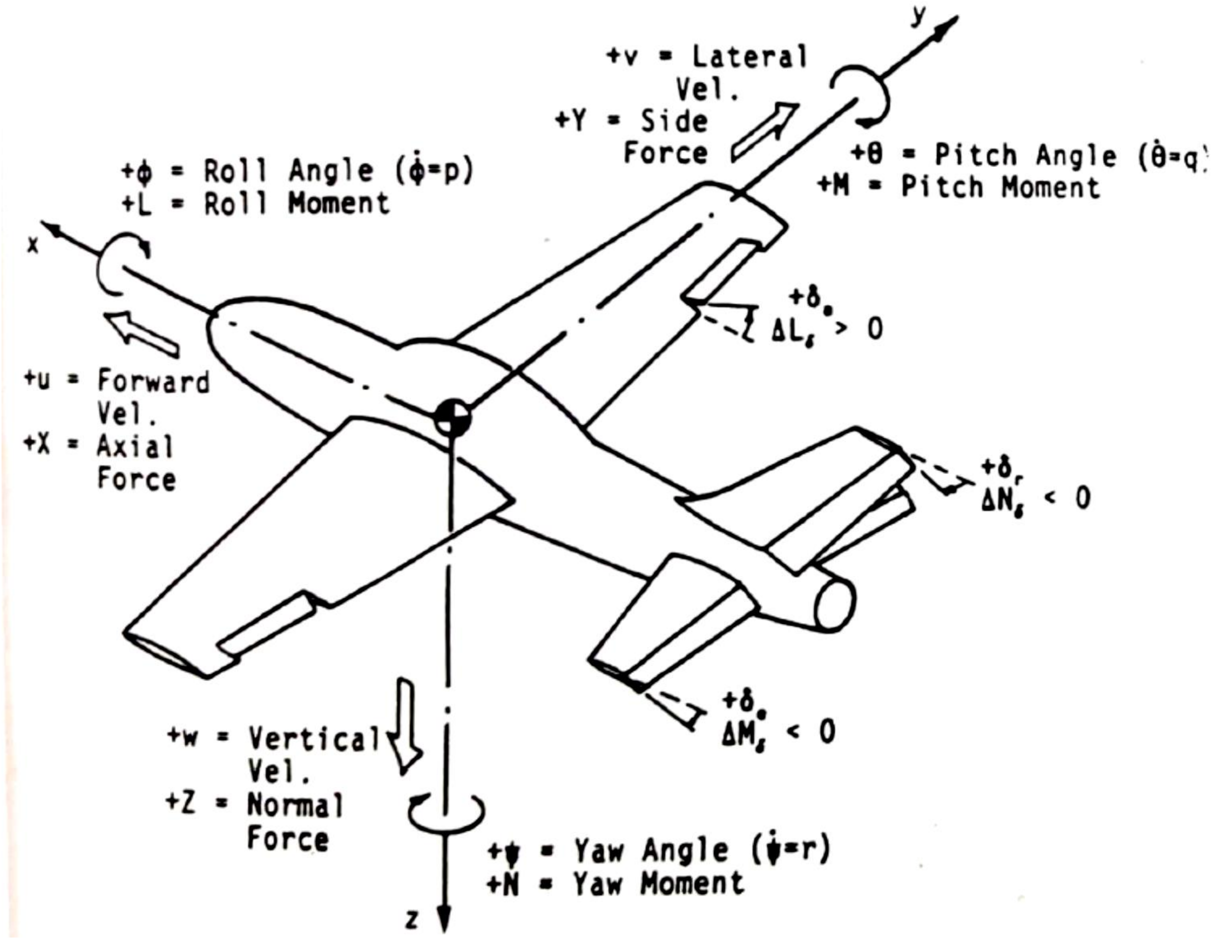
Chapter 2 : Static Stability and Control

Lecture 5

Topics

2.2. Lateral-Directional Stability and Control

From an operational sense, lateral-directional stability traits are related to the aircraft's control requirements when demonstrating straight, steady sideslips. A stable sideslip behavior implies that sideslips to the right (β) require the concurrent application of left-rudder ($+\delta_r$) and right-aileron ($+\delta_a$) control, where the control sign conventions are shown in below figure. Furthermore, It is desirable that the variation of control angles should be approximately linear for sideslip angles between ± 15 deg or $\pm\beta_{max}$. Sideslip capabilities on the order of ±10 deg are associated with maximum crosswind landings and takeoffs, which usually involve the aircraft in a deflected wing flap configuration.



lateral-directional control system design encompasses providing safe aircraft handling qualities by a competent pilot of engine-out thrust asymmetries, crosswind takeoffs and landings, trim of yaw and roll moment asymmetries, and correcting of aircraft transient dynamic responses during gusts, maneuvers, etc. These comments relate to static stability and control considerations for the lateral-directional axes of the aircraft. The following two sections will illustrate the principles involved in estimating the sideslip and engine-out capabilities for a representative aircraft.

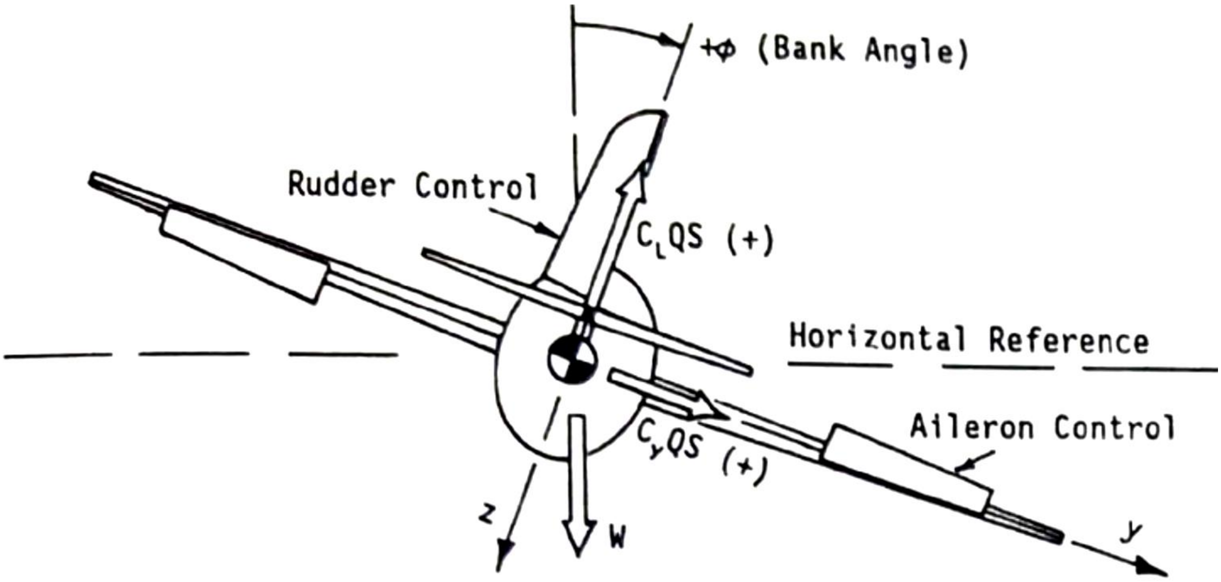


Fig. 2. 7 Rear view of aircraft in a steady bank angle.

2.2.1. Aircraft in Sideslip

An aircraft in a steady sideslip requires concurrent equilibrium in side force and the roll and yaw moments. Linear equations will be considered to illustrate the principles, especially since experience has shown them as reasonable to describe modest sideslips during normal flight situations. The three equilibrium equations can be solved to find the values of lateral and directional control (δ_r and δ_a), and bank angle (ϕ) .for a given sideslip angle (β).

Figure 2.7 shows the rear view of an aircraft in unaccelerated flight at a bank angle (ϕ) . Unlike the α and β angles, bank angle does not produce restorative moments. However, the aerodynamic forces in the y and z body-axis directions will be in balance with the corresponding dead-weight components.

In the y axis direction,

$$W \sin\phi + C_y QS = 0$$

and in the z axis direction,

$$W \cos\phi - C_L Q S = 0$$

These two static force relations provide an equivalent statement that

$$C_y = -C_L \tan\phi$$

The three static equilibrium relations may be stated in a linear form as

$$\begin{aligned} -C_L \tan\Delta\phi &= C_{y_\beta}\Delta\beta + C_{y_{\delta_r}}\Delta\delta_r + C_{y_{\delta_a}}\Delta\delta_a \\ 0 &= C_{\ell_\beta}\Delta\beta + C_{\ell_{\delta_r}}\Delta\delta_r + C_{\ell_{\delta_a}}\Delta\delta_a \\ 0 &= C_{n_\beta}\Delta\beta + C_{n_{\delta_r}}\Delta\delta_r + C_{n_{\delta_a}}\Delta\delta_a \end{aligned} \qquad \ldots(2.31)$$

where the Δ notation has been used with all of the variables to indicate a change from the trimmed, wings-level flight condition.

The teens in Eq. (2.31) may be arranged, along with a small angle assumption on bank angle, to yield

$$\begin{bmatrix} +C_L & C_{y_{\delta_r}} & C_{y_{\delta_a}} \\ 0 & C_{\ell_{\delta_r}} & C_{\ell_{\delta_a}} \\ 0 & C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{bmatrix} \begin{Bmatrix} \Delta\phi \\ \Delta\delta_r \\ \Delta\delta_a \end{Bmatrix} = - \begin{Bmatrix} C_{y_\beta} \\ C_{\ell_\beta} \\ C_{n_\beta} \end{Bmatrix} \Delta\beta \qquad \ldots(2.32)$$

Dividing both sides of Eq. (2.32) by Δβ and letting Δβ tend to a zero limit provides the following **<u>sideslip sensitivity derivatives</u>**:

$$\begin{Bmatrix} \dfrac{d\phi}{d\beta} \\ \dfrac{d\delta_r}{d\beta} \\ \dfrac{d\delta_a}{d\beta} \end{Bmatrix} = -[A]^{-1} \begin{Bmatrix} C_{y_\beta} \\ C_{\ell_\beta} \\ C_{n_\beta} \end{Bmatrix} \qquad \ldots(2.33)$$

Where

$$[A] = \begin{bmatrix} +C_L & C_{y_{\delta_r}} & C_{y_{\delta_a}} \\ 0 & C_{\ell_{\delta_r}} & C_{\ell_{\delta_a}} \\ 0 & C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{bmatrix}$$

The sideslip sensitivity derivatives of Eq. (2.33) may also be expressed in terms of dimensional stability derivatives, which should be evident when each row of Eq. (2.31) is multiplied by (QS/m), $(QSb\,/I_x)$, and $(QSb\,/I_z)$, respectively, i.e.,

$$\begin{Bmatrix} \frac{d\phi}{d\beta} \\ \frac{d\delta_r}{d\beta} \\ \frac{d\delta_a}{d\beta} \end{Bmatrix} = -[B]^{-1} \begin{Bmatrix} Y_\beta \\ L_\beta \\ N_\beta \end{Bmatrix} \quad \ldots(3.34)$$

Where

$$[B] = \begin{bmatrix} g & Y_{\delta_r} & Y_{\delta_a} \\ 0 & L_{\delta_r} & L_{\delta_a} \\ 0 & N_{\delta_r} & N_{\delta_a} \end{bmatrix}$$

Example 2.5

Estimate the sideslip sensitivity derivatives for the Lockheed Jetstar aircraft in the power approach condition described in below

$Y_\beta = -9.53\ m/s^2$ $L_\beta = -3.539\ s^{-2}$ $N_\beta = 1.598\ s^{-2}$

$Y_{\delta_r} = 2.314\ m/s^2$ $Y_{\delta_\alpha} = 0.0\ m/s^2$ $L_{\delta_r} = 0.887\ s^{-2}$

$L_{\delta_\alpha} = 2.148\ s^{-2}$ $N_{\delta_r} = -0.715\ s^{-2}$ $N_{\delta_\alpha} = -0.147\ s^{-2}$

Solution:

The sideslip sensitivity derivatives, according to Eq. (2.34), may be found by premultiplying the sideslip stability derivative column vector by the negative inverse of the [B] matrix. The use of a matrix-oriented computer program is convenient for obtaining the following solution:

$$\begin{Bmatrix} \frac{d\phi}{d\beta} \\ \frac{d\delta_r}{d\beta} \\ \frac{d\delta_a}{d\beta} \end{Bmatrix} = \begin{Bmatrix} 0.483 \\ 2.072 \\ 0.792 \end{Bmatrix} \text{ rad/rad (or deg/deg)}$$

MatLab code

```
B=[9.81  2.314  0 ; 0  0.887  2.148 ; 0  -0.715  -0.147];
A=[-9.53 ;  -3.539 ; 1.598];
ans =B^-1*A;
ans =
   -0.4827
   -2.0722
   -0.7919
```

The results represent stable lateral-directional stability traits as described in Sec. 2.2. If one assumes that the lateral-directional controls behave in an approximate linear manner, the results for the example aircraft in a power approach flight condition indicate that a crosswind landing requiring $\beta = 10\ deg$ would be satisfied when bank angle $\phi = 4.8\ deg$, rudder control $\delta_r = 21$ deg, and aileron control $\delta_a = 7.9\ deg$.

It should be recognized that maximum lateral control deflection is approximately $20\ deg$ and the analysis would indicate that there is an ample reserve of lateral control remaining to handle gust generated bank angle transients when encountering severe crosswind landing conditions.

2.2.2. Aircraft with Thrust Asymmetry

An important design consideration of the rudder control system for multiengined aircraft relates to establishing the minimum-control speed (V_{mc}). Takeoff performance (i.e., field length, climb gradients, etc.) is dependent on both stall speed and minimum-control speed with factors (e.g., 1.1 V_{mc}) applied to these speeds in order to ensure safe flight operation in case of an enginefailure.

The yawing moment due to the indicated thrust asymmetry from an engine failure (Fig. 2.8) may be expressed in dimensionless coefficient form as

$$\Delta C_{n_{eng}} = \frac{T \eta_{eng}}{2QS} = \frac{1}{2}\left(\frac{T}{W}\right)\eta_{eng} C_L \qquad \ldots(2.35)$$

Where

T= thrust force of a single propulsion unit.

η_{eng} = dimensionless spanwise location of the failed engine,$2y_{eng}/b$.

As in Sec. 2.2.1, linear equations will be considered to illustrate the principles. The thrust asymmetry may be included in the yaw moment equilibrium relation Eq. (2.31) when representing an engine-out condition. Propeller-driven aircraft would also introduce a rolling moment term in Eq. (2.31) due to propeller torque induced asymmetries. This roll moment term will be neglected in the subsequent material; consequently, the discussion will be more representative of jet-powered aircraft that are experiencing a thrust asymmetry due to engine failure. Static equilibrium in the lateral-directional equations using the stated assumptions can

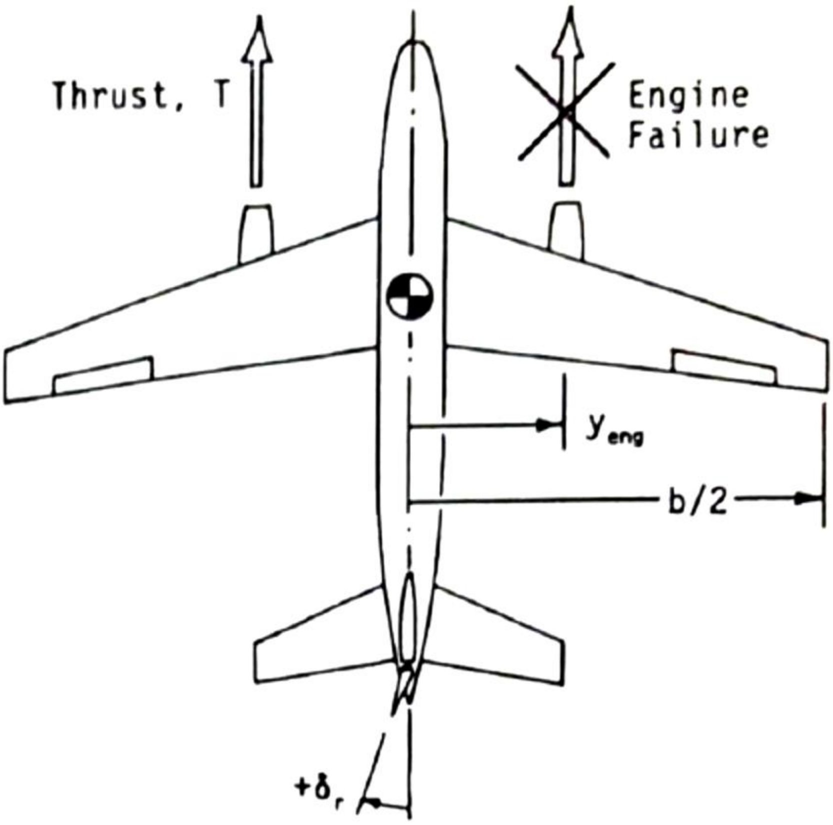


Fig. 2.8 Sketch of aircraft with a thrust asymmetry

be represented as

$$-C_L \tan \Delta\phi = C_{y_\beta}\Delta\beta + C_{y_{\delta_r}}\Delta\delta_r + C_{y_{\delta_a}}\Delta\delta_a$$
$$0 = C_{\ell_\beta}\Delta\beta + C_{\ell_{\delta_r}}\Delta\delta_r + C_{\ell_{\delta_a}}\Delta\delta_a$$
$$-\tfrac{1}{2}(T/W)\eta_{\text{eng}}C_L = C_{n_\beta}\Delta\beta + C_{n_{\delta_r}}\Delta\delta_r + C_{n_{\delta_a}}\Delta\delta_a \quad \text{….(2.36)}$$

or, alternatively, in a matrix format as

$$-\begin{Bmatrix} C_L \tan \Delta\phi \\ 0 \\ \tfrac{1}{2}(T/W)\eta_{\text{eng}}C_L \end{Bmatrix} = \begin{bmatrix} C_{y_\beta} & C_{y_{\delta_r}} & C_{y_{\delta_a}} \\ C_{\ell_\beta} & C_{\ell_{\delta_r}} & C_{\ell_{\delta_a}} \\ C_{n_\beta} & C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{bmatrix} \begin{Bmatrix} \Delta\beta \\ \Delta\delta_r \\ \Delta\delta_a \end{Bmatrix} \quad \text{…(2.37)}$$

The terms in Eq. (2.37) may be rearranged to solve for $\Delta\beta\ , \Delta\delta_r\ and\ \Delta\delta_a$ as a function of C_L and $\Delta\phi$ as will be shown. The small-angle assumption will be used on bank angle such that $\tan \Delta\phi \cong \Delta\phi$. Here

$$\begin{Bmatrix} \Delta\beta \\ \Delta\delta_r \\ \Delta\delta_a \end{Bmatrix} = -[A]^{-1} \begin{Bmatrix} \Delta\phi \\ 0 \\ \frac{1}{2}\left(\frac{T_{eng}}{W}\right)\eta_{eng} \end{Bmatrix} C_L \quad \text{….(2.38)}$$

where the matrix [A] corresponds to

$$[A] = \begin{bmatrix} C_{y\beta} & C_{y\delta_r} & C_{y\delta_a} \\ C_{l\beta} & C_{l\delta_r} & C_{l\delta_a} \\ C_{n\beta} & C_{n\delta_r} & C_{y\delta_a} \end{bmatrix}$$

It will be noted from Eq. (2.38) that when an aircraft lift coefficient is selected, which corresponds to a given weight and airspeed, the solution for $\Delta\beta\ , \Delta\delta_r\ and\ \Delta\delta_a$ will depend on an assumed value for the bank angle change from trim, $\Delta\phi$. This observation implies that a pilot, during an engine-out simulation at a given weight and airspeed, finds $\Delta\beta\ , \Delta\delta_r\ and\ \Delta\delta_a$ to be functions of the selected bank angle.

The minimum-control speed is established by decreasing the airspeed while maintaining the thrust asymmetry. V_{mc} is reached when a 5-deg bank angle occurs concurrently with the application of maximum rudder control while maintaining straight flight. If the selected aircraft weight is such that the stall speed (V_s) is equal to or greater than V_{mc}, unusual aircraft motions can occur due to the loss of lateral-directional control.

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Example 2.6

Estimate the static equilibrium conditions for the Lockheed Jetstar aircraft with a critical engine out in the power approach flight condition (W = 10843kg). Use the dimensionless coefficient data information in making the estimates. Assume that the outboard jet engine on the right-hand side (engine number 4) has failed and the remaining three engines are at an estimated takeoff thrust setting.

$The\ maximum\ takeoff\ weight\ = 18144\ kg$ $\quad S = 50.4m^2$

$Typical\ takeoff\ ground\ acceleration = g/3$ $\quad b = 30.48m$

$C = 1.0154m$ $\quad V = 67.9704m/s$ $\quad y_{eng_4} = 4.572m$

$\rho = 1.225\text{kg/m}^3$ $\quad C_{y\beta} = -0.722$ $\quad C_{l\beta} = -0.087$

$C_{n\beta} = 0.148$ $\quad C_{y\delta_r} = 0.175$ $\quad C_{l\delta_r} = 0.022$ $\quad C_{n\delta_r} = -0.066$

$C_{y\delta_a} = 0.0$ $\quad C_{l\delta_a} = 0.053$ $\quad C_{n\delta_a} = -0.014$ $\quad \Delta\phi = -3.0\ deg$

Solution

$$\eta_{eng} = \frac{2y_{eng}}{b} = 2 \times \frac{4.572}{30.48} = 0.3$$

Thrust $T_{total} = takeoff\ weight\ \times acceleration =\ 18144 \times \frac{9.81}{3}$

$= 59330.9N$

$$T_{for\ each\ eng} = \frac{59330.9N}{4} = 14832.71N$$

$$\frac{1}{2}\left(\frac{T_{eng}}{W}\right)\eta_{eng} = \frac{1}{2} \times \left(\frac{14832.71}{10843 * 9.81}\right) \times 0.3 = 0.0209$$

$$C_L = \frac{W}{QS} = \frac{10843 * 9.81}{0.5 * 1.224 * 67.9704^2 * 50.4} = 0.746$$

$$\begin{Bmatrix} \Delta\beta \\ \Delta\delta_r \\ \Delta\delta_a \end{Bmatrix} = -[A]^{-1} \begin{Bmatrix} \Delta\phi \\ 0 \\ \frac{1}{2}\left(\frac{T_{eng}}{W}\right)\eta_{eng} \end{Bmatrix} C_L$$

$$[A] = \begin{bmatrix} C_{y\beta} & C_{y\delta_r} & C_{y\delta_a} \\ C_{l\beta} & C_{l\delta_r} & C_{l\delta_a} \\ C_{n\beta} & C_{n\delta_r} & C_{y\delta_a} \end{bmatrix} = \begin{bmatrix} -0.722 & 0.175 & 0.000 \\ -0.087 & 0.022 & 0.053 \\ 0.148 & -0.066 & -0.014 \end{bmatrix}$$

$$\begin{Bmatrix} \Delta\beta \\ \Delta\delta_r \\ \Delta\delta_a \end{Bmatrix} = -[A]^{-1} \begin{Bmatrix} -0.0524 \\ 0 \\ 0.0209 \end{Bmatrix} 0.746 = \begin{Bmatrix} 0.0174 \\ 0.2942 \\ -0.0935 \end{Bmatrix} rad = \begin{Bmatrix} 1.00 \\ 16.86 \\ -5.36 \end{Bmatrix} deg$$

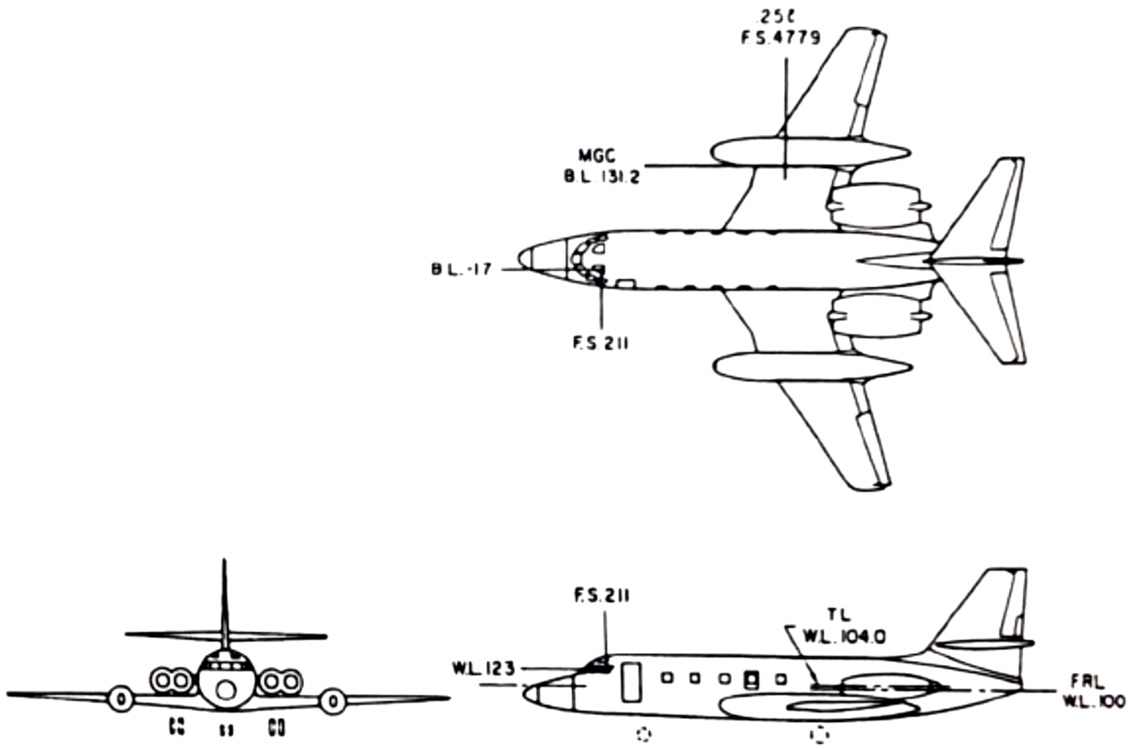

Fig Example 2.6 :The Lockheed Jetstar aircraft

The result states that with the assumed aircraft configuration, weight, and airspeed, an outboard engine out, and the remaining three engines at an estimated takeoff thrust, static equilibrium in straight flight with $\Delta\phi = -3.0\, deg$ (dead-engine high) will occur with a sideslip of $\Delta\beta = +1.0\, deg$ (nose to the left), rudder control of $\Delta\delta_r = 16.86\, deg$ (left-rudder pedal input), and aileron control of $\Delta\delta_a = -5.4\, deg$ (left-control wheel input).

Increasing the negative bank angle from -3.0 to -4.0 deg alters the results by reducing the amount of rudder control needed while allowing the nose to turn slightly into the dead-engine side, i.e., when $\Delta\phi = -4.0\, deg$ $(-0.0698\, rad)$, Eq. (2.38) yields (Home work)

$$\begin{Bmatrix} \Delta\beta \\ \Delta\delta_r \\ \Delta\delta_a \end{Bmatrix} = \begin{Bmatrix} -0.0188 \\ 0.2191 \\ -0.1217 \end{Bmatrix} rad = \begin{Bmatrix} -1.07 \\ 12 \\ -6.97 \end{Bmatrix} deg$$

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Bank angles for the engine-out scenario were considered for an angular range of $-6.0 < \Delta\beta < 0$ deg, and results are shown in Fig. 3.9. In addition, the airspeed was reduced by 10 KEAS corresponding to $C_L = 0.873$, and similar static equilibrium results are also shown in Fig. 2.9.

All data are based on linearized stability a derivative coefficient, which implies that control deflections greater than about $\pm 20\, deg$ are not valid. When engine-out estimates are made using best available airframe stability and control data (e.g., wind-tunnel test data), iterative procedures are frequently required to solve the nonlinear control system traits

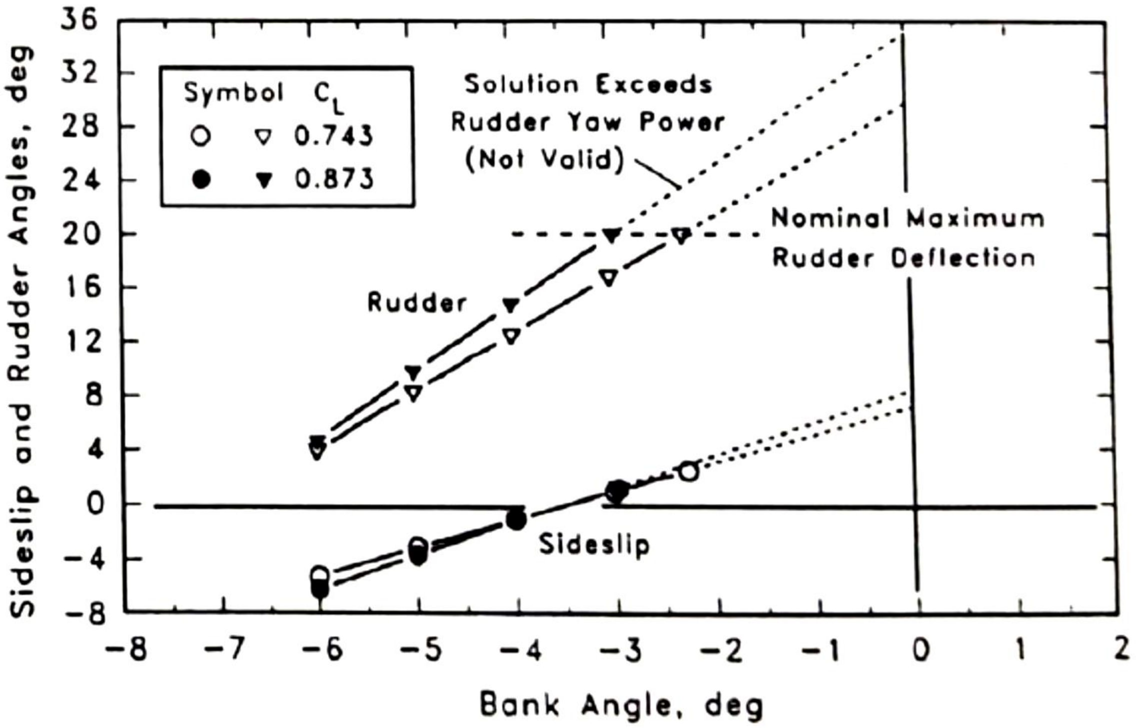


Fig. 2.9 Engine-out results for Example 2.6.

Figure 2.9 shows the tradeoff between rudder control and sideslip angle to establish static equ1hbnum for the assumed engine-out condition in Example 2.6. It is important to note that static equilibrium with an assumed nominal maximum rudder control can be maintained while speed is decreased by increasing the bank angle. A $10 - kn$ airspeed decrease altered bank angle from $\Delta\beta = -2.3\ to\ -3.0\ deg$ while applying full rudder control. At no time during the estimation process did the demands on the lateral control system near a limiting control situation.

The minimum-control speed determination for an aircraft is normally conducted in a takeoff configuration at the most critical weight and c.g. conditions. Although Fig. 2.9 would indicate that increasing the bank angle during a demonstration reduces the demands on directional control, certification requirements define the limit bank angle as being $5\ deg$. The rationale for the selection of $5\ deg$ as a bank angle limit during V_{mc} (minimum control speed) demonstrations appears arbitrary; however, its use does provide a reasonable ground clearance limit for low-wing, multi engined aircraft. The use of $1.1V_{mc}$ as a takeoff speed at lighter airplane weights provides the flight operation with an added safety margin, which time has proven to be prudent.

Chapter 3 : Equations of Motion

Lecture 6

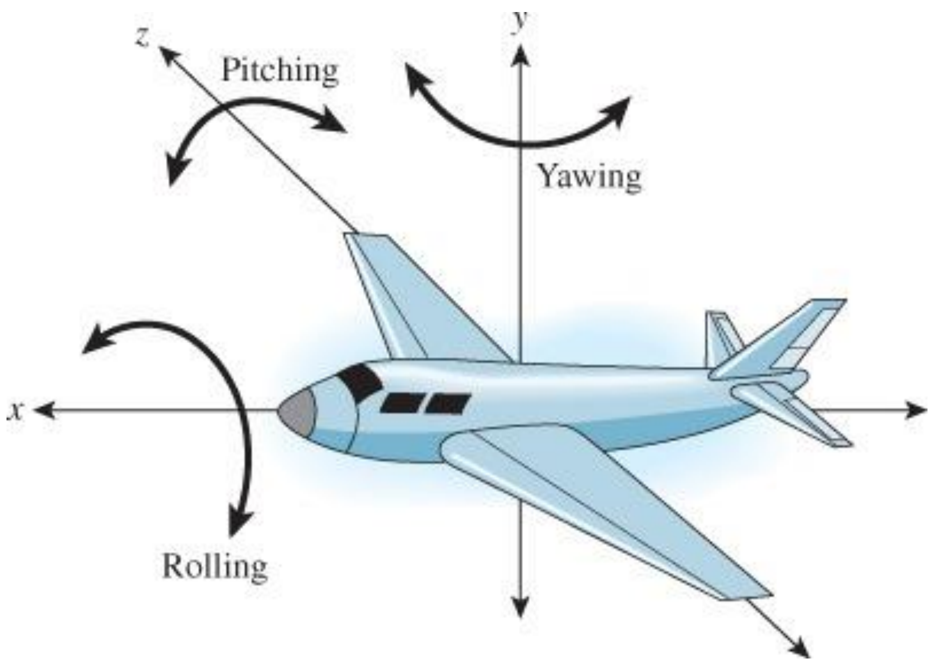
Topics

3.1 Airframe Equations of Motion

3.2 Linearized Equations of Motion

3.1. Airframe Equations of Motion

The full set of nonlinear equations describing the motion for an airframe having rotation in a body-fixed coordinate system. The relations between external forces and moments and the corresponding momentum conservation laws are given by.



$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = m \begin{Bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \\ \frac{dw}{dt} \end{Bmatrix} + m \begin{Bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{Bmatrix} \ldots\ldots(3.1)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} I_x\dot{p} - I_{xz}\dot{r} \\ I_y\dot{q} \\ I_z\dot{r} - I_{xz}\dot{p} \end{Bmatrix} + \begin{Bmatrix} qr(I_z - I_y) - I_{xz}pq \\ pr(I_x - I_z) + I_{xz}(p^2 - r^2) \\ pq(I_y - I_x) + I_{xz}qr \end{Bmatrix} \ldots\ldots(3.2)$$

3.2. Linearized Equations of Motion

The aircraft can change its flight configuration quite rapidly (i.e., transition from climb to cruise, decelerate from descent to approach, deflect wing flaps and extend landing gear), the linearization process will be considered relative to a fixed flight condition. The subsequent linearized equations of motion will allow a determination of an aircrame's small amplitude dynamic behavior about a trim point. Important assumptions for the perturbation analysis include the following.

1. Small motion dynamics will be initiated from wings-level flight with the aircraft controls and thrust set for a trim, static equilibrium flight condition.
2. The body axes will be aligned with the initial velocity vector, and as such, will become known as the airframe's stability axes.

The forces acting on the airframe will be due both to the aerodynamic and gravitational terms and may be expressed as

$$\begin{aligned} F_x &= X_o + \Delta X(t) - mg(\sin\Theta) \\ F_y &= Y_o + \Delta Y(t) + mg(\cos\Theta)(\sin\Phi) \qquad (3.3) \\ F_z &= Z_o + \Delta Z(t) + mg(\cos\Theta)(\cos\Phi) \end{aligned}$$

mg: related terms are due to body-weight components.

Φ and Θ : Euler angles.

X_o : The equilibrium value of the corresponding force component.

$\Delta X\,(t)$: The force term due to the time-varying aerodynamic forces caused by aircraft attitude, motion, and control deflections from trim

The moments acting on the aircraft relative to the e.g. and oriented in stability axes will follow the L, M, and N . The moment components are defined as

$$M_x = L = L_o + \Delta L(t)$$
$$M_y = M = M_o + \Delta M(t) \qquad (3.4)$$
$$M_z = N = N_o + \Delta N(t)$$

o subscript refers to initial trim conditions.

$\Delta L(t)$: type of term will correspond to the time-varying aerodynamic moments that arise due to aircraft attitude, motion, and control deflections from trim.

Before linearization, the set of six coupled equations of motion as contained in Eqs. (3.1 - 3.4) will appear as

$$X_o + \Delta X(t) - mg(\sin\Theta) = m(\dot{u} + qw - rv)$$
$$Y_o + \Delta Y(t) + mg(\cos\Theta)(\sin\Phi) = m(\dot{v} + ru - pw)$$
$$Z_o + \Delta Z(t) + mg(\cos\Theta)(\cos\Phi) = m(\dot{w} + pv - qu)$$
$$L_o + \Delta L(t) = I_x\dot{p} - I_{xz}\dot{r} + qr\left(I_z - I_y\right) - I_{xz}pq \qquad (3.5)$$
$$M_o + \Delta M(t) = I_y\dot{q} + pr(I_x - I_z) + I_{xz}(p^2 - r^2)$$
$$N_o + \Delta N(t) = I_z\dot{r} - I_{xz}\dot{p} + pq\left(I_y - I_x\right) + I_{xz}qr$$

The force and moment statements, linearization of the timedependent variables will start with a definition of an initial equilibrium value plus a small perturbation term

$$\begin{aligned} u &= u_o + u(t); & p &= p_o + p(t) = p(t); \\ v &= v_o + v(t) = v(t); & q &= q_o + q(t) = q(t); \\ w &= w_o + w(t) = w(t); & r &= r_o + r(t) = r(t); \\ \Theta &= \Theta_o + \Theta(t); & \Phi &= \Phi_o + \Phi(t) = \Phi(t); \end{aligned} \qquad (3.6)$$

It should be noted in Eq. (3.6) that the initial condition assumption of the motion starting from wings-level flight with body axes corresponding to the airframe stability axes allows the elimination of terms such as v_o, w_o, p_o, q_o, r_o, *and* Φ_o. In addition, $u_o = V_o$ by the assumptions.

At time $t=0$, when the small perturbation motions are to begin, the relations of Eq. (3.5) simplify to the static equilibrium conditions on the airframe force and moments, i.e.,

$$\begin{aligned} X_o - mg(\sin\Theta_o) &= 0 \\ Y_o = 0 \quad (since\ \Phi_0 &= 0) \\ Z_o + mg(\cos\Theta_o) &= 0 \\ L_o = M_o = N_0 &= 0 \end{aligned} \qquad (3.7)$$

The force relations involve trigonometric terms that can be linearized using the small perturbation angle assumptions, i.e.,

By using trigonometric relationships

$$\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$$

$$\cos(u+v) = \cos(u)\cos(v) - \mathrm{sn}(u)\sin(v)$$

$$\sin\Theta = \sin\Theta_o + (\cos\Theta_o)\ \theta(t) \quad for\ \theta(t) \ll 1$$
$$\cos\Theta = \cos\Theta_o - (\sin\Theta_o)\,\theta(t) \qquad (3.8)$$
$$\sin\Phi = \phi(t)\ \ for\ \Phi_o = 0 \quad and\ \ \phi(t) \ll 1$$
$$\cos\Phi = 1$$

As an example of the linearization process, consider the X force relation of Eq. (3.5). Subtracting the initial condition for X_0 from Eq. (3.7) along with use of the identity concerning $\cos(\Theta_o + \theta(t))$ leaves

$$\Delta X(t)\ - mg\,(\cos\Theta_o)\,\theta(t) = m(\dot{u} + q(t)w(t) - r(t)v(t)) \qquad (3.9)$$

The higher-order nonlinear terms such as $q(t)w(t)$are also dropped during the linearization process based on order of magnitude considerations. The end result for the three linearized force equations is

$$\Delta X(t)\ - mg\,(\cos\Theta_o)\,\theta(t) = m\dot{u} = mV_o(\dot{u}/V_o)$$
$$\Delta Y(t) + mg(\cos\Theta_o)\phi(t)\ = m(\dot{v} + rV_o) = mV_o(\dot{\beta} + r) \qquad (3.10)$$
$$\Delta Z(t) - mg(\sin\Theta_o)\theta(t) = m(\dot{w} - qV_o) = mV_o(\dot{\alpha} - q)$$

Where:

$\alpha(t)$ = angle of attack perturbation, $w(t)/V_o$.

$\beta(t)$ = sideslip angle perturbation, $v(t)/V_o$.

The linearized roll., pitch, and yawing moment equations are obtained more rapidly for the following reasons.

1) The initial conditions on L_o, M_o, *and* N_o are all zero due to the airframe being initially at trim.

2) Higher-order terms such as $p(t)r(t)$ *and* $p(t)^2$ are dropped by order of magnitude considerations.

The linearized, small perturbation equations of motion are:

$$\Delta X(t) - mg(\cos\Theta_o)\,\theta(t) = mV_o(\dot{u}/V_o)$$
$$\Delta Y(t) + mg(\cos\Theta_o)\phi(t) = mV_o(\dot{\beta} + r)$$
$$\Delta Z(t) - mg(\sin\Theta_o)\theta(t) = mV_o(\dot{\alpha} - q) \qquad (3.11)$$
$$\Delta L(t) = I_x\dot{p} - I_{xz}\dot{r}$$
$$\Delta M(t) = I_y\dot{q}$$
$$\Delta N(t) = I_z\dot{r} - I_{xz}\dot{p}$$

The aerodynamic force and moment terms remaining in Eq. (3.11) will be established using Taylor series expansions of the perturbation variables as typically indicated by

$$\Delta X(t) = \Delta X[u/V_o, \quad \alpha, \quad \dot{\alpha}, \quad q, \quad \delta, \quad \beta, \quad \dot{\beta}, \quad p, \quad r, \ldots]$$

Chapter 3 : Airframe Equations Of Motion

Lecture 7

Topics

3.3 Longitudinal Dynamics

3.4 Lateral-Directional Dynamics

3.3. Longitudinal Dynamics

The partial derivatives used in the Taylor series expansions of $\Delta X(t)$, $\Delta Z(t)$, and $\Delta M\ (t)$ will include terms such as

$$\frac{u(t)}{V}, \qquad \alpha(t) = \frac{w(t)}{V}, \qquad \dot{\alpha} = \frac{d\alpha}{dt}, \qquad q = \frac{d\theta}{dt}, \quad and \qquad \delta(t)$$

The subscript o on the initial freestream velocity will be dropped in the following presentations because it should be clear that the use of stability axes for the onset of the motion implies

$$Velocity = [u_o^2 + v_o^2 + w_o^2]^{\frac{1}{2}} = V_0 = V$$

The axial velocity perturbation $u(t)$ is normalized relative to the freestream velocity V so that the perturbation derivative has a similar order of magnitude to the $\alpha(t)$ derivative when making the Taylor series expansions. The $\alpha(t)$ perturbation in the following derivations is the same as the $\Delta\alpha(t)$ term described in the previous chapter (i.e., the Δ prefix is being dropped for the sake of notational simplicity). The expansion for the aerodynamic axial force may be expressed as

$$\Delta X(t) = \Delta X_u[u(t)/V] + \Delta X_\alpha \alpha(t) + \Delta X_{\dot{\alpha}} \dot{\alpha} + \Delta X_q q(t) + \Delta X_\delta \delta(t) \qquad (3.12)$$

Where

ΔX_u = change of ΔX per unit $\left(\frac{u}{V}\right), \frac{\partial \Delta X}{\partial\left(\frac{u}{V}\right)}$, N

ΔX_α = change of ΔX per unit $\alpha, \frac{\partial \Delta X}{\partial \alpha}$, $N. rad^{-1}$

$\Delta X_{\dot{\alpha}}$ = change of ΔX per unit $\dot{\alpha}, \frac{\partial \Delta X}{\partial \dot{\alpha}}$, $N. s. rad^{-1}$

ΔX_q = change of ΔX per unit $q, \frac{\partial \Delta X}{\partial q}$, $N. s. rad^{-1}$

ΔX_δ = change of ΔX per unit $\delta, \frac{\partial \Delta X}{\partial \delta}$, $N. rad^{-1}$

The $\Delta X_{\dot{\alpha}}$ and ΔX_q partial derivative terms are frequently neglected because of their small value relative to the other terms in the expansion. The ΔX_α term has a strong dependence on the aircraft lift coefficient CL. Eq. (3.12) becomes

$$\Delta X(t) = \Delta X_u[u(t)/V] + \Delta X_\alpha \alpha(t) + \Delta X_\delta \delta(t) \qquad (3.13)$$

The expansion for the aerodynamic normal force may be expressed as

$$\Delta Z(t) = \Delta Z_u[u(t)/V] + \Delta Z_\alpha \alpha(t) + \Delta Z_{\dot{\alpha}} \dot{\alpha} + \Delta Z_q q(t) + \Delta Z_\delta \delta(t) \qquad (3.14)$$

where, typically $\Delta Z_\alpha = \frac{\partial \Delta Z}{\partial \alpha}$ is the change of ΔZ_α force per unit α, $N. rad^{-1}$

The ΔZ_α term has a strong dependence on the aircraft lift-curve slope. The $\Delta Z_{\dot{\alpha}}$ and ΔZ_q terms in Eq, (3.14) will occasionally be given in listings of aircraft derivatives, but will not be dominant 1n their influence on airframe dynamics.

Similarly, the expansion of the pitching moment may be expressed as

$$\Delta M(t) = \Delta M_u[u(t)/V] + \Delta M_\alpha \alpha(t) + \Delta M_{\dot{\alpha}} \dot{\alpha} + \Delta M_q q(t) + \Delta M_\delta \delta(t) \qquad (3.15)$$

The preceding force and moment perturbation terms relate to a change in a force or moment due to a small change in a physical quantity. It is convenient to simplify the expressions of Eqs. (3.13-3.15) by using the following definitions of dimensional stability derivatives. Typically, let us define for subscript $i\ = \alpha, \dot{\alpha}, q, and\ \delta$

$$X_i = \Delta X_i/m$$

$$Z_i = \Delta Z_i/m$$

$$M_i = \Delta M_i/I_y$$

for subscript u

$$X_u = \Delta X_u/(mV)$$

$$Z_u = \Delta Z_u/(mV)$$

$$M_u = \Delta M_u/(I_yV)$$

After making these substitutions, the longitudinal portions of Eq. (3.11) take the form of dimensional stability derivatives for the coefficients, i.e.,

$$V\left(\frac{\dot{u}}{V_o}\right) = VX_u\left(\frac{u}{V}\right) + X_\alpha\alpha - g\cos\Theta_o\ \theta + X_\delta\delta$$

$$(V - Z_{\dot{\alpha}})\ \dot{\alpha} = VZ_u\left(\frac{u}{V}\right) + Z_\alpha\alpha + \left(V + Z_q\right)q - g\sin\Theta_o\,\theta + Z_\delta\delta \qquad (3.16)$$

$$-M_{\dot{\alpha}}\dot{\alpha} + \dot{q} = VM_u\left(\frac{u}{V}\right) + M_\alpha\alpha + M_q q + M_\delta\delta$$

To complete these expressions into a state vector format, also consider the identity relation

$$\dot{\theta} = q$$

If we define a longitudinal state vector by

$$\{x\} = \left[\frac{u}{V} \quad \alpha \quad q \quad \theta \right]^T$$

along with a single control term by δ, then the coupled set of four, first-order, ordinary differential equations that describe the linearized airframe longitudinal dynamics become

$$[I_n]\{\dot{x}\} = [A_n]\{x\} + \{B_n\}\delta \qquad (3.17)$$

Where

$$[I_n] = \begin{bmatrix} V & 0 & 0 & 0 \\ 0 & (V - Z_{\dot{\alpha}}) & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = inertial\ matrix$$

$$[A_n] = \begin{bmatrix} VX_u & X_\alpha & 0 & -g\cos\Theta_o \\ VZ_u & Z_\alpha & (V + Z_q) & -g\sin\Theta_o \\ VM_u & M_\alpha & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

And

$$[B_n] = [X_\delta \quad Z_\delta \quad M_\delta \quad 0]^T$$

Because the inertial matrix $[\mathrm{I_n}]$ is nonsingular, premultiplication of both sides of Eq. (3.17) by $[I_n]^{-1}$ will yield the recognized form of the governing state equations as

$$\{\dot{x}\} = [A\,]\{x\} + [B]\delta \qquad (3.17)$$

$[\mathrm{A}\,] =$ longitudinal airframe plant matrix, $[\mathrm{I_n}]^{-1}[\mathrm{A_n}]$

$[\mathrm{B}\,] =$ longitudinal airframe control matrix, $[\mathrm{I_n}]^{-1}[\mathrm{B_n}]$

In Eq. (317) both the plant matrix and the control vector have been altered so that the time derivative of the state vector $\{x\}$ is presented in an uncoupled form. It may be shown that the plant matrix $[A]$ and control vector $\{B\}$ become

$$[A]=\begin{bmatrix} X_u & X_\alpha/V & 0 & -(g/V)\cos\Theta_o \\ VZ_u/(V-Z_{\dot\alpha}) & Z_\alpha/(V-Z_{\dot\alpha}) & (V+Z_q)/(V-Z_{\dot\alpha}) & -g\sin\Theta_o/(V-Z_{\dot\alpha}) \\ VM_u+M_{\dot\alpha}VZ_u/(V-Z_{\dot\alpha}) & M_\alpha+M_{\dot\alpha}Z_\alpha/(V-Z_{\dot\alpha}) & M_q+M_{\dot\alpha}(V+Z_q)/(V-Z_{\dot\alpha}) & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$[B]=\left[X_\delta/V \quad Z_\delta/\left(V-Z_{\dot\alpha}\right) \quad M_\delta+M_{\dot\alpha}Z_\delta/\left(V-Z_{\dot\alpha}\right) \quad 0\right]^T$$

3.4. Lateral-Directional Dynamics

The partial derivatives used in the Taylor series expansions of $\Delta Y(t), \Delta L(t) and\ \Delta N(t)$ will include terms such as

$$\beta(t) = \frac{v(t)}{V}, \qquad p = \frac{d\phi}{dt} \qquad and \qquad r = \frac{d\psi}{dt}$$

as well as the control deflection term δ, which could represent either a lateral control, such as provided by ailerons and/or spoilers, or a directional control, such as a rudder input. Unlike the case of longitudinal dynamics where both a and a terms were considered, $\dot{\beta}(t) = d\beta/dt$ influences will be neglected under the assumption of quasi-steady aerodynamics.

There will be no aerodynamic perturbation terms due to bank angle ϕ inasmuch as bank angle by itself does not result in an aerodynamic force or moment. In addition, there are no aerodynamic perturbation terms developed due to heading angle change ψ.

The side-force aerodynamic terms may be expressed in a small perturbation form by

$$\Delta Y(t) = \Delta Y_\beta \beta(t) + \Delta Y_p p(t) + \Delta Y_r r(t) + \Delta Y_\delta \delta(t) \qquad (3.18)$$

where

$\Delta Y_\beta = \Delta Y$ force change per unit $\beta, \dfrac{\partial \Delta Y}{\partial \Delta \beta}, \mathrm{N.rad^{-1}}$

$\Delta Y_p = \Delta Y$ force change per unit $p, \dfrac{\partial \Delta Y}{\partial \Delta p}, \mathrm{N.s.rad^{-1}}$

$\Delta Y_r = \Delta Y$ force change per unit $r, \dfrac{\partial \Delta Y}{\partial \Delta r}, \mathrm{N.s.rad^{-1}}$

$\Delta Y_\delta = \Delta Y$ force change per unit $\delta, \dfrac{\partial \Delta Y}{\partial \Delta \delta}, \mathrm{N.rad^{-1}}$

The roll moment aerodynamic terms may be stated as

$$\Delta L(t) = \Delta L_\beta \beta(t) + \Delta L_p p(t) + \Delta L_r r(t) + \Delta L_\delta \delta(t) \tag{3.19}$$

The yaw moment aerodynamic expansion is similarly stated as

$$\Delta N(t) = \Delta N_\beta \beta(t) + \Delta N_p p(t) + \Delta N_r r(t) + \Delta N_\delta \delta(t) \tag{3.20}$$

As was done with the longitudinal stability derivatives, dimensional stability derivatives are defined for the lateral-directional aerodynamic perturbation terms by typical relations of

$$Y_i = \Delta Y_i / m$$

$$L_i = \Delta L_i / I_x$$

$$N_i = \Delta N_i / I_z$$

for subscript $i = \beta, p, r, or\ \delta$. In this manner, the lateral-directional portions of Eq. (3.11) become

$$\begin{aligned} V\dot{\beta} &= Y_\beta \beta + Y_p p + g \cos \Theta_o \phi + (Y_r - V) r + Y_\delta \delta \\ \dot{p} - (I_{xz}/I_x)\, \dot{r} &= L_\beta \beta + L_p p + 0 + L_r r + L_\delta \delta \\ -(I_{xz}/I_z)\, \dot{p} + \dot{r} &= N_\beta \beta + N_p p + 0 + N_r r + N_\delta \delta \end{aligned} \tag{3.21}$$

And as in the longitudinal equation formulation, the state vector format requires the following identity statement:

$$\dot{\phi} = p$$

If we define a lateral-directional state vector by

$$\{x\} = [\beta \quad p \quad \phi \quad r]^T \tag{3.22}$$

and a control term δ, which could represent either aileron (δ_a) or rudder (δ_r) deflection, then the coupled set of four, first-order, ordinary differential equations becomes, in matrix notation,

$$[I_n]\{\dot{x}\} = [A_n]\{x\} + \{B_n\}\delta \tag{3.23}$$

Where

$$[I_n] = \begin{bmatrix} V & 0 & 0 & 0 \\ 0 & 1 & 0 & -(I_{xz}/I_x) \\ 0 & 0 & 1 & 0 \\ 0 & -(I_{xz}/I_z) & 0 & 1 \end{bmatrix}$$

$$[A_n] = \begin{bmatrix} Y_\beta & Y_p & g\cos\Theta_o & (Y_r - V) \\ L_\beta & L_p & 0 & L_r \\ 0 & 1 & 0 & 0 \\ N_\beta & N_p & 0 & N_r \end{bmatrix}$$

And

$$[B_n] = [Y_\delta \quad L_\delta \quad 0 \quad N_\delta]^T$$

The matrix relation, Eq. (3.23), may be premultiplied by $[\mathrm{I_n}]^{-1}$ to obtain a standard form for the governing state equation, i.e.,

$$\{\dot{x}\} = [A\,]\{x\} + [B]\delta \tag{3.24}$$

$[\mathrm{A}\,] =$ lateral − directional airframe plant matrix, $[\mathrm{I_n}]^{-1}[\mathrm{A_n}]$

$[\mathrm{B}\,] =$ lateral − directional airframe control vector, $[\mathrm{I_n}]^{-1}[\mathrm{B_n}]$

Equation (3.24) corresponds to a single-input/multiple-output (SIMO) system.

The inertial matrix in Eq. (3.23) contains off-diagonal elements such as (I_{xz}/I_x) and (I_{xz}/I_z) due to the cross product of inertia terms. If the body axes were the principal axes for the aircraft, these terms would vanish. The alternate form for the governing state equation Eq. (3.24) shows the time derivative of the state vector as uncoupled, which corresponds to being premultiplied by a unit diagonal matrix. The matrix of the plant and the control vector become

$$[\boldsymbol{A}] = \begin{bmatrix} Y_\beta/V & Y_p/V & (g/V)\cos\Theta_o & (Y_r - V)/V \\ L'_\beta & L'_p & 0 & L'_r \\ 0 & 1 & 0 & 0 \\ N'_\beta & N'_p & 0 & N'_r \end{bmatrix}$$

where

$$L'_\beta = G\left[L_\beta + N_\beta(I_{xz}/I_x)\right], \quad L'_p = G\left[L_p + N_p(I_{xz}/I_x)\right], \quad L'_r = G[L_r + N_r(I_{xz}/I_x)],$$

$$N'_\beta = G\left[N_\beta + L_\beta(I_{xz}/I_z)\right], \quad N'_p = G\left[N_p + L_p(I_{xz}/I_z)\right], \quad N'_r = G[N_r + L_r(I_{xz}/I_z)],$$

$$G = 1/[1 - (I_{xz})^2/(I_x\, I_z)],$$

and

$$[\boldsymbol{B}\,] = [Y_\delta/V \quad L'_\delta \quad 0 \quad N'_r]^T$$

where

$$L'_\delta = G[L_\delta + N_\delta(I_{xz}/I_x)], \quad N'_\delta = G[N_\delta + L_\delta(I_{xz}/I_z)], \quad G = 1/[1 - (I_{xz})^2/(I_x\, I_z)],$$

Chapter 4: Dynamic Systems Response

Lecture 8

Topics

4.1. System Response

4.2. Response of First-Order Systems

4.3. Response of Second-Order Systems

4.1. System Response

The system response depends on the order of the system. The order of the system refers to the order of the differential equation representing the physical system or degree the denominator of the corresponding transfer function. For example,

$$m\ddot{x} + c\dot{x} + kx = u(t)$$

Is the second-order differential equation.

If

$$G_1(s) = \frac{k}{(s+a)}$$

$$G_2(s) = \frac{k}{s^2 + as + b}$$

Then $G_1(s)$ Is a first order and $G_2(s)$ is a second order system.

Generally, the output or response of a system consists of two parts: 1) the natural or free response and 2) forced response, In the following we discuss the unit-step response of typical first- and second-order systems

4.2. Response of First-Order Systems

We now discuss first-order systems without zeros to define a performance specification for such a system. A first-order system without zeros can be described by the transfer function shown in Figure 4.1(a). If the input is a unit step, where $R(s) = \frac{1}{s}$, the Laplace transform of the step response is $C(s)$, where

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)} \tag{4.1}$$

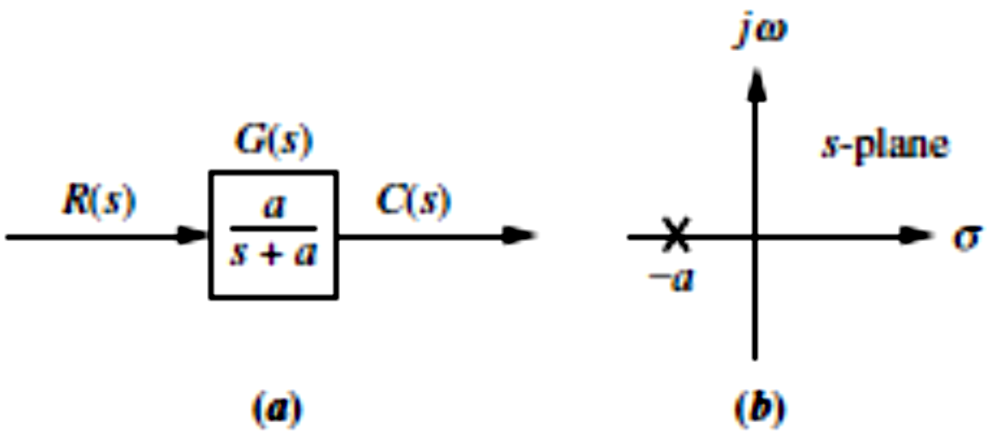


Fig 4.1 a. First-order system; b. pole plot

Taking the inverse transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at} \tag{4.2}$$

where the input pole at the origin generated the forced response $c_f(t) = 1$, and the system pole at $-a$, as shown in Figure 4.1(b), generated the natural response $c_n(t) = -e^{-at}$ Equation (4.2) is plotted in Figure 4.2.

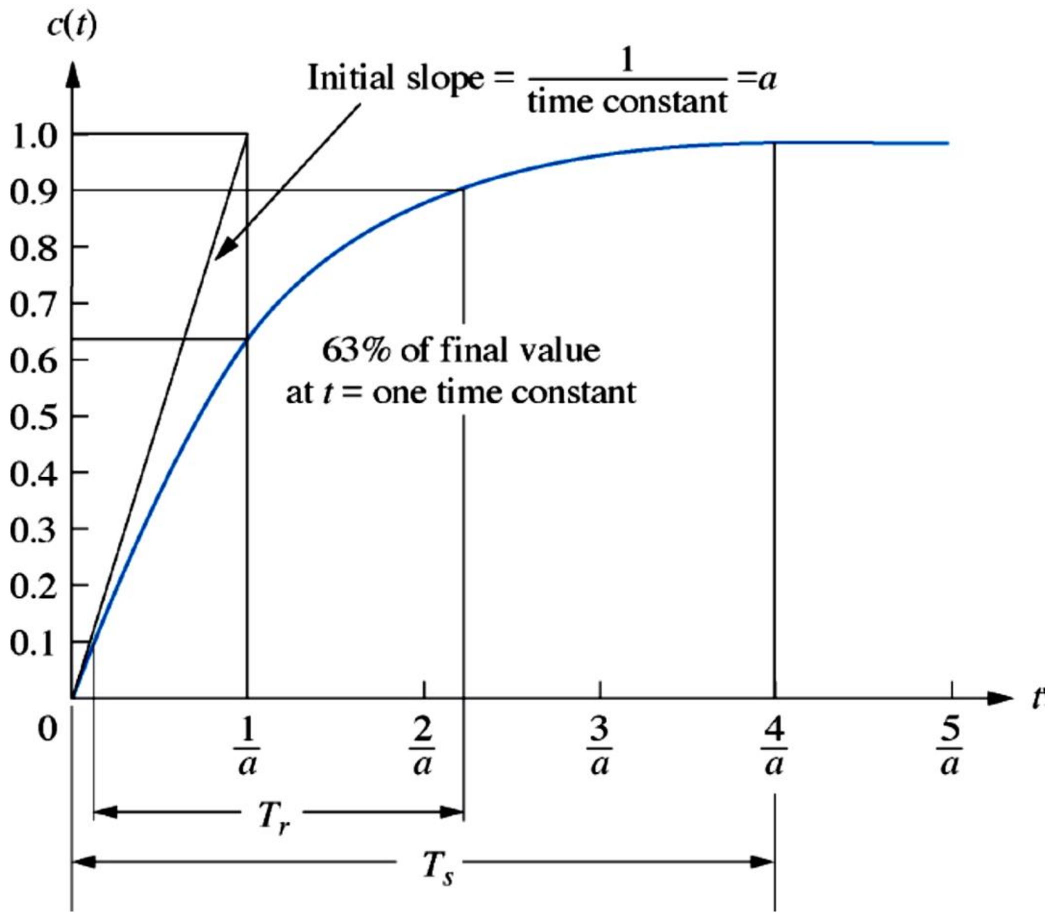


Fig 4.2 First-order system response to a unit step

Let us examine the significance of parameter a, the only parameter needed to describe the transient response. When $t = 1/a$,

$$c(t)|_{t=1/a} = 1 - e^{-at}|_{t=1/a} = 1 - 0.37 = 0.63 \qquad (4.3)$$

We now use Eqs. (4.2), (4.3), and (4.4) to define three transient response performance specifications.

Time Constant, $1/a$

We call $1/a$ the time constant of the response, from Eq. (4.3) the time constant is the time it takes for the step response to rise to **63%** of its final value (see Figure 4.2).

Rise Time, T_r

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value. Rise time is found by solving Eq. (4.2) for the difference in time at $c(t) = 0.9 \; and \; c(t) = 0.1$. Hence,

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a} \tag{4.4}$$

Settling Time, T_s

Settling time is defined as the time for the response to reach, and stay within, 2% of its final value. Letting $c(t) = 0.98$ in Eq. (4.2) and solving for time, t, we find the settling time to be

$$T_s = \frac{4}{a} \tag{4.5}$$

First-Order Transfer Functions via Testing

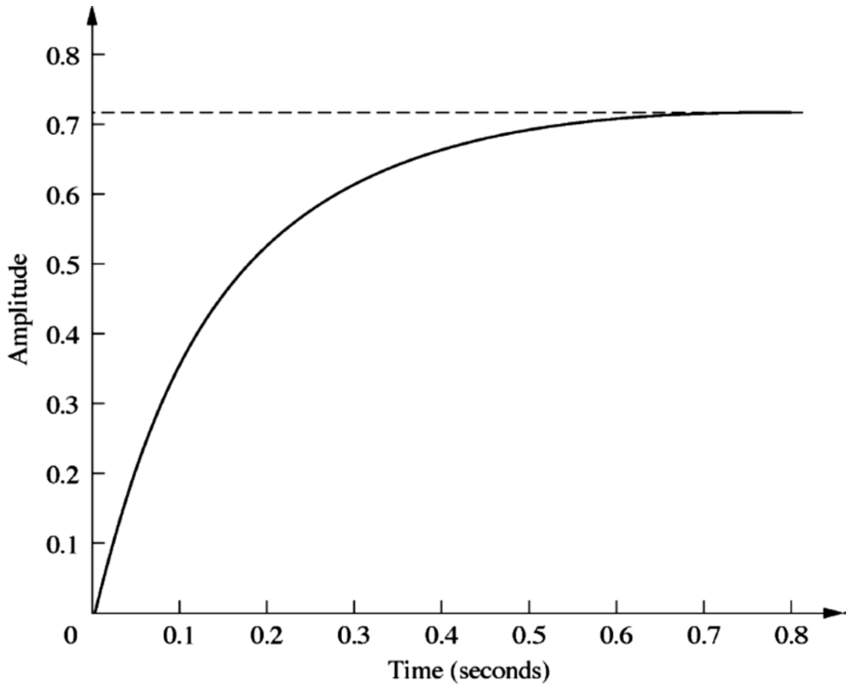
Sometimes we will know the form the response but will be required to derive the parameters experimentally.

Example 4.1:

Assume the step response has the following form and time-domain plot:

$$c(s) = \frac{K}{s(s+a)}$$

Determine a and K.



Solution:

$$c(s) = \frac{K}{s(s+a)} = \frac{K}{a}\frac{a}{s(s+a)}$$

This is a same stander form

The asymptote yields an estimate of $K/a = 0.72$.

63% of this is 0.45 which is reached around $t\ =\ 0.15s$

Hence $time\ constant\ t = \frac{1}{a} \rightarrow a = \frac{1}{0.15} = 6.67$ and;

$$\frac{K}{a} = 0.72 \rightarrow K = 0.72 * 6.67 = 4.8$$

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4.3. Response of Second-Order Systems

Second-order systems (systems described by second-order DE's) have transfer functions of the following form:

$$G(s) = \frac{b}{s^2 + as + b}$$

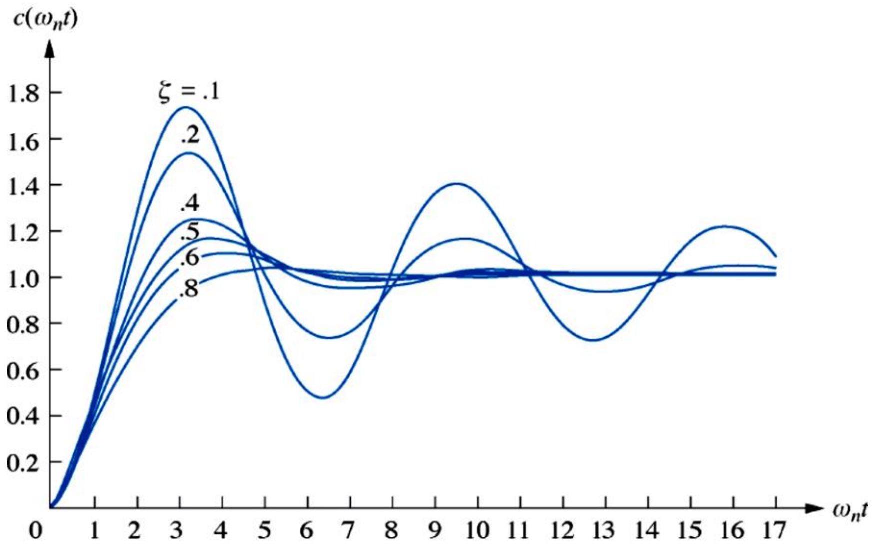
(This TF may also be multiplied by a constant K, which affects the exact constants of the time-domain signal, but not its form).

Depending upon the factors of the denominator we get four categories of responses (see figure 4.3). If the input is the unit step, a pole at the origin will be added which yields a constant term in the time-domain

We can characterize the response of second-order systems using two parameters: ω_n and ζ

Natural Frequency, ω_n: This is the frequency of oscillation without damping. For example, a mechanical system without dampers. An undamped system is described by its natural frequency.

Damping Ratio, ζ: This measures the amount of damping. For underdamped systems ζ lies in the range [0, 1]:



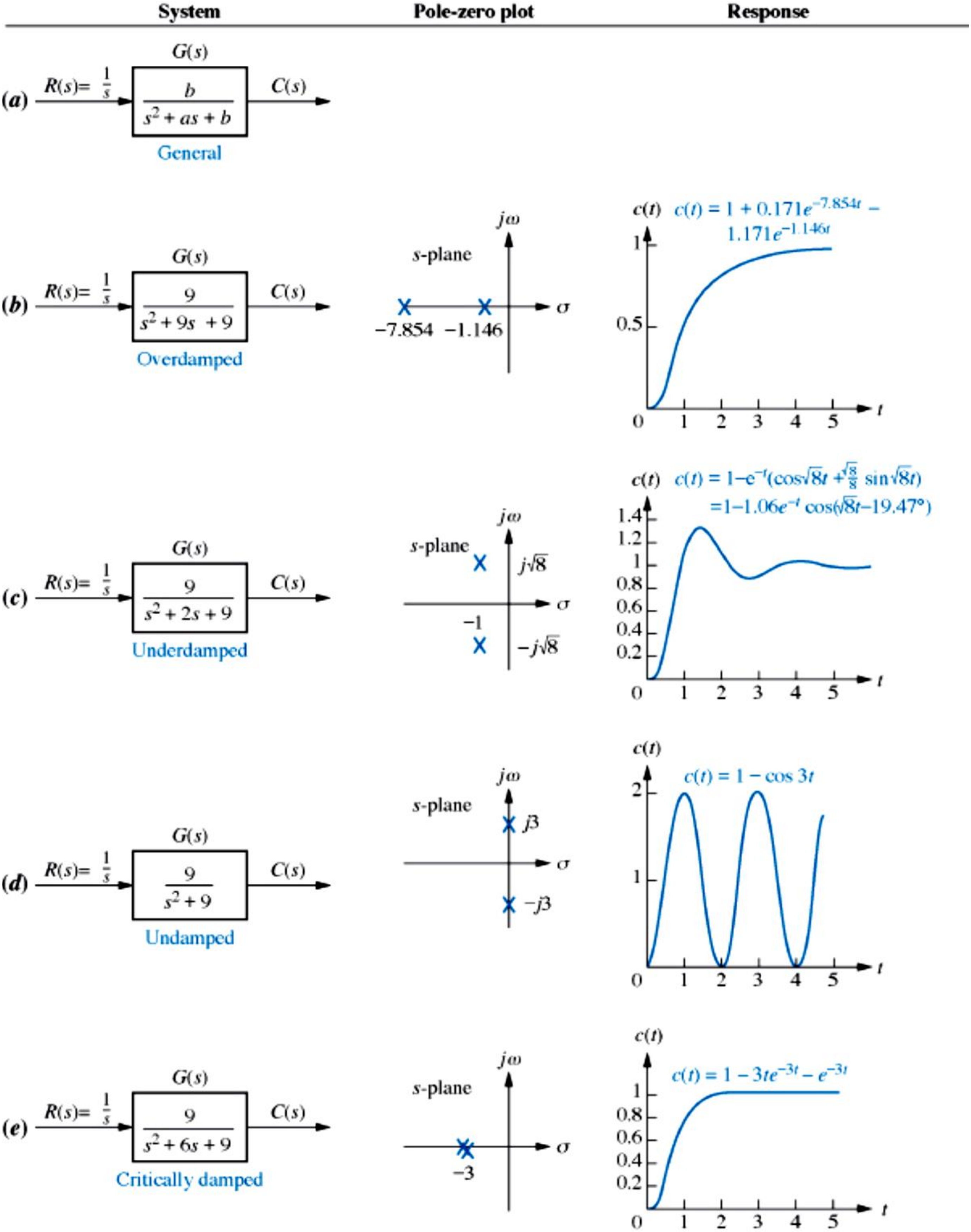


Fig 4.3 Second-order systems, pole plots, and step responses

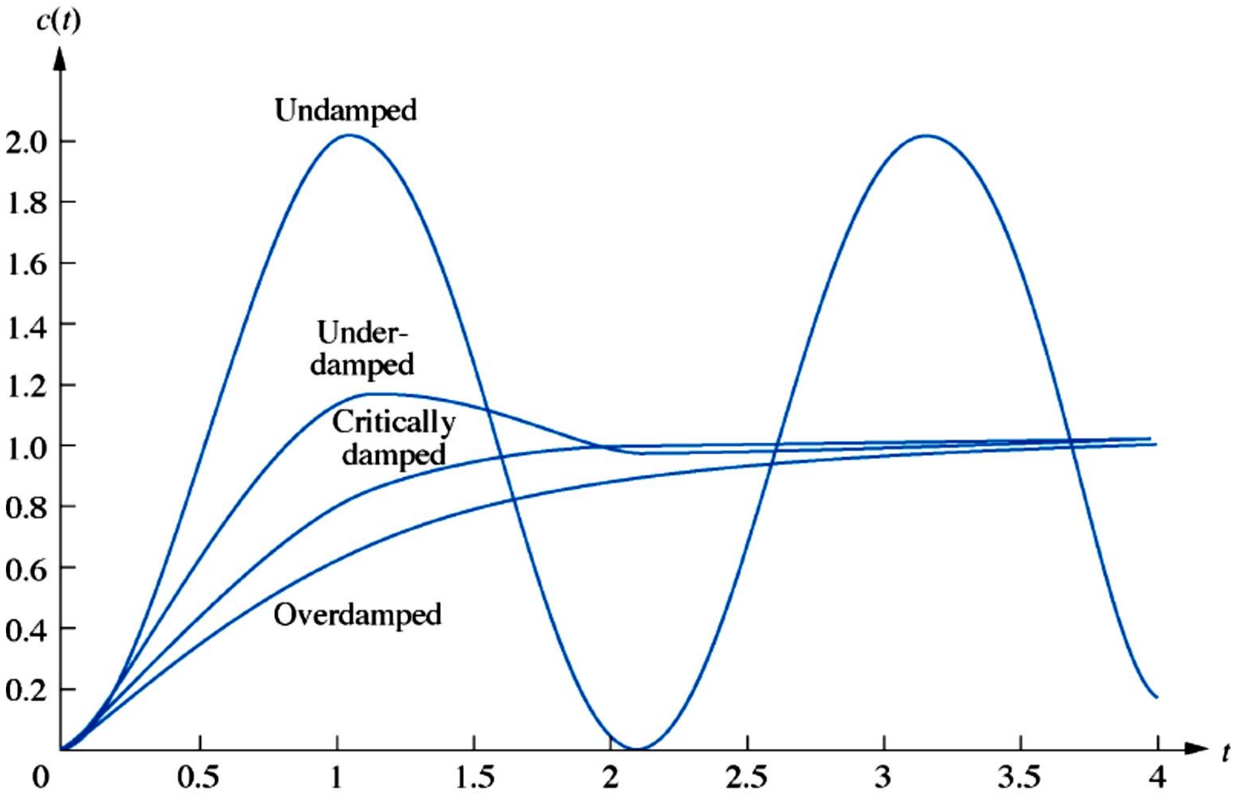


Fig 4.4 Step responses for second-order system damping cases

Category	Poles	$c(t)$
Overdamped	Two real: $-\sigma_1, -\sigma_2$	$K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$
Underdamped	Two complex: $-\sigma_d \pm j\omega_d$	$Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$
Undamped	Two imaginary: $\pm j\omega_n$	$A\cos(\omega_n t - \phi)$
Critically damped	Repeated real: $-\sigma_d$	$K_1 e^{-\sigma_d t} + K_2 t e^{-\sigma_d t}$

Damping ratio is dened as follows:

$$\zeta = \frac{Exponential\ decay\ frequency}{Natural\ frequency} = \frac{|\sigma_d|}{\omega_n}$$

The exponential decay frequency σ_d is the real-axis component of the poles of a critically damped or underdamped system.

We now describe the general second-order system in terms of ω_n and ζ.

$$G(s) = \frac{b}{s^2 + as + b}$$

In other words we want to get the relationships from ω_n and ζ to a and b. Why? Because ω_n and ζ are more meaningful and useful for design.

If there were no damping, we would have a pure sinusoidal response. Thus, the poles would be on the imaginary axis and the TF would have the form,

$$G(s) = \frac{b}{s^2 + b}$$

The poles are at $s = \mp j\sqrt{b}$. The natural frequency is governed by the position of the poles on the imaginary axis. Therefore,

$$\omega_n = \sqrt{b}$$

$$b = \omega_n^2$$

Consider an underdamped system with poles $-\sigma_d \mp j\omega_d$. The exponential decay frequency is σ_d . For a general second-order system the denominator is $s^2 + as + b$ and the roots have real part $\sigma_d = -\frac{a}{2}$.

We apply the denition for ζ:

$$\zeta = \frac{Exponential\ decay\ frequency}{Natural\ frequency} = \frac{|\sigma_d|}{\omega_n} = \frac{a/2}{\omega_n}$$

Thus, $a = 2\zeta\omega_n$. We can now describe the second-order system as follows:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.6}$$

Poles: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Although the two parameters ω_n and ζ completely characterize the form of the underdamped response, we usually specify the response with the following derived parameters:

Peak time, T_p: The time required to reach the first (maximum) peak.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Percent overshoot, $\%OS$: The amount that the response exceeds the final value at T_p.

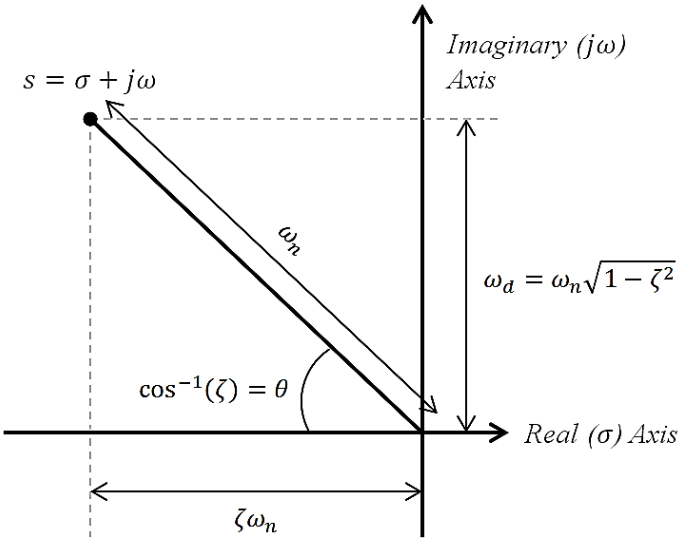
$$\%OS = e^{(-\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

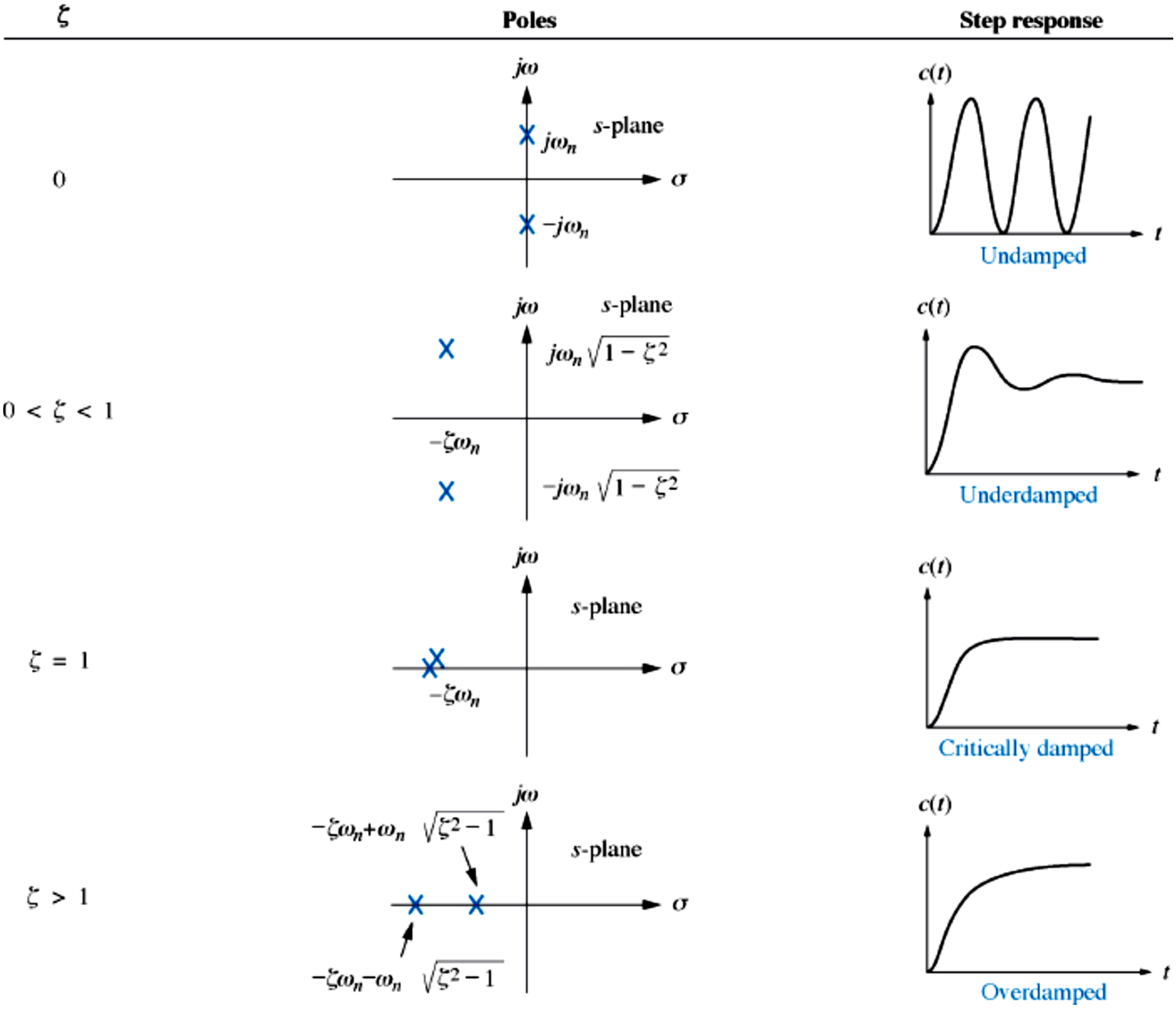
Settling time, T_s : The time required for the oscillations to die down and stay within 2% of the final value.

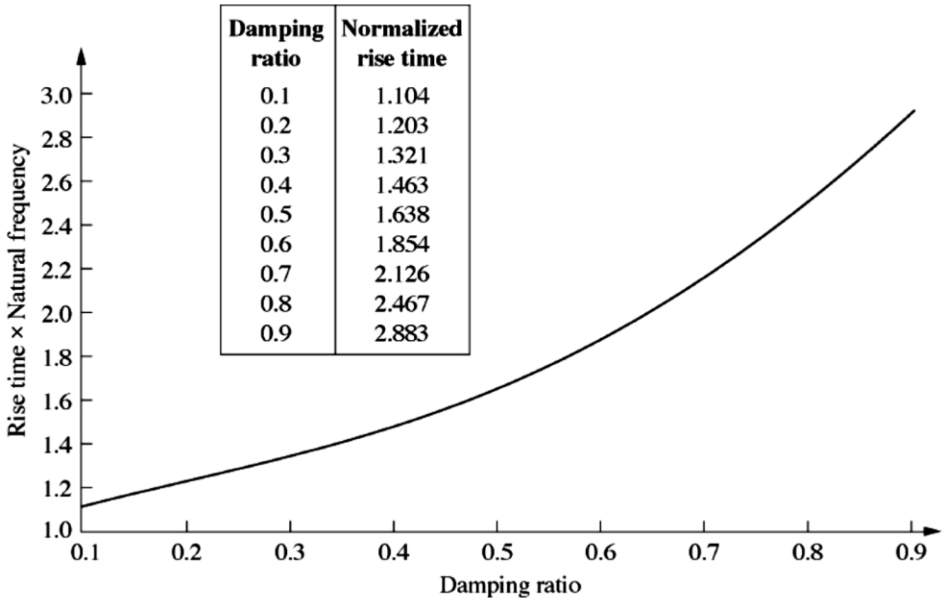
$$T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

Rise time, T_r : The time to go from 10% to 90% of the final value.

There is no analytical form for T_r (time to go from 10% to 90% of final value). This value can be calculated numerically and has been formed into a table:


$s = \sigma + j\omega$
Imaginary ($j\omega$) Axis
ω_n
$\omega_d = \omega_n\sqrt{1-\zeta^2}$
$\cos^{-1}(\zeta) = \theta$
Real (σ) Axis
$\zeta\omega_n$


ζ
Poles
Step response
0
$j\omega$
s-plane
$j\omega_n$
$-j\omega_n$
σ
$c(t)$
t
Undamped
$0 < \zeta < 1$
$j\omega_n\sqrt{1-\zeta^2}$
$-\zeta\omega_n$
$-j\omega_n\sqrt{1-\zeta^2}$
Underdamped
$\zeta = 1$
$-\zeta\omega_n$
Critically damped
$\zeta > 1$
$-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}$
$-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}$
Overdamped



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

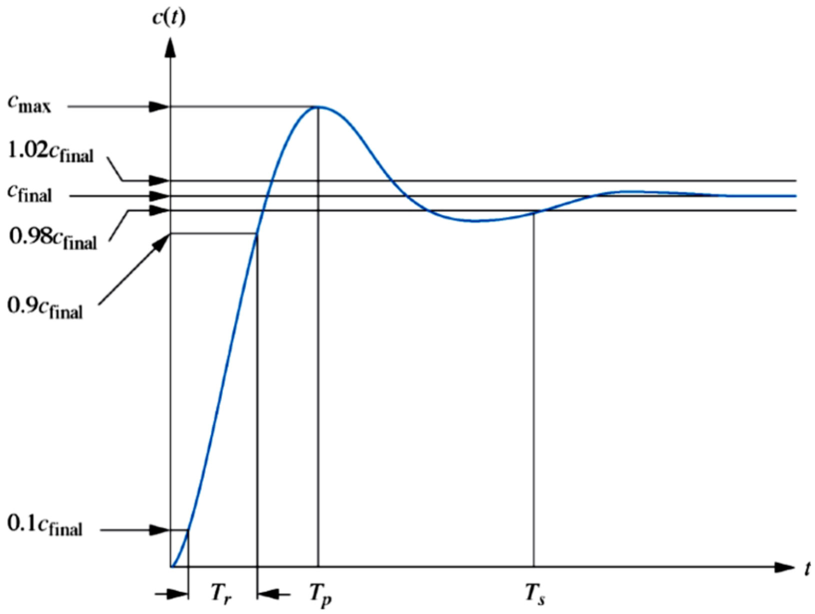


Fig 4.5 Second-order underdamped response specifications

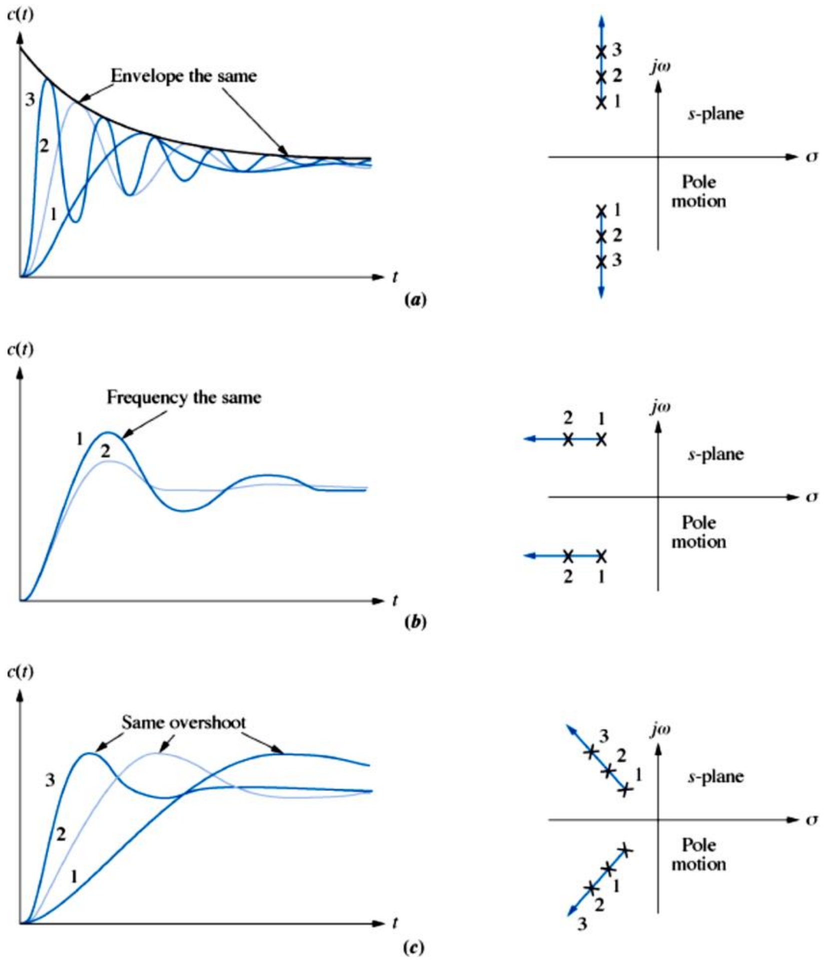


Fig 4.6 Step responses of second-order underdamped systems as poles move: a. with constant real part; b. with constant imaginary part; c. with constant damping ratio

Example 4.2:

Find the mode shape of displacement and velocity state for SODE damped harmonic oscillator

Sol:

The governing differential equation was described as

$$\ddot{y} + 2\zeta\omega_n\,\dot{y} + \omega_n^2\,y = 1/m\;\;f(t)$$

Assume that $\dot{y} = x; \ddot{y} = \dot{x};$

$$\dot{x} + 2\zeta\omega_n\,x + \omega_n^2\,y = 1/m\;\;f(t)$$

$$\dot{y} = x \ldots\ldots..(1)$$

$$\dot{x} = -\omega_n^2\,y - 2\zeta\omega_n\,x + 1/m\;\;f(t)\ldots\ldots(2)$$

where
$y = displasement$
$x = velocity$

Put it in state space form

$$\begin{Bmatrix}\dot{y}\\ \dot{x}\end{Bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}\begin{Bmatrix}y\\ x\end{Bmatrix} + \begin{Bmatrix}0\\ 1/m\end{Bmatrix}f(t)$$

Now, we find eigenvalues of the damped harmonic oscillator

$$|I\lambda - A| = \begin{vmatrix}\lambda & -1\\ \omega_n^2 & (\lambda + 2\zeta\omega_n)\end{vmatrix} = \lambda^2 + 2\zeta\omega_n + \omega_n^2 = 0$$

which yields

$$\lambda_{1,2} = -\zeta\omega_n \pm i\sqrt{1-\zeta^2}\omega_n = -\zeta\omega_n \pm i\omega_d \quad for\; 0 < \zeta < 1$$

To find eigenvalue must be $[A - \lambda_n]\{x^n\} = 0$

$$\begin{vmatrix} 0-(-\zeta\omega_n + i\omega_d) & 1 \\ -\omega_n^2 & -2\zeta\omega_n - (-\zeta\omega_n + i\omega_d) \end{vmatrix} \begin{Bmatrix} y \\ x \end{Bmatrix}^1 = 0$$

$$(\zeta\omega_n - i\omega_d)y^1 + x^1 = 0$$

$$(-\zeta\omega_n + i\omega_d)y^1 = x^1$$

$$\begin{Bmatrix} y \\ x \end{Bmatrix}^1 = \begin{Bmatrix} 1 \\ -\zeta\omega_n + i\omega_d \end{Bmatrix}^1 = \begin{Bmatrix} 1 \\ -\zeta\omega_n + i\sqrt{1-\zeta^2}\omega_n \end{Bmatrix}^1 \quad \ldots\ldots(4.7)$$

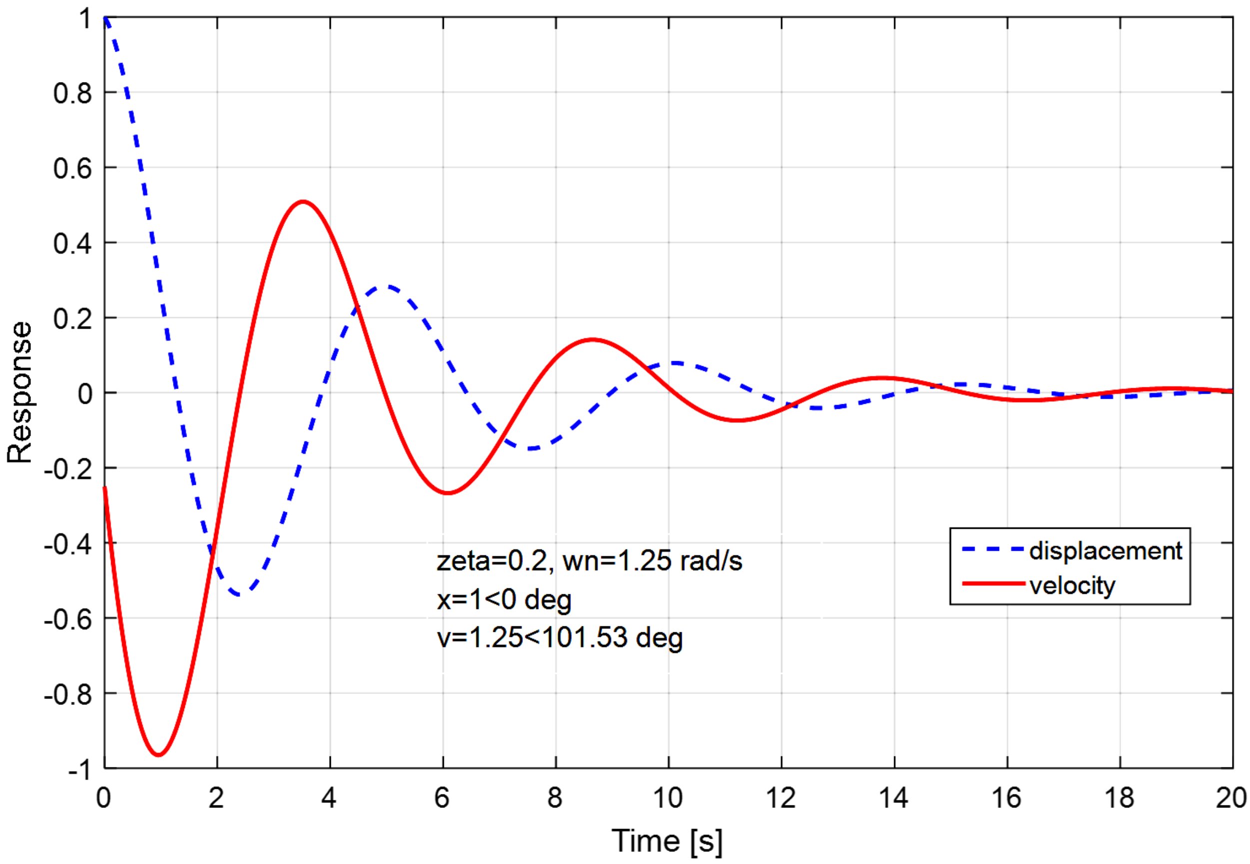


Fig 4.7 Plot the results of example 4.2

Chapter 5: Longitudinal Dynamics

Lecture 9

Topics

5.1. Aircraft Longitudinal Dynamics

5.2. Short-Period Approximation

5.3. Long-Period (Phugoid) Approximation

5.1. Aircraft Longitudinal Dynamics

The homogeneous form of the linearized equations of motion from Eq. (3.17) (see lecture 6) is given by

$$[I_n]\{\dot{x}\} = [A_n]\{x\} \tag{5.1}$$

This set of first-order linear differential equations will be solved by example using computer tools available to many engineers and students, i.e., MATLAB.

Example 5.1:

Consider the A-4D jet attack aircraft in level flight at $M = 0.6, h = 4572\ m$, and c.g. at 0.25c . Solve the eigenvalue problem and identify the mode shapes. The MATLAB listing is as follows:

```
>> disp(In)
  634.0000         0         0         0
         0  634.0000         0         0
         0    0.3530    1.0000         0
         0         0         0    1.0000
>> disp(An)
   -8.1790   -3.7210         0  -32.1740
  -65.9400 -518.9000  634.0000         0
    0.2500  -12.9700   -1.0710         0
         0         0    1.0000         0
```

```
% The plant matrix [A] in x-dot = Ax is:
A=inv(In)*An;
>> disp(A)
   -0.0129   -0.0059         0   -0.0507
   -0.1040   -0.8185    1.0000         0
    0.2867  -12.6811   -1.4240         0
         0         0    1.0000         0

% Find the characteristic polynomial
>> P=poly(A); disp(P)
    1.0000    2.2554   13.8749    0.1940    0.0788
%   (s^4)      (s^3)    (s^2)     (s^1)      (s^0)

% Find the roots of the characteristic polynomial
>> R=roots(P); disp(R)
  -1.1211 ∓ 3.5472i % Short-period mode (eigenvalue)
  -0.0065 ∓ 0.0752i % Long-period mode (Phugoid) (eigenvalue)

% Find modal damping and undamped natural frequencies
>> [Wn,Z]=damp(R);
>> disp(Z);
    0.3014 % Short-period modal damping
    0.0867 % Long-period modal damping
>> disp(Wn);
    3.7202 % Short-period natural frequency, rad/sec
    0.0755 % Long-period .natural frequency, rad/sec
% Find the eigenvectors
>> [V, D] =eig (A);
>> disp(V); % Display the conjugate paired eigenvectors
  Columns 1 through 3

   0.0032 - 0.0021i   0.0032 + 0.0021i  -0.0473 + 0.5549i
  -0.0222 - 0.2608i  -0.0222 + 0.2608i  -0.0001 + 0.0056i
   0.9321 + 0.0000i   0.9321 + 0.0000i  -0.0054 + 0.0623i
  -0.0755 - 0.2389i  -0.0755 + 0.2389i   0.8282 + 0.0000i
Column 4

  -0.0473 - 0.5549i
  -0.0001 - 0.0056i
  -0.0054 - 0.0623i
   0.8282 + 0.0000i
% Find magnitude and phasing for the short-period mode
>> MAG1=abs(V(:,1)); PHASE1=(180./pi)*angle(V(: ,1));
disp(MAG1'/MAG1(2)); % Normalized to a component
    0.0146    1.0000    3.5614    0.9573
%   (u/V)       α         q         θ

disp(PHASE1'-PHASE1(2)); % Phase relative to α, deg
   61.3327      0       94.8636  -12.6760
```

% (u/V) α q θ

% Find magnitude and phasing for the long-period mode
>> MAG2=abs(V(:,3)); PHASE2=(180./pi)*angle(V(: ,3));
>> disp(MAG2'/MAG2(1)); % Normalized to (u/V) component
1.0000 0.0101 0.1122 1.4870
% (u/V) α q θ

>> disp(PHASE2' - PHASE2(1)); % Phase relative to (u/V), deg
0 -3.9081 0.1014 -94.8749
% (u/V) α q θ

The MATLAB commands used to solve the eigenvalue problem of Example 5.1 included:
disp(A) = display matrix [A]
inv(In) = invert nonsingular matrix $[I_n]$
poly(A) = find polynomial coefficients from $|\lambda I - A|$
roots(P) = find roots of polynomial from row vector P
damp(R) = find natural frequencies and modal damping from R
eig(A) = eigenvectors and eigenvalues from square matrix A
abs(V(:, 1)) = absolute number value in first column of matrix V
angle(V(:,1)) = phase angle of complex numbers in first column of V

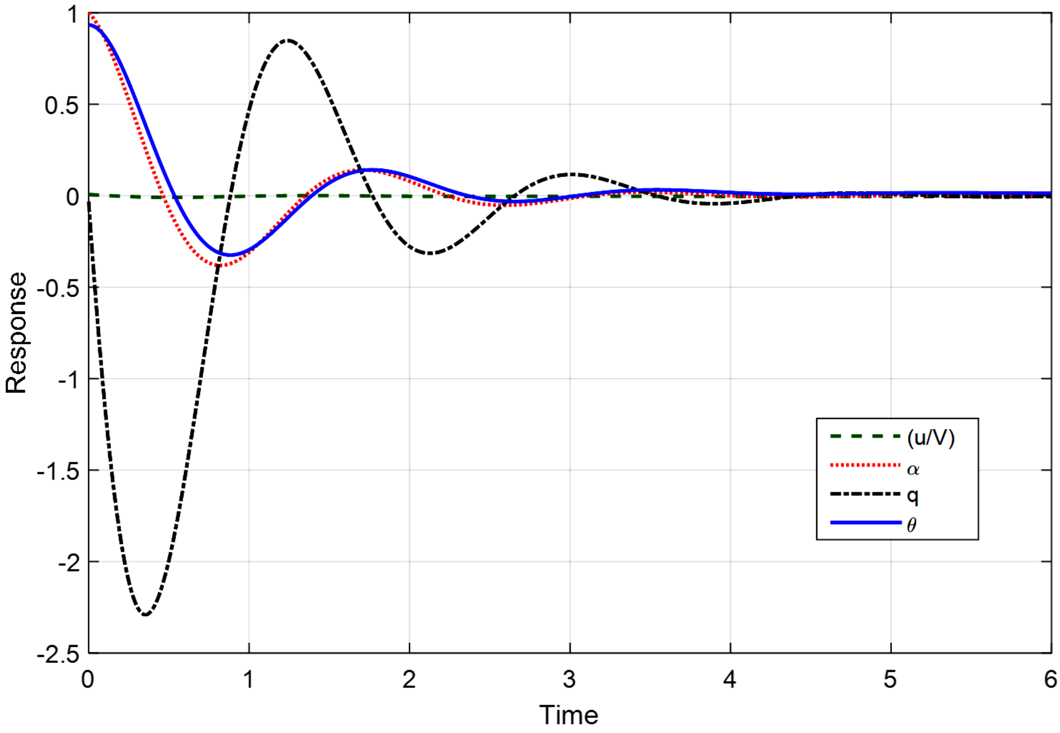


Fig 5.1 Short-period response to phasor initial condition.

A time-history trace of the short-period response due to a unit initial condition of the eigenvector is shown in Fig. 5.1. The plot represents the projection of the exponentially decaying rotating phasors on the real axis. The plot was obtained by using the initial condition of

```
>> disp(Xo)
   0.0070 - 0.0128i
   1.0000 + 0.0000i
  -0.0317 - 3.5486i
   0.9340 + 0.2101i
```

in conjunction with the MATLAB ***initial*** command and a time (row) vector that extended from 0 to 6 s by 0.05s intervals

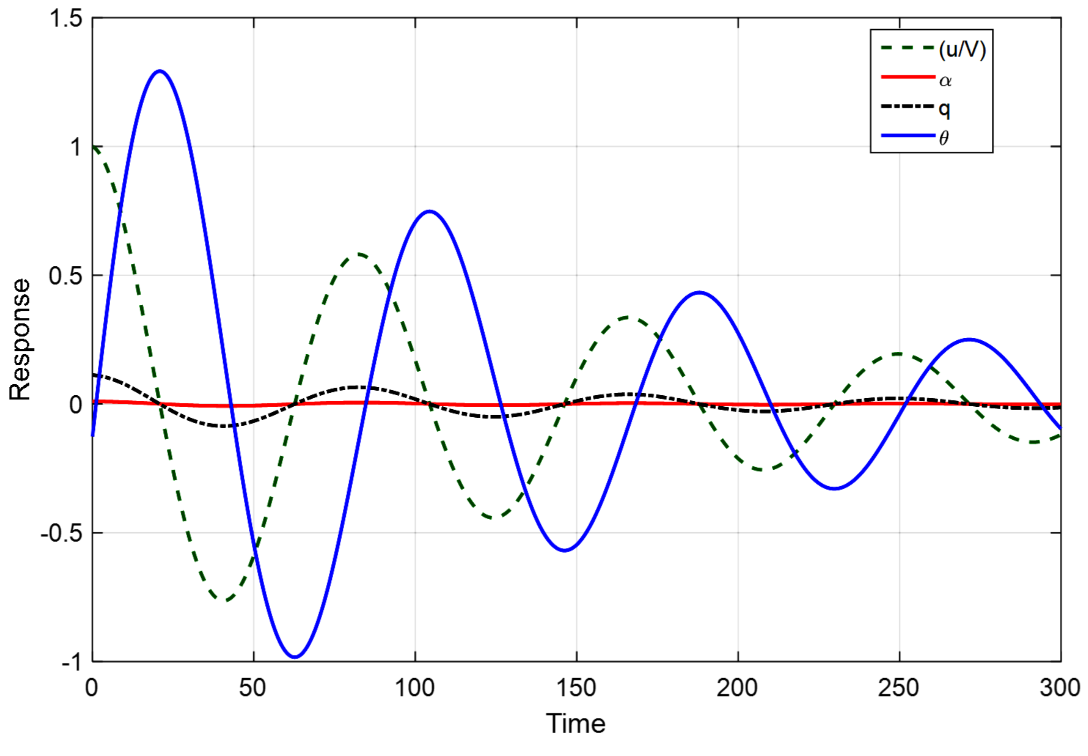


Fig 5.2 Short-period response to phasor initial condition

The time-history response of the phugoid mode is shown in Fig. 5.2 in a manner similar to the preceding illustration for the short-period mode. The influence of the light damping on the oscillation amplitude decay is clearly evident.

```
>> disp(Xo)
   1.0000 + 0.0000i
   0.0100 + 0.0007i
   0.1122 - 0.0002i
  -0.1263 + 1.4816i
```

=========++++++++++++++++++++++++=================================++++++++++++++++++=========

Short Period Mode properties

- Quick, high frequency *("short period")*
- Well Damped (*few overshoots*)
- Pitch and α vary

Phugoid Mode properties

- Slow, low frequency ("*long period*")
- Lightly Damped (*many overshoots*)
- Pitch and V vary

– Trade altitude and airspeed
– Rollercoaster in the sky

5.2. Short-Period Approximation

An initial short-period approximation can be made based on the nature of the eigenvector as seen in Sec. 4.1; i.e., the velocity perturbation is negligible by comparison to the α and θ components. These observations are reflected in the following assumptions.

1) The e.g. maintains a constant forward velocity; i.e., $u(t)/V = 0$. Because longitudinal acceleration, du/dt, is governed by the dynamics embodied in the F_x equation, the assumed constraint of no change in forward velocity allows the F_x equation to be dropped.

2) The aircraft is free or unconstrained both to pitch rotation about the c.g. and to vertical motion of the c.g. (**z** direction). The M_y and F_z dynamic equations will be altered by the deletion of the $(u/\,V)$ related terms.

3) The perturbation will start from level flight $(i.e.\theta_o = 0)$ with the aircraft reference being the body's stability axes.

$$
\begin{aligned}
(V - Z_{\dot{\alpha}})\,\dot{\alpha} &= Z_\alpha \alpha + \left(V + Z_q\right) q + Z_\delta \delta \\
-M_{\dot{\alpha}}\dot{\alpha} + \dot{q} &= M_\alpha \alpha + M_q q + M_\delta \delta
\end{aligned}
\qquad (5.2)
$$

A further simplification can be made by recognizing that both $Z_{\dot{\alpha}}$ and Z_q are nearly zero in magnitude and most assuredly are negligible when compared to the freestream velocity V in the preceding equations. The short-period approximation in a commonly used form becomes

$$\begin{Bmatrix} \dot{\alpha} \\ \dot{q} \end{Bmatrix} = \begin{bmatrix} (Z_\alpha / V) & 1 \\ \left(M_\alpha + \dfrac{M_{\dot{\alpha}} Z_\alpha}{V}\right) & (M_q + M_{\dot{\alpha}}) \end{bmatrix} \begin{Bmatrix} \alpha \\ q \end{Bmatrix} + \begin{Bmatrix} \left(\dfrac{Z_\delta}{V}\right) \\ \left(M_\delta + \dfrac{M_{\dot{\alpha}} Z_\delta}{V}\right) \end{Bmatrix} \delta \tag{5.3}$$

Equation (5.3) is in the standard matrix form

$$\{\dot{x}\} = [A]\{x\} + \{B\}u \tag{5.4}$$

Example 5.2

Use the approximate form, Eq. (5.3), find characteristic equation and estimate the short-period modal properties for the A-4D aircraft considered in Example 5.1.

$Z_\alpha = -158.16\ m/s^2$ $\quad V = 193.23\ m/s^2$ $\quad M_\alpha = -12.97\ 1/s^2$

$M_{\dot{\alpha}} = -0.353\ 1/s$ $\quad M_q = -1.071\ 1/s$ $\quad Z_\delta = -30.43\ 1/s^2$

Sol:

Apply equation (5.3) *to find plant matrix* $[A]$

$$[A] = \begin{bmatrix} -0.8185 & 1.0 \\ -12.6811 & -1.424 \end{bmatrix}$$

To find eigenvalue must be $|A - I\lambda| = 0$

$$\begin{vmatrix} -0.8185-\lambda & 1.0 \\ -12.6811 & -1.424-\lambda \end{vmatrix} = 0$$

$$(-0.8185-\lambda)(-1.424-\lambda) - 1.0 * (-12.6811) = 0$$

$$\lambda^2 + 2.2425\lambda + 13.8466 = 0$$

$$\lambda^2 + 2\zeta_{sp}\omega_{sp}\,\lambda + \omega_{sp}^2 = 0$$

$$\lambda_{1,2} = -1.1213 \pm 3.5482i$$

$$\omega_{sp}^2 = 13.8466 \rightarrow \omega_{sp} = 3.7211\ rad/\sec$$

$$2\zeta_{sp}\omega_{sp} = 2.2425 \rightarrow \zeta_{sp} = \frac{2.2425}{2 * 3.7211} = 0.3013$$

Now, find the eigenvector

$$\lambda_n\{x^n\} = [A]\{x^n\}$$

$$[A - I\lambda_n].\{x^n\} = 0$$

For n=1, one finds that

$$\begin{bmatrix} -0.8185 + 1.1213 - 3.5482i & 1.0 \\ -12.6811 & -1.424 + 1.1213 - 3.5482i \end{bmatrix} \begin{Bmatrix} \alpha^1 \\ q^1 \end{Bmatrix} = 0$$

$$(0.3028 - 3.548i)\alpha^1 + q^1 = 0$$

$$q^1 = (-0.3028 + 3.548i)\alpha^1$$

The mode shape for the short-period approximation is

$$r_\alpha = \sqrt{1^2 + 0^2} = 1$$

$$phase\ angle_\alpha = \tan^{-1}\left(\frac{Ri}{Im}\right) = \tan^{-1}\left(\frac{1}{0}\right) = 0.0\ deg$$

$$r_q = \sqrt{(-0.3028)^2 + 3.548^2} = 3.561$$

$$phase\ angle_q = \tan^{-1}\left(\frac{Ri}{Im}\right) + 90° = \tan^{-1}\left(\frac{-0.3028}{3.548i}\right) + 90° = 94.88\ \ deg$$

$$\{x\}_{sp} = \begin{Bmatrix} \alpha \\ q \end{Bmatrix}_{sp} = \begin{Bmatrix} 1.0\ \angle 0.0\ deg \\ 3.561\ \angle 94.88\ deg \end{Bmatrix}$$

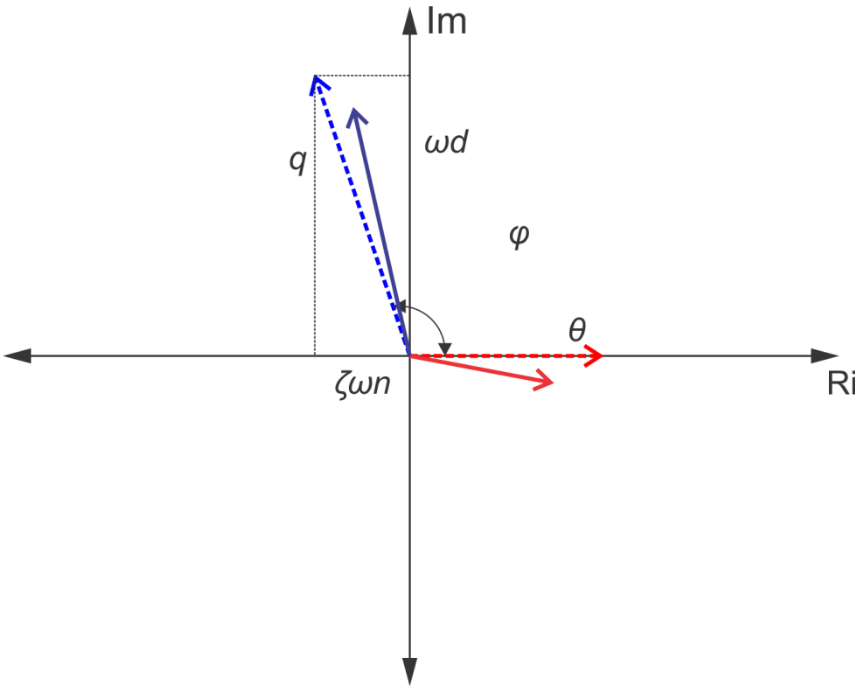
We can estimate the θ by Eq. (4.7) (see lecture 7).

$$\begin{Bmatrix} \theta \\ q \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\zeta\omega_n + i\sqrt{1-\zeta^2}\omega_n \end{Bmatrix}$$

$$if\ \theta = 1.0\ \angle 0.0\ deg;$$

$$q = -\zeta\omega_n + i\sqrt{1-\zeta^2}\omega_n = -1.1212 + i3.5482 = 3.7211\ \angle 107.54\ deg$$

$$then\ \theta = \frac{3.561}{3.7211}\ \angle(94.88 - 107.54)deg = 0.957\angle - 12.88\ deg$$



A comparison of the short-period approximations with the full solution for the example aircraft is shown on Table 5.1.

Table 5.1. A-4D short-period summary

Item	Airframe Example 5.1	Approximation Eq. (5.3)
$\boldsymbol{Eigenvalue}$	$-1.121 \pm 3.547i$	$-1.121 \pm 3.548i$
$\boldsymbol{Damping, \zeta}$	0.301	0.301
$\boldsymbol{Eigenvector}$		
$(\boldsymbol{u/V})$	$0.015 \angle\ 61.33\ deg$	n/a
$\boldsymbol{\alpha}$	$1.000\angle 0.0$ deg	$1.000\angle 0.0$ deg
$\boldsymbol{q}$	$3.561\angle\ 94.86\ deg$	$3.561 \angle\ 94.88$ deg
$\boldsymbol{\theta}$	$0.957 \angle -12.68$ deg	$0.957 \angle -12.66$ deg

5.3. Long-Period (Phugoid) Approximation

It was apparent in Sec. 4.1 that the velocity perturbation dominated the phugoid mode and that the angle-of- attack variation was orders of magnitude smaller. Consequently, the long-period approximation will assume that the aircraft's phugoid oscillation occurs at a constant angle of attack $(i.e., constant\ C_L)$. In addition to decreasing the number of time-varying quantities by one, this assumption will remove the influence of **c.g**. location because the M_a term (a function of static margin) in the moment equation will not make a contribution to the solution. From a constraint standpoint, the approximation will include the following.

1) The pitching moment equation is dropped based on the assumptions that angle of attack $\alpha(t)$ is not a dominant variable and compressibility effects are absent $(i.e., M_\alpha = 0)$.

2) Only the F_x and F_z equilibrium relations will be retained.

3) The phugoid perturbation will start from level flight with the body axes assumed as stability axes.

When these constraints are considered, the F_x and F_z equilibrium relations in Eq. (3.16) (see lecture 6) simplify to

$$\left(\frac{\dot{u}}{V}\right) = X_u\left(\frac{u}{V}\right) - (g/V)\ \theta + (X_\delta/V)\delta$$
$$q = -Z_u\left(\frac{u}{V}\right) - (Z_\delta/V)\delta \tag{5.5}$$

but $q = d\theta/dt$. Therefore, the phugoid mode approximation may be stated in a matrix form as

$$\begin{Bmatrix} (\dot{u}/V) \\ \dot{\theta} \end{Bmatrix} = \begin{bmatrix} X_u & -(g/V) \\ -Z_u & 0 \end{bmatrix} \begin{Bmatrix} (u/V) \\ \theta \end{Bmatrix} + \begin{Bmatrix} (X_\delta/V) \\ -(Z_\delta/V) \end{Bmatrix} \delta \tag{5.6}$$

which corresponds to the state-space relation

$$\{\dot{x}\} = [A]\{x\} + \{B\}u$$

The characteristic equation comes from $|\lambda I - A| = 0, i.e,$

$$\begin{vmatrix} (\lambda - X_u) & -(g/V) \\ -Z_u & \lambda \end{vmatrix} = \lambda^2 - X_u\lambda - (gZ_u/V) = 0 \tag{5.7}$$

which is recognized as having the form

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

Example 5.3

Estimate the properties of the phugoid mode for the A-4D airplane of Example 5.1 by use of the modal approximation from the homogeneous form of Eq. (5.6),

$$Z_u = -0.104\ m/s^2 \qquad V = 193.23\ m/s^2 \qquad X_u = -0.0129\ 1/s^2$$

Sol:

$$\begin{Bmatrix} (\dot{u}\ /V) \\ \dot{\theta} \end{Bmatrix} = \begin{bmatrix} -0.0129 & -0.0507 \\ 0.104 & 0 \end{bmatrix} \begin{Bmatrix} (u\ /V) \\ \theta \end{Bmatrix}$$

$$[A] = \begin{bmatrix} -0.0129 & -0.0507 \\ 0.104 & 0 \end{bmatrix}$$

To find eigenvalue must be $|A - I\lambda| = 0$

$$\begin{vmatrix} (-0.0129 - \lambda) & -0.0507 \\ 0.104 & -\lambda \end{vmatrix} = 0$$

$$(-0.0129 - \lambda)(-\lambda) - 0.104 * (-0.0507) = 0$$

$$\lambda^2 + 0.0129\lambda + 0.00527 = 0$$

$$\lambda^2 + 2\zeta_{sp}\omega_{sp}\ \lambda + \omega_{sp}^2 = 0$$

$$\lambda_{1,2} = -0.0065 \pm 0.0723i$$

$$\omega_{ph}^2 = 0.00527 \rightarrow \omega_{ph} = 0.0726\, rad/\sec$$

$$2\zeta_{ph}\omega_{ph} = 0.0129 \rightarrow \zeta_{ph} = \frac{0.0129}{2 * 0.0726} = 0.0888$$

Now, find the eigenvector

$$\lambda_n\{x^n\} = [A]\{x^n\}$$

$$[A - I\lambda_n].\{x^n\} = 0$$

For n=1, one finds that

$$\begin{bmatrix} (-0.0065+0.0723i)+0.0129 & -0.0507 \\ 0.104 & (-0.0065+0.0723i) \end{bmatrix} \begin{Bmatrix} (u/V)^1 \\ \theta^1 \end{Bmatrix} = 0$$

$$0.104(u/V)^1 - (-0.0065+0.0723i)\theta^1 = 0$$

$$\theta^1 = \frac{-0.104}{(0.0065-0.0723i)}(u/V)^1 = (-0.12828-1.427i)(u/V)^1$$

The mode shape for the short-period approximation is

$$r_{(u/V)} = \sqrt{1^2+0^2} = 1$$

$$phase\ angle_{(u/V)} = \tan^{-1}\left(\frac{Im}{Ri}\right) = \tan^{-1}\left(\frac{1}{0}\right) = 0.0\ deg$$

$$r_{(u/V)} = \sqrt{(-0.12828)^2+1.427^2} = 1.432$$

$$phase\ angle_{(u/V)} = -\tan^{-1}\left(\frac{Ri}{Im}\right) - 90° = -\tan^{-1}\left(\frac{0.12828}{1.427}\right) - 90° = -95.1\ deg$$

$$\{x\}_{ph} = \begin{Bmatrix} (u/V) \\ \theta \end{Bmatrix}_{ph} = \begin{Bmatrix} 1.0\ \angle 0.0\ deg \\ 1.432\ \angle -95.1\ deg \end{Bmatrix}$$

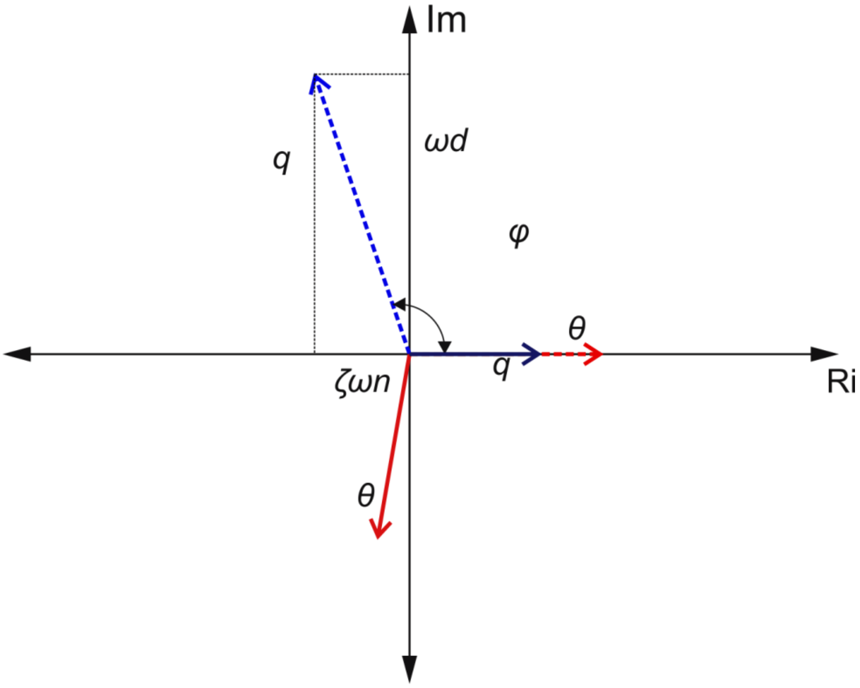
We can estimate the q by Eq. (4.7) (see lecture 7).

$$\begin{Bmatrix} \theta \\ q \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\zeta\omega_n + i\sqrt{1-\zeta^2}\omega_n \end{Bmatrix}$$

$$if\ \theta = 1.0\ \angle 0.0\ deg;$$

$$q = -\zeta\omega_n + i\sqrt{1-\zeta^2}\omega_n = -0.00644 + i0.0723 = 0.07259\ \angle 95.09\ deg$$

$$then\ q = 0.07259 * 1.432\ \angle\big((95.09)+(-95.1)\big)deg = 0.104\angle 0.0deg$$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Chapter 6: Lateral-Directional Dynamics

Lecture 10

Topics

6.1. Aircraft Lateral-Directional Dynamics

6.2. Dutch-Poll Approximation

6.1. Aircraft Longitudinal Dynamics

As a starting point for the (4 x 4) system, the relation, Eq. (4.24), from Sec. 3.4. (Lecture 6) will be used. The primed lateral-directional moment stability derivatives will be considered as defined previously in order to correct for the cross product of inertia terms. In this way, the inertial matrix [In] becomes a unit diagonal matrix.

$$\{\dot{x}\} = [A]\{x\} + [B]\{u\} \tag{6.1}$$

where the state vector is given as

$$\{x\} = [\beta \quad p \quad \phi \quad r]^T$$

and the control input, a column matrix of order (2 x 1), is

$$\{u\} = [\delta_r \quad \delta_a]^T$$

If only rudder or aileron control were under consideration, then the control input would simplify to a single (scalar) value.

The plant matrix, which involves primed dimensional moment derivatives, is expressed as

$$[\boldsymbol{A}] = \begin{bmatrix} Y_\beta/V & Y_p/V & (g/V)\cos\Theta_o & (Y_r - V)/V \\ L'_\beta & L'_p & 0 & L'_r \\ 0 & 1 & 0 & 0 \\ N'_\beta & N'_p & 0 & N'_r \end{bmatrix} \tag{6.2}$$

whereas the control sensitivity matrix is

$$[\boldsymbol{B}] = \begin{bmatrix} \frac{Y_{\delta_r}}{V} & \frac{Y_{\delta_a}}{V} \\ L'_{\delta_r} & L'_{\delta_a} \\ 0 & 0 \\ N'_{\delta_r} & N'_{\delta_a} \end{bmatrix} \tag{6.3}$$

The characteristic equation formed from Eq. (6.2) will be a quartic of the form

$$|\lambda I - A| = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

which in most cases will factor into a complex conjugate pair of roots and two distinct real roots. The factoring may be expressed as

$$(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)_{DR}(\lambda + \lambda_{roll})(\lambda + \lambda_{spiral}) = 0$$

Where

$$\lambda_{DR} = (-\lambda\omega_n \pm i\omega_d)_{DR}$$

The Dutch-roll mode shape comes from solving the eigenvalue problem, i.e.,

$$\{\dot{x}\}_{DR} = \lambda_{DR}\{x\}_{DR} = [A]\{x\}_{DR}$$

Because the Dutch-roll mode shape (eigenvector) involves solving four, complexnatured, simultaneous equations, computer methods (e.g., MATLAB) are suggested for convenience of evaluation.

It will be found useful, when considering the Dutch-roll mode shape, to include an estimate of the ψperturbation term even though ψ is not an eigenvector component. This estimation process involves using the relationship between an eigenvector and its derivative as explained in (**Lecture7; Example 4.2**) when describing a phasor representation.

Example 6.1:

Estimate the lateral-directional dynamic behavior of the DC-8 aircraft in a cruise-flight configuration which corresponds to $M = 0.84,\ h = 10000m$, and $V = 251.46\ m/s^2$. The plant matrix [A] according to Eq. (6.2) is

$$[\boldsymbol{A}] = \begin{bmatrix} \mathbf{-0.0869} & \mathbf{0.0} & \mathbf{0.039} & \mathbf{-1.0} \\ \mathbf{-4.424} & \mathbf{-1.184} & \mathbf{0.0} & \mathbf{0.335} \\ \mathbf{0.0} & \mathbf{1.0} & \mathbf{0.0} & \mathbf{0.0} \\ \mathbf{2.148} & \mathbf{-0.021} & \mathbf{0} & \mathbf{-0.228} \end{bmatrix}$$

1) The characteristic equation from $|\lambda I - A| = 0$ is

$$\lambda^4 + 1.4989\lambda^3 + 2.5477\lambda^2 + 2.8327\lambda + 0.0113 = 0$$

which may be factored as

$$(\lambda^2 + 0.2368\lambda + 2.2437\,)_{DR}(\lambda + 1.2580)_{roll}(\lambda + 0.0040)_{spiral} = 0$$

2) The four characteristic roots (eigenvalues) are as follows.
Dutch-roll roots,s^{-1}:

$$\lambda_{1,2} = -0.1184 \pm 1.4932i$$

Roll response root, s^{-1}:

$$\lambda_3 = -1.250$$

Spiral response root, s^{-1}:

$$\lambda_4 = -0.0040$$

3) The information contained in the Dutch-roll eigenvalue may be interpreted as

$$\zeta_{DR} = 0.0791\,, \qquad \omega_n = 1.4932\ s^{-1}\,, \qquad \omega_d = 1.4979\ s^{-1}$$

and Dutch-roll period

$$T_d = \frac{2\pi}{\omega_d} = 4.208\ s$$

4) The Dutch-roll mode shape may be identified by solving

$$\{\dot{x}\}_{DR} = \lambda_{DR}\{x\}_{DR} = [A]\{x\}_{DR}$$

and using λ_1 with the +value for ω_d to yield the Dutch-roll eigenvector of interest, normalized with respect to β , as

$$\{x\}_{DR} = \begin{Bmatrix} \beta \\ p \\ \phi \\ r \end{Bmatrix} = \begin{Bmatrix} 1.000 \angle 0.00\ deg \\ 2.412 \angle 131.84\ deg \\ 1.610 \angle 37.30\ deg \\ 1.457 \angle -86.79\ deg \end{Bmatrix}$$

An estimate of the iff perturbation term in degrees using the information contained in the eigenvector is

$$\psi = 0.973 \angle 178.67\ deg$$

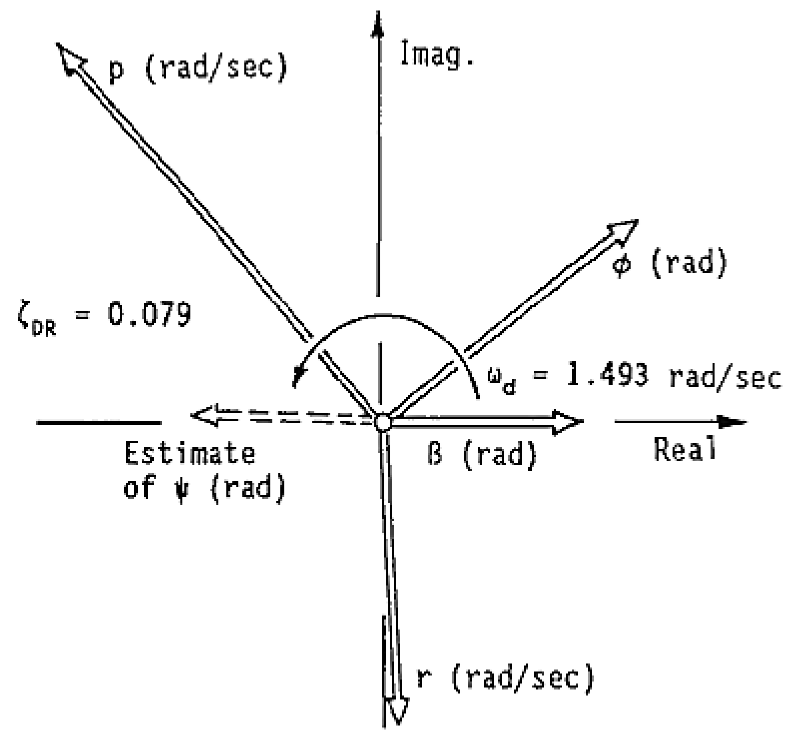


Fig 6.1 Dutch-roll phasor interpretation.

5) The roll response real root corresponds to a roll time constant of

$$\tau_{roll} = -\frac{1}{\lambda_3} = 0.795\ s$$

The mode shape, obtained from solving

$$\lambda_{roll}\{x\}_{roll} = [A]\{x\}_{roll}$$

gave, when normalized to the roll rate component,

$$\{x\}_{roll} = \begin{Bmatrix} \beta \\ p \\ \phi \\ r \end{Bmatrix} = \begin{Bmatrix} 0.016 \\ 1.000 \\ -0.795 \\ -0.013 \end{Bmatrix}$$

The roll mode is dominated by the p component with a minimal participation by the $\beta$and r components of the eigenvector

6) The spiral (real) root corresponds to a time constant of

$$\tau_{spiral} = -\frac{1}{\lambda_4} = 250.0\ s$$

The mode shape, when normalized with respect to bank angle ϕ, was estimated as

$$\{x\}_{spiral} = \begin{Bmatrix} \beta \\ p \\ \phi \\ r \end{Bmatrix} = \begin{Bmatrix} 0.004 \\ -0.004 \\ 1.00 \\ 0.039 \end{Bmatrix}$$

The spiral mode in Example 6.1, as illustrated by the eigenvector, showed that the roll angle component was dominant with a slight participation by the r (yaw rate) component. The estimated ψ component, when r is considered, becomes approximately −9.75 when compared to the ϕ component. The minus sign is due to the example situation of a stable spiral mode. This observation implies that the aircraft heading angle is a major participant in the spiral mode while sideslip angle perturbations remain negligible. Because the example mode had a quite large (stable) time constant, its impact on the airplane dynamics would normally be considered as imperceptible by the pilot.

6.2. Dutch-Roll Approximation

It was observed during the identification of the Dutch-roll mode in Example 6.1 that, although the heading perturbation was not included in the equations of motion, it was possible to estimate a heading perturbation term from the yaw rate (r) component. The Dutch-roll approximation will be based on the observation that the heading angle is nearly the same as the sideslip β perturbation with almost a 180-deg phase difference.

It will be assumed that the aircraft, while experiencing a Dutch-roll type of modal oscillation, is traveling in a straight flight track. The c.g.'s lack of lateral motion imposes a side-force constraint on the dynamic system. This assumption is equivalent to stating 1) $\psi = -\beta \; and \; \dot{\psi} = +r = -\dot{\beta}$ and 2) the linearized roll and yaw moment equations are retained while the side-force equation is omitted. Removal of the side-force equation in the complete set of lateral-directional equations, Eq. (6.2), simplifies the relations to

$$\begin{Bmatrix} \dot{p} \\ \dot{r} \end{Bmatrix} = \begin{bmatrix} L'_p & L'_r & L'_\beta \\ N'_p & N'_r & N'_\beta \end{bmatrix} \begin{Bmatrix} p \\ r \\ \beta \end{Bmatrix}$$

which is almost in the form of

$$\{\dot{x}\} = [A]\{x\}$$

When the assumption of $r = — \dot{\beta}$ is included in the formulation, the resulting approximation, including the control terms, becomes

$$\begin{Bmatrix} \dot{p} \\ \ddot{\beta} \\ \dot{\beta} \end{Bmatrix} = \begin{bmatrix} L'_p & -L'_r & L'_\beta \\ -N'_p & +N'_r & -N'_\beta \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} p \\ \dot{\beta} \\ \beta \end{Bmatrix} + \begin{bmatrix} L'_{\delta_r} & L'_{\delta_a} \\ -N'_{\delta_r} & -N'_{\delta_a} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta_r \\ \delta_a \end{Bmatrix} \qquad (6.4)$$

Where

$$\{x\} = [p \quad \dot{\beta} \quad \beta]^T$$

is the state vector for the approximation?

The characteristic equation can be obtained from Eq. (6.4) in the cubic form as

$$|\lambda I - A| = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

which factors into a form to provide estimates for both the Dutch-roll and roll response modes, i.e.,

$$(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2\,)_{DR}(\lambda + \lambda_{roll}) = 0$$

Example 6.2

Use the Dutch-roll approximation to estimate the lateral-directional dynamics of the DC-8 aircraft in the cruise-flight configuration. Applicable dimensional stability derivatives from Example 6.1 will be applied to Eq. (6.4) for the analysis.

Solution

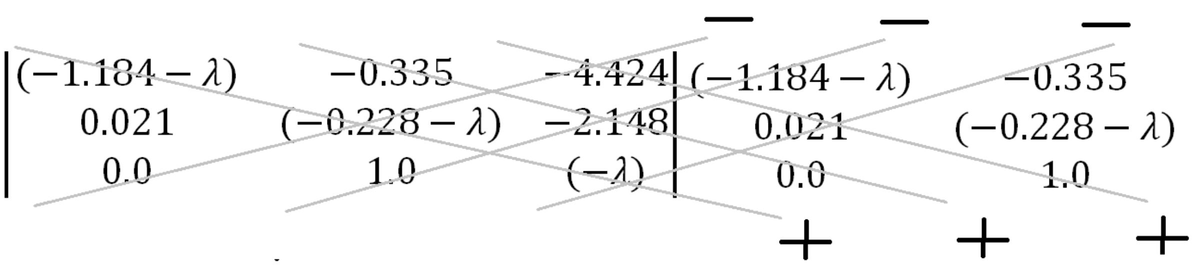
The plant matrix is

$$[A] = \begin{bmatrix} -1.184 & -0.335 & -4.424 \\ 0.021 & -0.228 & -2.148 \\ 0.0 & 1.0 & 0.0 \end{bmatrix}$$

1) The cubic characteristic equation from $|A - \lambda I| = 0$ is

$$\begin{vmatrix} (-1.184 - \lambda) & -0.335 & -4.424 \\ 0.021 & (-0.228 - \lambda) & -2.148 \\ 0.0 & 1.0 & (-\lambda) \end{vmatrix} = 0$$

$$\begin{vmatrix} (-1.184 - \lambda) & -0.335 & -4.424 \\ 0.021 & (-0.228 - \lambda) & -2.148 \\ 0.0 & 1.0 & (-\lambda) \end{vmatrix} \begin{matrix} (-1.184 - \lambda) & -0.335 \\ 0.021 & (-0.228 - \lambda) \\ 0.0 & 1.0 \end{matrix}$$



$$(-1.184 - \lambda)(-0.228 - \lambda)(-\lambda) + 0 + (-4.424 * 0.021 * 1.0) - 0$$
$$- (1)(-2.148)(-0.228 - \lambda) - (-\lambda)(0.021)(-0.335) = 0$$

$$-0.26995\lambda - 1.184\lambda^2 - 0.228\lambda^2 - \lambda^3 - 0.0929 - 2.5432 - 2.148\lambda - 0.00704 = 0$$

$$-\lambda^3 - 1.4120\lambda^2 - 2.4250\lambda - 2.6361 = 0$$

$$\lambda^3 + 1.4120\lambda^2 + 2.4250\lambda + 2.6361 = 0$$

which may be factored as

$$(\lambda^2 + 0.2026\lambda + 2.1798\,)_{DR}(\lambda + 1.2093_{roll}) = 0$$

2) The three characteristic roots (eigenvalues) are as follows.
Dutch-roll roots,s^{-1}:

$$\lambda_{1,2} = -0.1013 \pm 1.4730i$$

Roll response root, s^{-1}:

$$\lambda_3 = -1.2093$$

The information contained in the Dutch-roll eigenvalue may be interpreted as

$$\zeta_{DR} = 0.0686\,, \qquad \omega_n = 1.4730\ s^{-1}\ , \qquad \omega_d = 1.4764\ s^{-1}$$

The eigenvector for the Dutch-roll approximation, when normalized with respect toβ , is

$$\begin{vmatrix} -1.184-(-0.1013+1.4730i) & -0.335 & -4.424 \\ 0.021 & -0.228-(-0.1013+1.4730i) & -2.148 \\ 0.0 & 1.0 & -(-0.1013+1.4730i) \end{vmatrix} \begin{Bmatrix} p \\ \dot{\beta} \\ \beta \end{Bmatrix} = 0$$

$$(-1.0827 - 1.4730i)p - 0.335\dot{\beta} - 4.424\beta = 0 \ldots\ldots(1)$$

$$0.021p + (-0.1267 - 1.4730i)\dot{\beta} - 2.148\beta = 0 \ldots\ldots(2)$$

$$\dot{\beta} + (0.1013 - 1.4730i)\beta = 0 \ldots\ldots(3)$$

$$\dot{\beta} = (-0.1013 + 1.4730i)\beta \ldots\ldots(*1)$$

$$0.021p + (-0.1267 - 1.4730i)(-0.1013 + 1.4730i)\beta - 2.148\beta = 0$$

$$0.021p + (0.03456 - 0.037414i)\beta = 0$$

$$p = (-1.6457 + 1.7816i)\beta \ldots\ldots\ldots(*2)$$

From Eq. (*1) and (*2)

$$\{x\}_{DR} = \begin{Bmatrix} p \\ \dot{\beta} \\ \beta \end{Bmatrix} = \begin{Bmatrix} 2.425\ \angle 132.73\ deg \\ 1.476\ \angle 93.93\ deg \\ 1.000\ \angle 0.00\ deg \end{Bmatrix}$$

3) The roll response root corresponds to a time constant of

$$\tau_{roll} = -\frac{1}{\lambda_3} = 0.826\ s$$

The mode shape for the roll mode, when normalized relative to the roll rate component, is

$$\{x\}_{roll} = \begin{Bmatrix} p \\ \dot{\beta} \\ \beta \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -0.008 \\ 0.006 \end{Bmatrix}$$

The roll angle component can be estimated is

$$\{\dot{x}\}_{roll} = \lambda_{roll}\{x\}_{roll} \rightarrow \phi = \frac{1}{\lambda_3} p = -0.826$$